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(23 TO 26)

CONTENTS

VOLUME I

CHAPTER I

ELEMENTARY PROPERTIES OF FUNCTIONS

CHAPTER II

DIFFERENTIATION OF FUNCTIONS OF A SINGLE VARIABLE. APPLICATION TO ALGEBRAIC FUNCTIONS

CHAPTER III

SUCCESSIVE DERIVATIVES. APPLICATION TO ALGEBRAIC FUNCTIONS

CHAPTER IV

PROPERTIES OF FUNCTIONS ASSOCIATED WITH THE EXISTENCE OF DERIVATIVES. APPLICATION TO ALGEBRAIC FUNCTIONS

CHAPTER V

CURVATURE OF A PLANE CURVE. CARTESIAN COORDINATES

CHAPTER VI

THE TRIGONOMETRICAL FUNCTIONS

CHAPTER VII

THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS. ASSOCIATED FUNCTIONS (HYPERBOLIC, EPICENE, EXPOCYCLIC)

CHAPTER VIII

CURVES VARIOUSLY SPECIFIED. POLAR COORDINATES, LINE COORDINATES, RADIUS-RECEDENT RELATION

VOLUME II

CHAPTER IX

INTEGRAL CALCULUS. INTEGRATION

CHAPTER X

SYSTEMATIC METHODS OF INTEGRATION

CHAPTER XI

APPROXIMATE INTEGRATION, NUMERICAL AND MECHANICAL

CHAPTER XII

LENGTHS OF ARCS OF PLANE CURVES

CHAPTER XIII

AREAS OF PLANE FIGURES WITH CURVILINEAR BOUNDARIES

CHAPTER XIV

VOLUMES OF SOLID FIGURES BOUNDED BY CURVED SURFACES

CHAPTER XV

AREAS OF SURFACES OF SOLID FIGURES

CHAPTER XVI

MOMENTS AND MEAN DISTANCES. WORKING NOTIONS FOR DIMETRIC AND TRIMETRIC INTEGRALS

CHAPTER XVII

APPROXIMATE REPRESENTATION OF FUNCTIONS BY THE METHODS OF MOMENTS AND LEAST SQUARES. CURVE FITTING

CHAPTER XVIII

TAYLOR'S THEOREM WITH INTEGRAL FORM OF REMAINDER. APPLICATIONS

CHAPTER XIX

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

CHAPTER XX

LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER. CASE OF CONSTANT COEFFICIENTS. OUTLINE FOR HIGHER ORDERS

CHAPTER XXI

ARITHMETIC BASIS OF THE INFINITESIMAL CALCULUS

CHAPTER I

ELEMENTARY PROPERTIES OF FUNCTIONS

1. Introduction.

The primary objects of thought, or elements, with which the Differential Calculus [1] and the Integral Calculus deal, are the same as those of Arithmetic and Algebra, namely, numbers and the fundamental operations [2] with numbers. The characteristic feature of the Differential Calculus is the systematic development of the notion of *instantaneous rates of change of continuously varying magnitudes*. In the Integral Calculus the notion of *mean values* takes the place of the notion of rates in the Differential Calculus, but the two subjects are found to be intimately connected so that, from one point of view, the Differential Calculus dominates the whole. The two subjects together are usually described as *The Infinitesimal Calculus*, or more shortly as *The Calculus*.

The notion of an instantaneous rate is a familiar one. Thus in the case of a body in motion we are accustomed to distinguish between the instantaneous rate of motion (velocity at the moment) and the average rate of motion (average velocity) during an interval. So in the case of a motor we distinguish between the instantaneous rate of work (horse-power at the moment) and the average rate of work (average horse-power) during an interval.

A rate, whether average or instantaneous, involves a comparison of the changes of two quantities. Thus in the case of motion, rate of motion involves a comparison of distance traversed with time elapsed. In speaking of a rate of change of any quantity, therefore, a second quantity with respect to which the rate is considered, (that is, with whose change the change of the first quantity is compared), must be stated or implied. The adjective *instantaneous*, which is

[1] The adjective *differential* refers to the indefinitely small quantities (differentials or infinitesimals) in terms of which the subject was expounded by Leibniz, (1646-1716).

A *Calculus* is a mathematical method of investigation or the subject which results from the application of the method.

[2] These are usually enumerated as addition, subtraction, multiplication, division, potentiation, radication and logarithmation.

naturally used where the second quantity is time elapsed, is also applied, by analogy, in cases where the time is not involved. It is often convenient to prefix the name of the second quantity to the word 'rate.' Thus *velocity* is described as *the time-rate of increase of distance traversed.* Other common examples of rates are time-rate of work (power developed), distance-rate of work (force exerted), mileage-rate of consumption of oil (oil per mile), petrol-rate of miles traversed (miles per gallon), and time-rate of transformation of matter (for example, rate of decay of radium).

An *average rate of increase* is *measured* by the ratio of the increase of the measure of the quantity whose rate is considered to the corresponding increase of the measure of the quantity with respect to which the rate is considered. The corresponding *average rate of decrease* is the negative of the average rate of increase.

Thus if a body (point) is moving along a straight line and x_1 is its distance from a fixed point O on the line at time t_1, while x is its distance at another time t, the respective increments of distance and time are $x - x_1$, $t - t_1$, and the average velocity in the interval from t_1 to t is measured by $(x - x_1)/(t - t_1)$. To make the formula general we must suppose the measurement x to be algebraic, that is, x (or x_1) is positive or negative according as the moving point is on one side (the positive side) or the other (the negative side) of the origin O of measurement at the time t (or t_1). If the time t is subsequent to t_1, the average rate of increase or the average velocity is positive or negative according as x has increased or decreased (algebraically) in the time $t - t_1$. The measure of the average velocity will, of course, depend on the units employed for the measures of distance and time.

The average rate will depend on the interval, $t - t_1$, of time chosen unless the motion is such that the distance described is *always* proportional to the time elapsed. When this proportionality exists, the motion is said to be *uniform*. The average rate for *any* interval is then equal to that during a unit of time, that is, it is measured by the distance described in unit time. For this reason the measure of an average velocity is commonly described as the distance per unit time. The corresponding language is applied to any other rate when the increments compared are always in the same ratio.

If, now, we pass from average to *instantaneous* rates, we pass from a number which we could think of as calculated from direct observation of changes in the measures of the two quantities concerned, to a number which measures an instantaneous property

associated with those changes, and which, therefore, implies an additional conception or assumption as to the manner in which the changes are taking place. Going back to the measure $(x - x_1)/(t - t_1)$ of average velocity, a good working notion of the conception involved in the introduction of instantaneous velocity is conveyed by the statement that we suppose all the average velocities to close up to equality when the intervals of time for which they are reckoned lie in a period which is continually and indefinitely diminished, or more shortly, that the average velocities are equal in the ultimate detail of the motion. Otherwise said, the assumption is that the motion is *uniform in ultimate detail.* The common value to which the average velocities close up is called the *limit* of those velocities. It is also called the *ultimate ratio* of the increments $x - x_1$, $t - t_1$.

Exactly similar remarks apply to any other rate. The existence of an instantaneous rate of change of one quantity with respect to another implies that the change of the first quantity is *ultimately uniform* with respect to the second. The instantaneous rate is then the limit to which all the average rates close up when the changes of the second quantity (from a fixed value) are all continually and indefinitely diminished.

The formal exposition and systematic development of the familiar notions here introduced constitute the subject of Differential Calculus, of which we commence the discussion in Chap. II. In the present Chapter we consider a number of preliminary questions, most of which are concerned with the methods employed to deal rigorously with properties of ultimate detail or limits.

1. 1. Two general remarks may here be interpolated. The first concerns the introduction of the conception of an instantaneous rate in scientific theories. So far as Pure Mathematics is concerned, the Differential Calculus is developed from a consideration of those variable mathematical quantities, defined by formulae or otherwise, whose variations can be demonstrated to have the character required for the existence of an instantaneous rate. The incitement to such a development, however, arises from the impression that the conception of such rates is useful in scientific investigations. This impression is associated with a general tendency in science to think of phenomena as simpler in detail than in gross. Thus in the case of motion, the tendency is to suppose the motion to be more and more uniform during a smaller and smaller interval of time, and, generally, a change of any kind to take place uniformly in ultimate detail. But modern Physics is by no means bound by such a conception, and it is, in fact, led to regard many phenomena that seem to support the notion of ultimate uniformity as arising from statistical averaging, over very small intervals of time, of events whose laws of change are as complicated as those for whose explanation the conception of ultimate

uniformity was introduced. The usefulness of the Calculus in Physics remains, but it must, like other theoretical aids to the investigation of natural phenomena, be employed with a consciousness of the difference between a logical conclusion from the mathematical theory and a demonstration of the existence of phenomena suggested by that conclusion. All theories, mathematical and other, have limits of applicability which can only be discovered by actual observation or experiment.

The second remark is of a more practical character but has the same general trend. The Differential Calculus is quite commonly employed in Engineering in the discussion of problems in which the assumption of uniform variation in ultimate detail is in obvious conflict with the physical data. As a simple example we may take the question of the form assumed by a chain (consisting of links of any kind) when suspended by its ends. A solution which is of sufficient accuracy for most purposes is obtained by replacing the chain by a continuous linear system (an ideal perfectly flexible rope), whose form is specified by a mathematical curve. The problem in its modified form permits of the use of the Calculus and is easily solved, whereas the original problem would require a consideration of the equilibrium of each link. The Calculus is thus, from the practical point of view, on some occasions an instrument of simplification, on others, an instrument of refinement.

Delicate questions may evidently arise concerning the practical availability of the solutions obtained by smoothing-out processes of the kind indicated. We are not here concerned with such questions, which, on the mathematical side, belong to the theory of statistical averaging. It is, however, important to note that a set of measures of a physical quantity, however numerous they may be, can never be in conflict with the assumption of the existence of an instantaneous rate of variation of the quantity. In other words, a law of variation can always be found which is consistent both with the set of records and with the existence of an instantaneous rate.

2. Physical and Abstract Numbers.

We now proceed to consider the character and generality of the system of numbers employed in the Calculus. In the applications of the subject the data involve *physical numbers*, that is, numerical measures of the physical quantities that enter into the discussion. As soon, however, as the problem to be dealt with is put into mathematical form, with the aid of the laws to which the variation of the physical quantities are subject, those numbers which have physical associations are treated exactly like any other numbers which enter, and we are concerned only with *abstract numbers* until the mathematical investigation is completed or until it is interrupted by further reference to the physical data.

Throughout the present Volume we shall be concerned only with theorems involving the ordinary numbers of Arithmetic and Algebra, with which the student is familiar. These numbers are called

simplex (or *real* [1]) to distinguish them from *complex* (or *imaginary*) numbers. A complex (more properly *duplex*) number is so called because two ordinary (or simplex) numbers are required for its specification. The word 'number' is henceforth to be taken to refer to a simplex number, unless the contrary is indicated.

3. The Number-System of the Calculus of Real Variables.

The word *number* has received many extensions of meaning since the time when it had reference solely to counting, and it has consequently become necessary, in any mathematical subject, to explain what number-system is employed. The reader will be familiar with the successive extensions which have resulted in the number-system employed in the Differential and Integral Calculus of real variables. We enumerate them, without regard to historical order, partly for the sake of explaining the nomenclature, but chiefly in order to emphasize certain points on which the rigour of the subsequent discussion depends.[2]

Starting with the natural numbers 1, 2, 3, ... of enumeration, we proceed to the following extensions :

3. 1. *Integers, positive and negative.* The importance of negative integers, from the point of view of abstract mathematics, is due to the fact that they render subtraction always possible, so that there is always a solution of the equation $a+x=b$, namely $x=b-a$, where a, b, x are integers. Otherwise said, the inversion of the process of addition has become always possible by the introduction of negative integers, whereby not only is there always a definite sum of two integers, but there is always a definite integer, positive or negative, which on addition to a given integer a produces a given integer b. It is convenient to include 0 amongst the integers ; it is neither positive nor negative.

3. 2. *Rational* [3] *fractions* (*vulgar fractions*), *positive and negative.* Rational fractions were introduced into the number-system for the measurement of continuous or quasi-continuous quantities by means of the division of a unit. Their importance in abstract mathematics is due to the fact that they render division

[1] The adjective *real* is still commonly used while the correlative *imaginary* has been for the most part abandoned. These adjectives go back to a time when complex numbers were regarded as a mysterious excrescence on ordinary numbers.

[2] The present discussion is merely introductory. The subject is dealt with in a more complete manner in Chap. XXI.

[3] The adjective *rational* in this connection has reference to the fact that a vulgar fraction is the *ratio* of two integers.

always possible, so that there is always a solution of the equation $ax = b$, namely $x = b/a$, where a, b are integers or fractions. Otherwise said, the inversion of the process of multiplication has become always possible by the introduction of rational fractions, whereby not only is there always a definite product of two numbers (integral or fractional), but there is always a definite number which on multiplication with a given number a produces a given number b. To this statement, however, there is an exception, which is due to the inclusion of 0 amongst the integers. There is no definite number which on multiplication by 0 produces a given number b (other than 0), while any number on multiplication by 0 produces 0. Thus $b/0$ is not a number and $0/0$ may be *any* number, or is indeterminate. We shall have many opportunities of explaining the method of handling these special forms when they arise.

For the sake of generality of statement it is often convenient to regard the integers as a particular class of rational fractions. Thus we may suppose the integer a replaced by the rational fraction $a/1$.

3. 3. *Irrational* [1] *numbers, or irrationals.* We may look on the introduction of *irrationals* as primarily due to the attempt to imitate in Arithmetic the *continuity* involved in our conception of a straight line. It is easy to shew that, according to this conception, if the length of a certain line is measured by an integer, or a rational fraction, in terms of a unit, that line can be divided into two parts, by a geometrical construction, such that neither part is measured by any rational fraction, its length being incommensurable with the unit. If, for example, we construct an isosceles right-angled triangle on the line as hypotenuse and strike off on it a part equal to one of the sides, that part will be incommensurable with the whole or with the unit, and will be measured not by a rational fraction but by an *irrational* number.

From the standpoint of abstract mathematics the introduction of irrationals is required to generalize the process of radication [2] (taking roots). Otherwise said, irrationals are required to invert the process of potentiation (taking powers), whereby there shall not only be a definite integral power of any number, but also a definite number which, when raised to a given integral power, produces a given number, (which must be positive if the index is even).

The process of converting rational fractions into decimals may be

[1] The word *irrational* here implies merely that the numbers in question are not rational fractions (or integers).

[2] This operation is cited as the most familiar case.

quoted as providing a clue to the simplest method of defining irrationals. This process gives rise to a decimal of a finite number of places only when the denominator [1] is a power of 10, or a factor of such a power. In other cases it gives a periodic non-terminating decimal in which a set of digits is repeated indefinitely. Conversely, if a decimal of this character is written down, a method is known for finding a vulgar fraction which will generate it by applying the conversion-process. On account of the invariable association of a vulgar fraction with such a periodic decimal, we easily get to associate the notion of a definite number with the latter without using the support of the former.

We have next to associate the same notion of number with any non-periodic non-terminating decimal, on the understanding that there is given or implied some law or criterion which determines its successive digits. Such a decimal is called an irrational number. The propriety or feasibility of introducing such decimals as part of the number-system depends on two conditions,

(i) that they can be assigned a definite *order* with respect to the rational fractions and to each other,

(ii) that they can be subjected to the same operations, (addition, multiplication, etc.), and in such a way as to obey the same fundamental laws [2] of those operations, as the integers and rational fractions.

In considering these conditions it is convenient to suppose all integers and rational fractions expressed in the form of periodic non-terminating decimals, so that *all* numbers have a common form of expression. Thus we write $7{\cdot}999\ldots$ for 8, $\cdot 33999\ldots$ for $\cdot 34$, $\bar{1}{\cdot}999\ldots$ for 0, $\bar{3}{\cdot}999\ldots$ for -2, and so on.[3] This convention often dispenses with the consideration of special cases in the proof of theorems concerning the existence of a number possessing a certain property, and it will be adopted henceforth in such proofs.

We can now lay down the rule for the relative order (of magnitude) of two numbers by means of which the above condition (i) is satisfied. Of two numbers which agree up to a certain number of digits but which differ in the next digit, (or place of decimals), that one precedes (or is less than) the other which has the (algebraically)

[1] The fraction is supposed in its lowest terms.

[2] That is, the associative, commutative, distributive and index laws.

[3] The decimal parts of the numbers so written are positive, as in the case of common logarithms.

smaller digit in that place. The complete discussion of the condition (ii) must be left to Chap. XXI, but the student has already obtained from Arithmetic a general knowledge of the manner in which the successive digits of the number arising from one of the fundamental operations are to be obtained. In each case, the law for the successive digits in each of the numbers operated with leads to a law for the successive digits of the number to be obtained as the result of the operation. The resulting number thus belongs either to the rationals or to the irrationals and no new class of numbers is introduced by any of the operations. From this point of view the rational and irrational numbers together form a complete system, commonly called the system of *real* numbers.

4. Representation of Numbers by Letters.

The representation of numbers by letters is applied to all members of the system of real numbers, without any distinction being made between rational and irrational numbers, as they all obey the same laws. Since, however, irrational numbers cannot be expressed by a finite number of digits, the practice has been adopted of representing those of frequent occurrence by special letters permanently associated with them; thus the Greek letter π is reserved for the ratio of the circumference of a circle to its diameter, e is reserved for the base of natural logarithms, and so on. The same practice could, of course, be adopted for special rational numbers expressed by a large number of digits, but numbers of this kind which are of theoretical importance scarcely occur in Pure Mathematics. In Physics, fundamental constants are in the same way represented by special letters. A familiar case is the use of g for the acceleration of gravity at the place of observation. These constants are deduced from observations and experiments, and are only known approximately. It would be very much better to use special symbols, or letters of alphabets not otherwise drawn on, for the representation of special numbers, since ordinary letters can ill be spared for this purpose, but the general practice is here followed.

5. Modulus of a Real Number.

Since a letter, a say, may represent either a positive or a negative number and it is often necessary to consider its magnitude,[1] it is convenient to have a symbol for it, and $|a|$ has been introduced

[1] The word *magnitude* will, in the sequel, be used for *numerical value* as contrasted with *algebraic value*.

for the purpose. Thus if a is a positive number, a and $|a|$ are the same number, while if a is negative, $|a|$ has the same magnitude as it but is positive. As particular examples, $2=|2|$ but $-2=-|-2|$. The number $|a|$ is called the *modulus* of a or simply 'mod. a'.

Four very simple results which are conveniently expressed with the aid of the modulus sign should be noted, as they are of constant occurrence. If a, b are any two numbers, then

(i) $|a+b| \leqslant |a|+|b|$;

(ii) if, further, a, b are of the *same* sign, $|a+b|=|a|+|b|$;

$$\text{e.g. } |3+4|=7=|3|+|4|, \text{ and}$$
$$|-3-4|=|-7|=7=|-3|+|-4|;$$

(iii) if, however, a, b are of *different* signs, $|a+b|<|a|+|b|$;

$$\text{e.g. } |3-7|=|-4|=4 \text{ while } |3|+|-7|=10;$$

(iv) $|ab|=|a|.|b|$; e.g. $|-3\times 4|=|-12|=12=|-3|.|4|$.

These theorems can be immediately extended to three or more numbers.

Ex. 1. If $a_1, a_2, \dots, a_n$ are alternately positive and negative and if the modulus of each (except the first) is less than that of the next preceding, shew that

(i) the sum $a_1+a_2+\dots+a_n$, which we denote by s_n, has the same sign as a_1;

(ii) s_n lies between a_1 and a_1+a_2; also $|a_1-s_n|>|a_2+a_3|$, $(n>3)$, and $|s_n-(a_1+a_2)|>|a_3+a_4|$, $(n>4)$;

(iii) s_n lies between $a_1+a_2+\dots+a_r$, or s_r, and $a_1+a_2+\dots+a_r+a_{r+1}$, or s_{r+1}, where $1\leqslant r<n-1$; also $|s_n-s_r|>|a_{r+1}+a_{r+2}|$, $(n>r+2)$.

It is sufficient, in the proof, to suppose a_1 positive, the sign of every term being changed to obtain the results when a_1 is negative.

(i) If n is even, write $s_n=(a_1+a_2)+(a_3+a_4)+\dots+(a_{n-1}+a_n)$, and if n is odd, write $s_n=(a_1+a_2)+(a_3+a_4)+\dots+(a_{n-2}+a_{n-1})+a_n$. In these forms, each compound term is positive and so is a_n in the second form. Therefore s_n is positive, so that it has the sign of a_1. Also $s_n>a_1+a_2$ and $|s_n-(a_1+a_2)|>|a_3+a_4|$ if $n>4$.

(ii) If n is even, write $s_n=a_1+(a_2+a_3)+\dots+a_n$, and if n is odd, write $s_n=a_1+(a_2+a_3)+\dots+(a_{n-1}+a_n)$. In these forms, each compound term is negative and so is a_n in the first. Therefore s_n is less than a_1. Further, $|a_1-s_n|>|a_2+a_3|$ if $n>3$.

(iii) Writing s_n in the form $(a_1+a_2+\dots+a_{r-1})+(a_r+a_{r+1}+\dots+a_n)$, we can apply the results (ii) to the second sum and assert that it lies between a_r and a_r+a_{r+1}. It follows that s_n lies between $a_1+a_2+\dots+a_{r-1}+a_r$, or s_r, and $a_1+a_2+\dots+a_{r-1}+a_r+a_{r+1}$, or s_{r+1}. Further,

$$|s_n-s_r| = |a_{r+1}+a_{r+2}+\dots+a_n|>|a_{r+1}+a_{r+2}|$$

if $n>r+2$.

6. Correspondence of Real Numbers and the Points of a Line.

Let $X'X$ (Fig. 1) be any straight line and let the portion OA be taken as a unit line, so that O is associated with (or corresponds to)

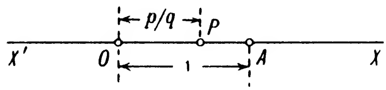


FIG. 1.

the number 0 and A with the number 1. Then we know how to associate with any proper fraction p/q a definite point on that portion. To do this we divide OA into q equal parts and take the first p parts, starting from O, to form a line OP whose end is P. The point P is then associated with, or corresponds to, the fraction p/q. We cannot reverse this process to find a proper fraction corresponding to any point P' of the line, for this would require all lines such as OP' to be commensurable with OA, and this we know not to be the case. We can, however, associate with P' a definite real number (in general irrational) by the following process. Let the line OA be divided into ten equal parts and let the point P' fall either inside the (p_1+1)th part, (reckoning from O), or at its end remote from O. Now divide this part into ten equal parts and let P' fall either inside its (p_2+1)th part or at its end remote from O, and so on. This process of successive subdivisions gives a criterion to determine in turn each of the digits $p_1, p_2, p_3, \ldots$ of a definite non-terminating decimal, namely $\cdot p_1 p_2 p_3 \ldots$, that is, to determine a real number (rational or irrational) which we associate with the point P'.

If we reverse the process and begin with a number $\cdot p_1' p_2' p_3' \ldots$, we can make the successive subdivisions and select the part in each in accordance with the successive digits of that number, but it is a matter of postulation or of our conception of a line, not a matter of proof, if we assert that a definite point of the line is defined by this process. If this postulate is granted, the conclusion is that the process explained associates a definite number with each point of OA and a definite point with each number between 0 and 1. In other words, the process is said to effect a *one-to-one*, or *one-one, correspondence* of the points on the line and the numbers from 0 to 1. The one-one correspondence can, of course, be extended beyond the chosen range in the same way.

No argument in the analytical theory is based on this supposed correspondence, but it has led to the use in that theory of words

which properly belong to Geometry. Thus it is quite usual to employ the word *point* instead of the word *value* and to call the real numbers from a to b the *range* from the point a to the point b. Another case is noted in the next Article. Such metonymy is sometimes very convenient, but it may, on occasion, be confusing and, unless carefully watched, may lead to arguments which are wanting in rigour.

7. The Arithmetic Continuum.

The property proved in the preceding Article that there is a real number lying in the range 0 to 1 corresponding to each point on the line OA, and its extension to any range, has led to the application of the word *continuum*, which properly belongs to our conception of a line, to the system of real numbers, which is hence said to form the *arithmetic continuum*. No difficulty arises from this transference of a name as long as it is remembered that it does not justify looking on the number-system as other than essentially *discrete*. Such 'continuity' as exists in the number-system arises from continued interpolation between isolated numbers, that is, in number-construction from without inwards, and it is not of the same character as the inherent continuity involved in our conception of a straight line. The connection between such a conception of inherent continuity and any conception (ancient or modern) of space based on experimental evidence, is a matter which does not concern us here.

8. The Symbols $+\infty$, $-\infty$.

The system of real numbers is said to be unbounded (or infinite) in the same sense as the system of integers is said to be unbounded (or infinite). A number always exists which is algebraically greater (less) than any assigned number; it is always possible to add to (subtract from) an assigned number another assigned number as often as we please, and in this way to arrive at a number which is greater (less) than any third assigned number. The word *infinite* and the symbols $+\infty$, $-\infty$ do not apply to a single number but to a *set* or *sequence of numbers* which, when considered in succession, attain values algebraically greater (less) than any assigned or assignable number. For example, when the system of positive integers is indicated by $1, 2, 3, \ldots, n, \ldots, \infty$, the symbol ∞ does not indicate the *last* number, but only that the system is unbounded. Again, if we write $a/0 = \infty$, we mean to indicate that if the denominator is at first a number other than zero and is then continually

decreased in magnitude, the fraction will attain a magnitude greater than any assigned number. We shall have occasion to use the symbol ∞ in various connections, but its general significance is always the same.

9. Aggregates, Barriers, Upper and Lower Bounds of Aggregates.

The term *aggregate* is applied in Mathematics to a finite or an infinite set of elements *regarded as a whole.* For the present we consider only numerical aggregates, each element or member of which is a single real number.[1] The identification of a number as an element of the aggregate will, in general, be by its possession of a property which characterizes the elements. For example, the reciprocals of the positive integers form an infinite aggregate.

We proceed to give a fundamental theorem concerning aggregates, (finite or infinite), which is evident when geometrical intuition is used in aid of arithmetical notions, but which requires formal proof as a purely arithmetic result.

If a number N exists which is algebraically greater than any (every) element of the aggregate, we say that the aggregate is *barred above* and that N is an *upper barrier* of the aggregate. In the same way, if a number N' exists which is algebraically less than any (every) element, the aggregate is *barred below* and N' is a *lower barrier.* The barriers N, N' are clearly not unique; any number greater than N is an upper barrier and any number less than N' a lower barrier. It would be natural to assume that the existence of an upper barrier implies the existence of a *definite smallest* upper barrier and, similarly, that a lower barrier implies a *definite greatest* lower barrier. These assumptions are not tenable in the form stated and the theorem which we are about to prove concerning upper and lower bounds replaces them by accurate conclusions. Two simple examples shew clearly why a modification is necessary.

(i) Consider first the infinite aggregate formed of the reciprocals of the positive integers, namely $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$. This aggregate is

[1] Aggregates may also be of a geometrical character. Thus with a number-aggregate we may associate an aggregate of plot-points on a straight line, so that an element x of the number-aggregate corresponds to a point of abscissa x on the line. The point-aggregate is then described as a *monometric* or *linear point-set,* and the same descriptive adjectives are applied to the numerical aggregate. On account of the postulate of one-one correspondence of real numbers and points on a line, the properties of monometric number-aggregates correspond exactly to those of monometric point-sets and the language appropriate to the latter is constantly applied to the former.

barred above, and any number, say 2, which is greater than 1 is an upper barrier. Further, there is a definite number, namely 1, to which one element is equal and which no element surpasses. The aggregate has thus a definite greatest element.[1] The number 1 is, however, not itself an upper barrier since one element of the aggregate is equal to it. There is thus no definite smallest upper barrier.

(ii) Now consider the aggregate formed by subtracting from 1 the reciprocals of the positive integers, namely $0, \frac{1}{2}, \frac{2}{3}, \dots, 1-\frac{1}{n}, \dots$. This is barred above and any number which is greater than 1 is again an upper barrier, but 1 is now itself an upper barrier since any (every) element of the aggregate is less than 1. There is in this case no definite greatest element; we can find elements as nearly equal to 1 as we please, but there is no element equal to 1. The number 1 is here the least upper barrier of the aggregate. In each of these cases the number 1 is called the upper bound of the aggregate.

The general definition of an *upper bound* of an aggregate is that it is a number such that there is no element of the aggregate which is greater than it but there is one element (at least) which is greater than any assigned number less than it. A *lower bound* is such that no element of the aggregate is less than it but there is one element (at least) which is less than any assigned number greater than it. It is clear that an aggregate cannot have two upper or two lower bounds.[2] We can now state the theorem of bounds.

9. 1. THEOREM OF BOUNDS. *An aggregate which is barred above has an upper bound, and an aggregate which is barred below has a lower bound.*

To prove the theorem for an aggregate which has a lower barrier, a say, we introduce a number b which is an upper barrier to some part of the aggregate, consisting of its elements which lie between a and b. For simplicity, a, b may be chosen to be integers. The method of successive decimal subdivisions, already illustrated in Art. 6, is now employed. The range extending from a to b is divided into ten equal parts and the lowest which has any element of the aggregate within it or at its upper end is selected. Let this be the

[1] Any *finite* aggregate has, of course, a definite greatest element and a definite smallest, when only unequal elements are regarded as distinct.

[2] Suppose that there are two upper bounds a, b and that b is greater than a. Then, since b is an upper bound, either there is an element of the aggregate equal to b or there is an element lying between b and a. In both cases there is an element greater than a, which is therefore not an upper bound, contrary to the supposition.

(p_1+1)th part, reckoning from a. Subdividing this selected part into ten equal parts, we again select the lowest which contains any element of the aggregate within it or at its upper end. Let this be the (p_2+1)th part, reckoning from the lower end of the first selected part. This process is repeated indefinitely. The set of lower ends of the selected parts defines a definite number, M say, by means of a non-terminating decimal. This number is $a+(b-a)(\cdot p_1 p_2 p_3 \ldots)$. We prove that M is a lower bound of the aggregate.

In the first place it is clear that no element of the aggregate is less than M, for the lower end of each selected part is below any element. In the second place the number M belongs to each of the selected parts, and therefore one of these can be chosen which is as small as we please, and which at the same time contains M and at least one element of the aggregate. Thus M satisfies the conditions for a lower bound.

In like manner we can prove that an aggregate which is barred above has an upper bound.

It should be observed that when one element of the aggregate is equal to the lower bound, that element may be the only one which is less than any assigned number greater than the bound, and this is always the case when the aggregate is finite.[1] If no element of the aggregate is equal to the lower bound, there will be an infinite set of elements between the lower bound and any number greater than it. For if there were only a finite set of such elements, a number could be found less than any of them and greater than the bound, and this is contrary to the nature of a bound. A similar statement applies to an upper bound.

It is now clear that there is a definite smallest upper barrier when, and only when, the upper bound is not an element of the aggregate.

10. Points of Crowding of an Infinite Barred Aggregate.

A theorem closely connected with that just proved holds of an infinite aggregate which lies between barriers a, b. It asserts that *there is at least one point in the range*[2] *from a to b such that an arbitrarily small range in which it lies contains an infinite number of elements of the aggregate.* A point satisfying this condition is called a *point of crowding*[3] of the aggregate. If there were an infinite number of elements each equal to a in the aggregate, a would plainly be a point of crowding, according to the definition, if these

[1] Equal elements here count as only one.

[2] In this description the points a, b are supposed included in the range.

[3] German *häufungspunkt*. The names *point of accumulation* and *limit-point* are also used.

elements were treated as distinct. We are usually concerned with the case in which equal elements count as only one.

The proof of the theorem[1] is based on the evident fact that if the range from a to b is divided into a finite number of sub-ranges, one of them at least must contain an infinite number of elements of the aggregate, since otherwise the total number would be finite. The process of successive decimal subdivisions accordingly applies. At each subdivision we choose that sub-range nearest the barrier a which contains an infinite number of elements. The set of lower ends of the successive sub-ranges so chosen defines a number by decimal approximation, as in Art. 9. 1, and this number lies in a sub-range which can be taken as small as we please while containing an infinite number of elements of the aggregate.

The number of points of crowding of an infinite aggregate may itself be infinite. For example, in the case of the aggregate of rational numbers in the range from a to b, each point of the range is a point of crowding.

The student will see that the applicability of the method of subdivision, used in the present Article, depends on the fact that the property of the range which is under consideration is necessarily also the property of one at least of any finite number of sub-ranges into which it is divided. The result of the application of the method may be described as a (not necessarily unique) *localization of the property at a point of the range.*

Ex. 1. A bound of an aggregate is certainly a point of crowding if it is not an element of the aggregate ; it may be if it is an element.

Ex. 2. The infinite aggregate of reciprocals of integers (which lies in the range -1 to $+1$) has the point of crowding 0.

Ex. 3. The infinite aggregate of values of $n/(n+1)$, where n is a positive integer, has a point of crowding at 1, which is an upper bound and a barrier of the aggregate.

11. Constants and Variables.

When, in an investigation, a letter represents the same number throughout, that letter is said to represent a *constant number* or, more shortly, to be a *constant.* It has become traditional in Pure Mathematics to use the earlier letters a, b, c, ... of the alphabet for constants. Sometimes the Greek letters α, β, γ, ... are used for the same purpose. The letters may also be accented or distinguished by means of other affixes.

When, in an investigation, a letter represents *some* number and the discussion involves the *change* of that number, either with or without restriction, that letter is said to represent a *variable number* or, more shortly, to be a *variable.* For example, when we are considering the motion of a particle during a certain time, if the letter t is used as the measure of the time elapsed up to *some* instant, t varies continually during the motion and is a mathematical variable.

[1] Known as the *Bolzano-Weierstrass Theorem.*

From a slightly different point of view, a mathematical variable may be described as a set or aggregate (finite or infinite) of numbers which are thought of as a whole and which are represented (in turn) by the same letter. It is usual to employ the later letters of the alphabet for variables, x, y, z being commonly introduced in order when there is no sufficient reason to the contrary. Sometimes it is convenient to use the Greek letters ξ, η, ζ in the same way when x, y, z have been already employed or are otherwise contra-indicated. The tradition in such matters can only be learnt with the subject and is, of course, not dictatorial. Particular values assigned to a variable, x say, are often denoted by that letter with the addition of affixes. Thus x_1, x_2, ... may be introduced for the particular values as they occur, the suffixes 1, 2, ... having no reference to the magnitudes of the numbers represented.

In Applied Mathematics there is a natural tendency to use for variables and constants, which represent measures of physical quantities, letters (or symbols) which in some way recall them; for example, the initial letter of the name of a physical quantity is often conveniently used as the symbol for its measure. The practice in Dynamics of using v for the measure of a velocity, F for that of a force, α for that of an acceleration, and so on, illustrates the tendency referred to. There is further a consensus of opinion in favour of a standardization of the letters to be employed, and considerable progress has been made in this direction both in Physics and in Engineering. We shall conform to the tradition explained above in all abstract investigations and when we proceed to applications change the letters to those which are more natural in the subjects concerned.

If the values of a variable x which are considered lie *between* two numbers a and b, x is said to lie *within* the range defined by the terminal values a, b. If the values of x considered *include* the terminal values, x is said to lie *in* the same range. In the latter case, the range of x is indicated by the symbol (a, b), which is described simply as 'the range of x'. A more complete symbolism for ranges is introduced later, Art. 22. 2.

12. Linear Plot of a Variable.

The one-one correspondence of real numbers and points on a line, which has already been discussed, is utilized in the specification of a point on a straight line by a number and in the converse representation of a number by a point on the line, which is called the

plot-point of the number. The process is now described in its practical form in terms of lengths and their measures. We take a straight line $X'X$, unlimited in both directions, and a point O, called the origin (of measurement), on it. We distinguish between the direction of description of the line from X' to X and that from X to X', the former being called the positive, the latter the negative, direction or sense of description. Points which are reached from O by proceeding in the positive (negative) direction are said to be on the positive (negative) side of O. See Fig. 1, p. 10.

We first choose the length, OA say, which is to be measured by 1, that is, the unit of length. A point P on the positive side of O is then specified by (the measure of) the length of OP, and a point P' on the negative side of O by minus the length of OP'. Conversely, the plot-point P of a positive number a is the end P of the line drawn from O in the positive direction and of length a, the plot-point P' of a negative number $-a$ is the end of the line drawn from O in the negative direction and of length a. The possibility of plotting irrational numbers is of theoretical but not of practical importance. In actual plotting, unavoidable inaccuracies make the retention of many places of decimals in the numbers plotted useless.

When the numbers plotted are values of the variable x, the line of representation is usually described as 'the line Ox', $x'Ox$ taking the place of $X'OX$. The point P will then be the plot-point of a value x, and a particular point P_1 the plot-point of x_1. It is customary to say shortly that '*P is the point x*', and to write P, (x) as a compound symbol assigning both point and number.

13. Function of a Real Variable.

13. 1. *Introduction.* We have seen in Art. 1 that rates, which are the primary subject of discussion in the Differential Calculus, involve a comparison of the simultaneous increments of two variables. Let us take again the familiar case of the velocity of a particle (or point) which is supposed to be moving along a straight line in a definite manner. The quantities brought into comparison are here an increment of distance and the interval of time in which it is described. When we say that the particle is moving in a definite manner, we imply that the (measure of the) distance, x say, of the particle from a fixed origin is related to the (measure of the) time elapsed, t say, from a fixed epoch, in such a way that to a specified value, t_1, of t there corresponds a definite value, x_1, of x. From

the standpoint of Pure Mathematics, x, t are two variables between which a correspondence of values is created by the mode of motion of the particle considered, the value x_1 corresponding to the value t_1, and so on. This property is described by saying that the variable x is a *function* of the variable t, and the law of motion of the particle is said to create or determine a *functional relation* between the two variables.

Let us now reverse the order of the steps and begin by supposing that by some purely mathematical means a correspondence is effected between the values of x and t, which are, for the present, mere number-variables without physical significance. When a value t_1 of t is chosen, the correspondence is supposed to determine a definite value x_1 of x. The variable x is again a function of t, the functional relation being now created by purely mathematical means. The most obvious means of creating such a relation is by a formula or expression which contains t and to which x is to be always equal. For example, the simple formula

$$x = at^2 + bt + c, \qquad \text{(1)}$$

where a, b, c are constants, effects a correspondence which makes x a function of t. To the particular value t_1 of t corresponds the value x_1 of x, given by

$$x_1 = at_1^2 + bt_1 + c. \qquad \text{(2)}$$

If, now, we introduce a particle P and suppose it to move along a straight line OX so that x is always its distance from the origin O at the time t, the functional relation between x and t determines the mode of motion. The problem of finding the velocity of the particle at any time is then a purely mathematical one.

Let us carry out the process for the simple functional relation (1) so as to find the velocity at the time t_1. The distance described in the interval from t_1 to t_2 is found from the equations

$$x_1 = at_1^2 + bt_1 + c, \qquad \text{(3)}$$

$$x_2 = at_2^2 + bt_2 + c, \qquad \text{(4)}$$

and is therefore given by

$$x_2 - x_1 = (t_2 - t_1)\{a(t_2 + t_1) + b\}. \qquad \text{(5)}$$

The average velocity during the time $t_2 - t_1$ is $(x_2 - x_1)/(t_2 - t_1)$, that is, $a(t_2 + t_1) + b$. The instantaneous velocity at time t_1 is the value of this average velocity when the interval $t_2 - t_1$ is indefinitely diminished so that t_2, t_1 are ultimately equal. The instantaneous velocity is therefore $2at_1 + b$.

In this description of a functional relation the variable t takes the lead as being that with respect to which the other variable x

is brought into correspondence. For this reason t is called the *independent* variable, while x is called the *dependent* variable. The rôle of the two variables can usually be interchanged, a point which will be discussed at length later.

This introduction will enable the reader to understand the purport and the importance of the present Article on functions of a real variable. We are here going to study the nature of functional relations as a preliminary to the discussion of rates, which is commenced in Chap. II.

13. 2. *General definition of a function of a real variable.* The word ' function ' was, in the earlier history of Mathematics, applied exclusively to a mathematical expression (or formula) which contained one or more variables. The expression was, and still is, said to be a function of *each* of the variables. Thus the expression

$$\frac{ax^2+bx+c}{a'x^2+b'x+c'}, \qquad (6)$$

is a function of the variable x in this sense of the word. The expression, or rather the aggregate of its values, may be regarded as forming a variable of which the particular value in any case is dependent on, or determined by, the value of the variable x. Let this variable be denoted by y; then the expression effects a correspondence between x and y such that when x is given, y is determinable. When the dependence of y on x is considered in this way, x is described as the independent variable and y as the dependent variable, and y is said to be a function of x.

The word ' function ' is still used in such a case, but following on the use of more and more elaborate expressions in extending the range of functions considered, it was realized that it is desirable to make the treatment of the general theory of functional relationship independent of the nature of its expression by formulae. To do this, we take as our starting point a variable x which consists of an aggregate of numbers, with whose prescription we are not for the moment concerned. This aggregate may, for example, consist of all numbers in a finite range or within an infinite range; on the other hand, it may consist only of the integers or the rational numbers in a certain range. We now suppose that with each value of x there is associated another number, (on grounds and by means not for the moment considered), so that corresponding to the aggregate of numbers x we have another aggregate which we denote by y. The aggregate y forms a second variable, and the association of values

described brings each value of x into correspondence with a value of y. The aggregate x is called the independent variable, the aggregate y is called the dependent variable, and y is said to be a function of x. In this definition it is not required that each number in y should occur once only, that is, in association with only one value of x. As an extreme case the same number may be associated with *each* value of x. The aggregate y then contains only a single number, but it is still described as a function of x.

To illustrate the general definition of a function, we give three examples which are covered by it although not of the usual type occurring in the Calculus.

(i) The variable x consists of all the positive integers and the value of the variable y associated with the integer n is n^2 if n is even and 0 if n is odd, that is, the formula for y is $\frac{1}{2}\{1+(-1)^n\}n^2$.

(ii) The variable x consists of the real continuum and the value of y associated with x is the greatest integer which is not greater than x; when x is positive, y is simply the integral part of x. The graph of this function is shewn in Fig. 11, Art. 17. There is no simple algebraic formula in this or the next case.

(iii) The variable x consists of the real continuum and the variable y has the value 1 corresponding to rational values of x and the value -1 corresponding to irrational values.

In the general definition of a function we have supposed that only *one* value of y is associated with each value of x. When this is so, y is said to be a *single-valued function* of x. In the general discussions it is necessary to suppose that y is single-valued. When, as often occurs, y is defined by a formula or equation which gives two (or more) values for one value of x, we begin by separating the values to form two (or more) single-valued functions, which are treated separately. For example, the equation $y^2=a^2-x^2$ does not define y as a single-valued function, for it gives $y=\pm\surd(a^2-x^2)$, and y may have one of two values in the range $(-a, +a)$. In this case we consider separately the two single-valued functions defined by[1] $y=\surd(a^2-x^2)$ and $y=-\surd(a^2-x^2)$.

The condition that for any (every) assigned value of the variable in a range there is a definite value of the function, does not ensure that the function is barred above and below in the range. A simple example is sufficient to illustrate this point. Consider the function which is defined by the equation $y=1/x$ when $x\neq 0$ and which is equal to 1 when $x=0$, the range of x being $(-1, +1)$, say.

[1] Here, and in the sequel, the symbol $\surd$ indicates the positive square-root.

For any assigned value of x other than 0, $1/x$ is a definite number, and for the value 0 of x the function is, by assumption, equal to 1. The function, however, has neither upper nor lower barrier in the range, since $1/|x|$ can be made as large as we please by decreasing $|x|$. The function is not *defined* throughout the range by the formula $1/x$ alone since $1/0$ is not a number.

14. The Functional Symbol $f(x)$.

It is desirable to have a symbol for the dependent variable y which indicates the dependence on the independent variable x. Such a symbol is $f(x)$, which has been arrived at by an abbreviation of the description *function of x*. The symbol implies a prescription of some kind which, for any value, x_1 say, of the variable x, assigns a number, y_1, associated with it. That number, y_1, is represented by $f(x_1)$. The *equation* $y=f(x)$ represents the correspondence of the values of the variable y with those of x, and is said to symbolize, or simply to be, the functional relation between x and y.

When functional symbols are required for two or more dependent variables, the usual practice is to replace f in the symbol $f(x)$ by other letters. Thus $g(x)$ or $\varphi(x)$ may be used to symbolize functions, the letter replacing f being here suggested solely by alphabetical or phonic nearness to it.

During a particular investigation the letter f in the symbol $f(x)$ will, of course, imply the *same* functional relation between y and x. If other letters are used instead of y and x, such as v and u, the equation $v=f(u)$ implies that the *same* functional relation exists between v and u as between y and x, that is, if the numbers x and u are equal, so are the numbers y and v. In particular, if a formula or expression in x is known for y, say

$$y=f(x)=\frac{ax+b}{cx+d}, \qquad \text{.............................(1)}$$

where a, b, c, d are constants, then also

$$v=f(u)=\frac{au+b}{cu+d}. \qquad \text{.............................(2)}$$

The above remarks apply to general symbols which may be employed for any function. When a particular function is of sufficient importance, it receives a special symbol in which an appropriate letter, or group of letters, replaces the letter f in $f(x)$. Thus the symbols $\sin x$, $\cos x$, etc., for the trigonometrical functions are to

be looked on as special cases of $f(x)$, in which f is replaced by sin, cos, etc. Many other cases of the same kind will occur later.[1]

15. Note on the Trigonometrical Functions.

As we shall find it convenient in this Chapter to illustrate various features of functions by means of the elementary properties of the trigonometrical functions, it may be well to make a definite statement here of the connection between the trigonometrical functions of Analysis and the trigonometrical ratios defined geometrically. In Trigonometry, the sine and cosine are defined in terms of an angle, and the symbols $\sin A$, $\cos A$ are used when that angle is A. When a measure of A, say α, is introduced in terms of a unit of angle, the same symbols are employed whatever the unit, so that $\sin\alpha$, $\cos\alpha$ are not definitely fixed until the unit of angle is known as well as the measure α. In Analysis, on the other hand, $\sin x$, $\cos x$ are definite functions of the variable x (which is the aggregate of real numbers). When a definite number is assigned as the value of x, the values of the functions $\sin x$, $\cos x$ are definitely fixed. If, then, we are to define $\sin x$, $\cos x$ by means of Trigonometry, we must know for what unit of angle these functions are the sine and cosine of the angle of measure x. The unit angle actually chosen is the *radian*, so that $\sin x$, $\cos x$ are respectively equal to the sine and cosine of an angle of x radians. If another unit of angle is employed on any occasion, the functional symbol should be modified; for example, if the unit is the *degree*, we may denote the sine of an angle of x degrees by $\sin^\dagger x$, say, and since x degrees are equal to $\pi x/180$ radians, we have $\sin^\dagger x = \sin\dfrac{\pi x}{180}$.

Several ways of defining the functions $\sin x$, $\cos x$ without reference to Trigonometry have been developed, but they are all too advanced, analytically speaking, to be used in an introductory course. See, for example, the definitions in terms of infinite series in Chap. VI, Art. 146. 5.

16. Plot-Point for Two Variables.

When, as in the case of a functional relation, two variables x, y are to be plotted, it is usually most convenient to take two lines, $x'Ox$, $y'Oy$ at right-angles to one another, plotting values of x on $x'Ox$, and values of y on $y'Oy$, as in Art. 12, the rule of signs applying so that points on the parts Ox, Oy represent positive values of x, y, respectively. In *theoretical discussions* it is supposed that the unit line is of the same length for plots along Ox and Oy, in other words, that the scale of representation is the *same* along the two lines; in *practical work* it may be much more convenient to choose different scales. Suppose now that M_1 on Ox is the plot-point of x_1 and that

[1] In an elementary case like $y = x^n$, the regular functional notation does not appear at all. It would, in fact, be absurd to introduce any *additional* notation for a function which is represented by a simple formula shewing its construction by elementary operations. A functional symbol, f say, can, of course, be used temporarily, when convenient, with the definition $f(x) \equiv x^n$.

18. Introduction of Oblique Coordinates.

In plotting the graph of a function it is sometimes convenient to use a pair of axes Ox, Oy which are not at right-angles, or, as is said, are *oblique* to one another, as in Fig. 12. The angle ω between the axes is called their *obliquity*. In this case the coordinate lines are drawn parallel to the axes. The plot-point P corresponding to a pair of values (x, y) of the variables is reached by measuring a distance x, (OM), along Ox and a distance y, (MP), parallel to Oy, or by measuring a distance y, (OL), along Oy and a distance x, (LP), parallel to Ox. The plot-point P is said to have *oblique coordinates* (x, y).

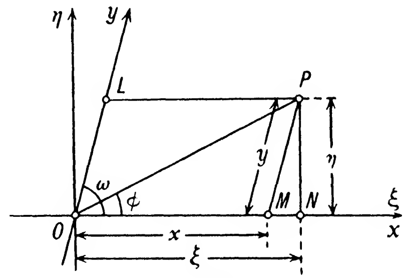


FIG. 12.

If we draw rectangular axes $O\xi$, $O\eta$, the former along Ox and the latter perpendicular to it through O, as in the Figure, the coordinates (ξ, η) of P relative to them are given by

$$\xi = x + y \cos \omega, \quad \eta = y \sin \omega, \quad \text{.....................(1)}$$

so that

$$x = \xi - \eta \cot \omega, \quad y = \eta \operatorname{cosec} \omega. \quad \text{.....................(2)}$$

Oblique axes are not frequently employed in this Book and attention will always be called to the fact when they are being used.

The specification of the position of a point in a plane by means of its distances from two lines Ox, Oy in the plane, either at right-angles or oblique to each other, is described as the *Cartesian* [1] specification, and the same adjective is applied to either system of coordinates.

Ex. 1. Shew that in Fig. 12,

$$\frac{y}{x} = \frac{\sin \varphi}{\sin(\omega - \varphi)} \quad \text{.................................(3)}$$

and

$$\tan \varphi = \frac{y \sin \omega}{x + y \cos \omega}. \quad \text{.................................(4)}$$

Ex. 2. Establish the following results :

$$\left.\begin{aligned} OP^2 &= x^2 + y^2 + 2xy \cos \omega, \\ PP'^2 &= (x - x')^2 + (y - y')^2 + 2(x - x')(y - y') \cos \omega, \\ \cos \angle POP' &= \frac{xx' + yy' + (xy' + x'y) \cos \omega}{OP \, . \, OP'}, \end{aligned}\right\} \text{............(5)}$$

where P' is the point (x', y'). Deduce the analytical condition that the lines OP, OP' should be at right-angles.

[1] After Descartes, (1596-1650), the inventor of Analytical Geometry.

19. One-way Functions (Monotone Functions).

A class of functions which is of importance in connection with many general theorems is that described as *one-way* (or *monotone* or *monotonic*). A single-valued function y is said to be one-way in a range (a, b) of x, (where we suppose $a < b$), if it never decreases or if it never increases as x increases from a to b. In the former case the function is said to be *up-way*, in the latter *down-way*. These definitions allow a one-way function to remain constant in a part or in the whole of the range ; they merely require that for an up-way function we shall have $y_2 \geqslant y_1$ if $x_2 > x_1$, and for a down-way function $y_2 \leqslant y_1$ if $x_2 > x_1$, where x_1, x_2 are any values of x in the range (a, b).

If a function is *continually increasing* as x increases, so that $y_2 > y_1$ if $x_2 > x_1$, it is said to be *strictly up-way* ; if *continually decreasing* as x increases, so that $y_2 < y_1$ if $x_2 > x_1$, it is said to be *strictly down-way*. In either case the function is said to be *strictly one-way*. It is clear that a strictly one-way function cannot take the same value twice in the range.

No sort of continuity or regularity in the values of one-way functions is to be inferred beyond that involved in the definition ; in particular, they need not be graphable functions so as to be adequately represented by a plot.

Ex. 1. *The parabola* $y = ax^2 + bx + c$, $(a, b, c$ *constants*$)$. Writing the equation in the form

$$y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right), \qquad \text{(1)}$$

y is expressed as the sum of two terms, the first of which is variable and the second constant. If $a > 0$, the first term is positive except when $x = -b/(2a)$, its value then being zero. Therefore y has, in this case, the minimum value $c - b^2/(4a)$ for this value of x. If $a < 0$, y has the maximum value $c - b^2/(4a)$ when $x = -b/(2a)$.

When $a > 0$, the form of the graph in the neighbourhood of the vertex is as shewn in Fig. 13. For values of x less than $-b/(2a)$, y is a strictly down-way function and for values of x greater than $-b/(2a)$, a strictly up-way function. These characteristics are interchanged when $a < 0$.

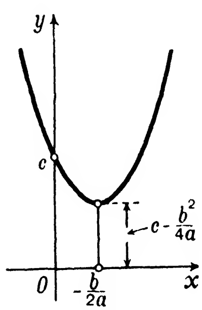


FIG. 13.
$y = ax^2 + bx + c, \quad (a > 0)$.

Ex. 2. Investigate ranges in which the following functions are one-way : (i) $y = k^2/x$, (ii) $y = \sqrt{(a^2 - x^2)}$, (iii) $y = \sin x$, (iv) $y = \cos x$, (v) $y = \tan x$, (vi) $y = I(x)$.

[Remarks on the solutions : $\sin x$ is strictly up-way when x lies in a range given by $(2n\pi - \frac{1}{2}\pi, 2n\pi + \frac{1}{2}\pi)$, ($n$ an integer) ; $I(x)$ is up-way but in no range strictly up-way.]

20. The Polynomial $\sum_{r=0}^{n} a_r x^r$ as a One-way Function.

We prove that the polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n, \quad (a_1 \neq 0), \quad \text{.................... (1)}$$

is strictly one-way in a range, $(-h, h)$ say,[1] of x having its centre at the origin. For this purpose we require the following lemma and corollary.

20. 1. LEMMA. *Let k, K be any two assigned positive numbers. Then the magnitude of the expression $a_1 x + a_2 x^2 + \dots + a_n x^n$ is less than K provided that $\frac{|x|}{k} \leqslant \frac{K}{M+K}$, where M is an upper barrier to the set of numbers $k|a_1|, k^2|a_2|, \dots, k^n|a_n|$.*

To prove this lemma we observe that

$$\begin{aligned} |a_1 x + a_2 x^2 + \dots + a_n x^n| &\leqslant |a_1 x| + |a_2 x^2| + \dots + |a_n x^n| \\ &= k|a_1| \frac{|x|}{k} + k^2 |a_2| \frac{|x|^2}{k^2} + \dots + k^n |a_n| \frac{|x|^n}{k^n} \\ &< M \frac{|x|}{k} \cdot \frac{1 - |x|^n/k^n}{1 - |x|/k} \\ &< M \frac{|x|/k}{1 - |x|/k} \text{ provided } |x|/k < 1. \quad \text{.................. (2)} \end{aligned}$$

Now

$$M \frac{|x|/k}{1 - |x|/k} \leqslant K \text{ according as } \frac{|x|}{k} \leqslant \frac{K}{M+K}. \quad \text{.................. (3)}$$

Since $K/(M+K) < 1$, this condition includes the previous proviso $|x|/k < 1$. The lemma is therefore proved.

In particular, if we take $k = 1$ the condition becomes

$$|x| \leqslant K/(M_1 + K), \quad \text{.................................. (4)}$$

where M_1 is an upper barrier to the set $|a_1|, |a_2|, \dots, |a_n|$.

COROLLARY. *A range, $(-h', h')$ say, can be found such that, for values of x in it, the polynomial $a_0 + a_1 x + \dots + a_n x^n$, $(a_0 \neq 0)$, has the sign of a_0.*

This follows since the magnitude of the terms following a_0 is less than $|a_0|$ if $|x| \leqslant |a_0|/(M_1 + |a_0|)$. We can thus take

$$h' = \frac{|a_0|}{M_1 + |a_0|}. \quad \text{...................................... (5)}$$

20. 2. We now return to the investigation of the nature of the polynomial (1). The difference $f(x_2) - f(x_1)$ between the values of $f(x)$ at the values x_1, x_2 of x, where $x_1 < x_2$, is given by

$$\begin{aligned} f(x_2) - f(x_1) &= a_1(x_2 - x_1) + a_2(x_2^2 - x_1^2) + \dots + a_n(x_2^n - x_1^n) \\ &= (x_2 - x_1)\{a_1 + a_2(x_2 + x_1) + \dots + a_n(x_2^{n-1} + x_2^{n-2}x_1 + \dots + x_1^{n-1})\}. \quad (6) \end{aligned}$$

[1] Here h is some positive number for which a suitable value is found in the course of the discussion. The determination of the *whole* range, containing the origin, within which the function is one-way, is, in general, most conveniently effected by means of the Calculus; see Chap. IV, Art. 98.

Now

$$|a_2(x_2+x_1)+\ldots+a_n(x_2^{n-1}+x_2^{n-2}x_1+\ldots+x_1^{n-1})|$$
$$\leqslant |a_2(x_2+x_1)|+\ldots+|a_n(x_2^{n-1}+x_2^{n-2}x_1+\ldots+x_1^{n-1})|$$
$$< 2|a_2|.|x^*|+3|a_3|.|x^*|^2+\ldots+n|a_n|.|x^*|^{n-1}, \qquad\ldots\ldots\ldots\ldots(7)$$

where $|x^*|$ is the greater of the numbers $|x_1|$, $|x_2|$. According to the above condition (4) we therefore have

$$|a_2(x_2+x_1)+\ldots+a_n(x_2^{n-1}+\ldots+x_1^{n-1})|<|a_1| \text{ if } |x^*|\leqslant \frac{|a_1|}{M+|a_1|}, \quad\ldots.(8)$$

where M is now an upper barrier of the set $2|a_2|$, $3|a_3|$, $\ldots$, $n|a_n|$.

Hence if x_1, x_2 lie in the range $(-h, h)$, where $h=|a_1|/(M+|a_1|)$, the expression $\quad a_1+a_2(x_2+x_1)+\ldots+a_n(x_2^{n-1}+x_2^{n-2}x_1+\ldots+x_1^{n-1}) \quad\ldots\ldots\ldots\ldots(9)$

is of the same sign as a_1 and hence $f(x_2)-f(x_1)$ is of the same sign as $(x_2-x_1)a_1$. Thus within the range $(-h, h)$ so determined, $f(x)$ is strictly up-way (down-way) if a_1 is positive (negative).

20. 3. Consider next an aggregate of polynomials of the form (1) arising from the variation of the coefficients $a_1, a_2, \ldots, a_n$. From the above demonstration it follows that these polynomials will all be strictly up-way in a common range $(-h, h)$ of x provided that a_1 is always positive and

$$h\leqslant a_1^*/(M^*+a_1^{**}), \qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

where

(i) a_1^* is a positive lower barrier of the values of a_1,

(ii) a_1^{**} is an upper barrier of the values of a_1, and

(iii) M^* is an upper barrier of the aggregate consisting of all the values of the numbers $2|a_2|$, $3|a_3|$, $\ldots$, $n|a_n|$.

21. Functions of Bounded Variation.

A function $f(x)$ which in a range (a, b), $(a<b)$, is equal to the sum [1] of an up-way function and a down-way function is called a *function of bounded variation* in that range. This statement includes the case of an up-way function (or a down-way function) alone, that is, one of the functions of the sum may be zero throughout.

The name given to the sum arises from a property which is easily proved. We suppose that the variable x includes all the numbers in the range (a, b) and subdivide this range in any manner by values $x_2, \ldots, x_{n-1}$, (x_1, x_n being supposed equal to a, b, respectively). The sum of the magnitudes of the increments of the function in the intervals (x_1, x_2), (x_2, x_3), $\ldots$, (x_{n-1}, x_n) is

$$|f(x_2)-f(x_1)|+|f(x_3)-f(x_2)|+\ldots+|f(x_n)-f(x_{n-1})|. \qquad\ldots\ldots\ldots\ldots(1)$$

Let $f(x)=\varphi(x)+\psi(x)$, where $\varphi(x)$ is up-way and $\psi(x)$ down-way. Then

$$f(x_2)-f(x_1)=\varphi(x_2)-\varphi(x_1)+\psi(x_2)-\psi(x_1) \qquad\ldots\ldots\ldots\ldots(2)$$

and $$|f(x_2)-f(x_1)|\leqslant|\varphi(x_2)-\varphi(x_1)|+|\psi(x_2)-\psi(x_1)|, \qquad\ldots\ldots\ldots\ldots(3)$$

[1] By the *sum* of two functions $\varphi(x)$, $\psi(x)$ we mean a function whose value for any value of the independent variable x is the sum of the values of the two functions for that value of x. The sum of the two functions is represented by $\varphi(x)+\psi(x)$.

with corresponding inequalities for the other differences. Summing each side of all these inequalities we get

$$\sum_{r=1}^{n-1} |f(x_{r+1}) - f(x_r)| \leqslant \sum_{r=1}^{n-1} |\varphi(x_{r+1}) - \varphi(x_r)| + \sum_{r=1}^{n-1} |\psi(x_{r+1}) - \psi(x_r)|. \quad \text{....(4)}$$

Since $\varphi(x)$ is up-way, all the increments of that function which do not vanish are positive. In like manner the increments of $\psi(x)$ which do not vanish are negative. Hence the sums on the right are, respectively, $\varphi(x_n) - \varphi(x_1)$ and $\psi(x_1) - \psi(x_n)$, simply. Therefore, writing a, b for x_1, x_n, respectively, we have

$$\sum_{r=1}^{n-1} |f(x_{r+1}) - f(x_r)| \leqslant \varphi(b) - \varphi(a) + \psi(a) - \psi(b). \quad \text{...............(5)}$$

The sum of the magnitudes of the increments of $f(x)$ for any mode of subdivision is thus not greater than a fixed finite number. The aggregate of these sums has therefore an upper barrier and consequently an upper bound. It is this property which is described by saying that the function $f(x)$ is of *bounded variation.*

It is now shewn, conversely, that any function which has this property in a range can be expressed as the sum of an up-way and a down-way function. In the first place, if on introducing subdivisions of the range, as above, we form the sum of those increments only which are positive, the aggregate of such sums will have an upper bound, $B^{(u)}$ say, since each of them is not greater than the sum of the magnitudes of *all* the increments for the same subdivision. In the same way the aggregate of the sums formed of the negative increments alone will have a lower bound, $B^{(l)}$ say. Now the total increment $f(b) - f(a)$ of the function is the algebraic sum of the positive and negative increments for any mode of subdivision. Therefore [1]

$$B^{(u)} + B^{(l)} = f(b) - f(a). \quad \text{.............................(6)}$$

This argument may be applied to any part (a, x) of the range (a, b), for the function $f(x)$ is easily seen to be of bounded variation for a part of the range if it is so for the whole. Denote the corresponding bounds for the part (a, x) by $B^{(u)}(x)$, $B^{(l)}(x)$, so that

$$B^{(u)}(x) + B^{(l)}(x) = f(x) - f(a), \quad \text{...........................(7)}$$

and hence

$$f(x) = f(a) + B^{(u)}(x) + B^{(l)}(x). \quad \text{..........................(8)}$$

Now from its definition as a sum of positive terms, $B^{(u)}(x)$ cannot decrease as x increases. Therefore $f(a) + B^{(u)}(x)$ is up-way, and similarly $B^{(l)}(x)$ is down-way, so that $f(x)$ is expressed as the sum of an up-way and a down-way function.

Functions of bounded variation are of considerable importance in the theory of Integration and elsewhere in the Calculus.

22. Continuous Functions.

The description and discussion of functions up to this point has included cases which, though important from the standpoint of

[1] For a subdivision can be found in which the sums of positive and negative increments are as nearly equal as we please to $B^{(u)}$ and $B^{(l)}$, respectively.

general theory, do not usually occur in the applications of the subject. We are now prepared to introduce restrictions as to the nature of the independent and the dependent variables which give to them the character of *continuity*, which they were assumed, without discussion, to possess in the earlier history of the subject.

22. 1. *Intrinsic continuity of the independent variable.* If the number-aggregate of a variable includes all the numbers, rational and irrational, in a range (a, b), the variable is said to be continuous *per se*, or *intrinsically*, in that range. If such a variable is thought of as taking in succession, and in natural order, all its values in the range, starting at a, it is described as *passing continuously from a to b*. This continuous passage plainly cannot be carried out, even mentally, in detail; a comparison may be made with continuous passage along a curve.

22. 2. *Specification of a range.* When a continuous variable may have any value between two numbers a, b we say that it has the range a to b, and the same description is used whether the end numbers or terminals a, b are included or not in the permissible values of the variable. When a terminal number, a say, is *included*, the range is said to be *closed* at the end a; otherwise the range is *open* at a.

It is convenient to introduce for ranges symbols which indicate whether terminals are included or not. As in Art. 11, we denote the range from a to b by (a, b) when the terminals are included. If a terminal is excluded we change the corresponding limb of the *parenthesis* to that of a *bracket*. Thus in $[a, b)$, a is excluded and b included, while in $[a, b]$, both a and b are excluded. When, for any reason, it is not desired to indicate whether a terminal is included or excluded, the corresponding limb of a *brace* is used. Thus in $(a, b\}$, the inclusion or exclusion of b is not indicated, and in $\{a, b\}$ the indecision applies to both ends.[1]

A range from a to b is described as *positive* (*negative*) if the difference $b - a$ between its upper and lower terminals is positive (negative), that is, if $b > a$, $(b < a)$.

If a variable may have any value, however large, which is greater than a, its range is said to extend from a to $+\infty$ and it is indicated

[1] For a closed range, in which $a < b$, the symbol $a \leqslant x \leqslant b$ can be employed. Similarly $a \leqslant x < b$ can be used to indicate a range closed at the lower end and open at the upper, and so on. This symbolism, however, is only applicable to a range the nature of whose terminals is known.

by $(a, +\infty]$, $[a, +\infty]$, or $\{a, +\infty]$ according as the inclusion of a is to be asserted, denied, or left undecided. A bracket-limb is used after $+\infty$, as $+\infty$ is not a number and there is no definite upper terminal to the range. The ranges indicated by symbols such as $[-\infty, a)$ and $[-\infty, \infty]$ will now be clear. A variable can, of course, have two or more ranges; they will be separately represented in the manner described.

In many theoretical discussions the range considered has merely to satisfy the condition that a given point P_1, (x_1), is *within* it, and so at a certain distance from each end. Such a range is commonly called a *neighbourhood* of x_1. Usually the point x_1 is supposed at the centre of the neighbourhood, so that it is specified by $(x_1 - h, x_1 + h)$, where h is some positive number. Sometimes a range, such as $(x_1, x_1 + h)$ or $(x_1 - h, x_1)$, terminating at x_1, is spoken of as a *one-sided neighbourhood of* x_1. We shall always understand the word 'neighbourhood', without qualification, to mean a two-sided neighbourhood, that is, as extending on both sides of x_1.

22. 3. *Continuity of a function relative to the independent variable.* We consider a single-valued function y, or $f(x)$, of an independent variable x which is continuous *per se* in a range (a, b). According to the general definition of a function, the variation of its values may be as great in any part of the range, however small, as in the whole range. A case in point is the function, already instanced, whose value is 1 for rational values of the variable and -1 for irrational values. Such a function does not conform to the notion of continuous variation, which requires that the values of the function should *close up to equality* as the part of the range considered is continually diminished. To put the matter in more accurate language, it is required for continuity that the part in question can always be taken so small that the magnitude of the variation of the values of the function in it is less than any assigned positive number.

We can now give the formal condition for the continuity of a function at a value x_1 of the range. Consider a neighbourhood $(x_1 - h, x_1 + h)$ of x_1, (h supposed positive), y_1, or $f(x_1)$, being the value of the function when $x = x_1$, and y', or $f(x')$, its value for another value x' of x within the neighbourhood. Then $y' - y_1$ is the difference of the values of the function at x' and x_1. The necessary and sufficient condition of continuity of the function at x_1 is that, having chosen a positive number ε, however small, it must be possible to choose a value for h, such that each of the differences $|y' - y_1|$, for

the different values of x', is less than ε. In symbolic form the condition is expressed by

$$|y' - y_1| < \varepsilon, \ (\varepsilon \text{ arbitrary}), \text{ if } |x' - x_1| < h, \ (h \text{ appropriate}).^1 \ldots(1)$$

This statement is quoted as *the ε-condition*[2] *for continuity* at x_1, or at a point x_1, (using 'point' for 'value'). The student may at first think the ε-condition cumbrous and unnecessary, but it only puts into formal language a property implied by the notion of continuity, and it represents the only satisfactory procedure that has hitherto been found for the purpose of dealing rigorously with the property.

A simple result that is constantly applied follows immediately from the ε-condition. If a function is continuous at x_1 and its value there is not zero, there exists a neighbourhood of x_1, say $(x_1 - h, x_1 + h)$, in which the function has throughout the same sign as at x_1. This is an evident consequence of the fact that we can choose h so that the variation of y from the value y_1 is as small as we please, and, in particular, so that the magnitude of the variation is less than $|y_1|$, the magnitude of the function at x_1.

22. 4. *Continuity in a range.* So far we have considered only continuity of the function at *one* value x_1. For practical purposes we require the function to possess the same property for *every* value of x in some range, (a, b) say. When this further condition is satisfied, we say that the function is *continuous in the range* (a, b).[3] Thus a function is continuous in a range of the independent variable when (and only when) it is continuous at every value in the range.

22. 5. *Continuity of* $|f(x)|$. If $f(x)$ is continuous at a value x_1, the function $|f(x)|$, obtained by taking the magnitude of the function $f(x)$ for each value of x in the range considered, is also continuous at x_1. For we have

$$\big| |f(x')| - |f(x_1)| \big| \leqslant |f(x') - f(x_1)|. \ \ldots\ldots\ldots\ldots\ldots(2)$$

[1] That is, to any positive value of ε there corresponds a positive number h such that the condition (1) is satisfied. It is plain that a positive number less than an appropriate h is also appropriate.

[2] When the letter ε is used in the sequel without special definition, it denotes a positive number which is arbitrarily assigned, as in the present case.

[3] For the present we suppose that the function is defined beyond the terminals a, b, so that the condition of continuity can be applied at the terminals as well as within the range (a, b). When this is not so, we must either confine ourselves to a range $[a, b]$ or take the continuity at the terminals as *one-sided*, in the sense explained in Art. 32. See also the remark at the end of Art. 33 with respect to the extension of a range.

Therefore, the sub-range $(x_1 - h, x_1 + h)$ being chosen so that we have $|f(x') - f(x_1)| < \varepsilon$, ($\varepsilon$ arbitrary), for $|x' - x_1| < h$, (h appropriate), we have also

$$\big|\,|f(x')| - |f(x_1)|\,\big| < \varepsilon. \quad \text{.........................(3)}$$

If, therefore, the ε-condition of continuity is satisfied by $f(x)$, it is also satisfied by $|f(x)|$. It follows at once that if $f(x)$ is continuous in a range, $|f(x)|$ is also continuous in the range.

The argument plainly cannot be reversed to prove the converse result, which is, in fact, not true. For example, if $f(x)$ is equal to $+1$ for rational values of x and to -1 for irrational values, $|f(x)|$ is always equal to $+1$ and is a continuous function, since its variation vanishes. But $f(x)$ is not continuous since its values do not close up to equality in a continually diminished range.

22. 6. *The property of intermediate values for continuous functions.* If the conditions which have been imposed for the continuity of a function are compared with the conditions for the continuity of a curve, it will be seen that the former apply in both cases, that is, the closing up to equality of the ordinates of the curve over a sub-range which is indefinitely diminished, is a feature of geometrical continuity. But it is by no means evident to what extent the conditions imposed for a function ensure that it possesses other properties corresponding to those which are associated with the continuity of a curve. We consider here the property of a continuous curve that if two points of it lie on opposite sides of an unlimited straight line, it must intersect the line at least once. Suppose that the straight line is taken as the axis Ox and that the two points have coordinates (x_1, y_1), (x_2, y_2), y_1, y_2 having opposite signs. Then if the curve passes continuously from one of the points to the other between the ordinates through them, it must intersect Ox in at least one point, x_3 say, in the range (x_1, x_2).

We now shew that the corresponding property is possessed by a function which satisfies the analytical conditions of continuity, namely, that if the function y has values y_1, y_2, of opposite signs, for the respective values x_1, x_2 of x, lying within the range of continuity, then the function has the value 0 for at least one value, x_3 say, intermediate between x_1 and x_2. Let the function be negative when $x = x_1$ and positive when $x = x_2$. In the following demonstration we use the method of decimal subdivision of ranges. Divide the range (x_1, x_2) into ten equal parts, each of which is regarded as closed, (that is, as including its terminal values). Since the function is

positive at the upper end of the last part, there must be a first part, counting from x_1, for which the value of the function at the upper end is positive or zero,[1] the value at its lower end being therefore negative. Select this part and subdivide it into ten equal parts. Of these last parts, select that one nearest x_1 for which the value of the function at the upper end is positive or zero. Continue this process indefinitely. The aggregate of lower ends of the selected parts defines a value of x, x_3 say, and we shew that the value of the function at x_3 is zero. By the nature of the process, x_3 lies in each of the successive selected parts, and we can therefore choose a part, as small as we please, such that x_3 lies in it, while the value of the function is negative at the lower end and positive or zero at the upper end. Now if the function has not the value zero at x_3, we can choose a neighbourhood $(x_3 - h, x_3 + h)$ in which it has the same sign as at x_3, and this is impossible according to the statement in the previous sentence. Hence the function vanishes at x_3 and the theorem is proved.

A simple generalization is proved in the same way. If the function has the values y_1, y_2 at x_1, x_2, it has any assigned value y_3, which is intermediate between y_1 and y_2, for at least one value of x, x_3 say, in the range (x_1, x_2). In this statement y_1, y_2 may be of the same or opposite signs. To establish this result the successive parts are selected so that, at the upper end of each, the function is either greater than, or equal to, the value y_3.

23. Two Simple Continuous Functions.

For the sake of the subsequent discussions it is convenient to assert, and examine formally, the continuity of the two functions defined by the relations $y = c$, (c a constant), and $y = x$.

(i) The function defined by $y = c$ has a constant value in the range of x considered. Its variation therefore vanishes in any sub-range, and the ε-condition

$$|y' - y_1| < \varepsilon, \ (\varepsilon \text{ arbitrary}), \text{ if } |x' - x_1| < h, \ (h \text{ appropriate}). \quad \ldots(1)$$

is always satisfied. The function is therefore continuous.

(ii) The value of the function defined by $y = x$ is always equal to that of the independent variable, and we have

$$|y' - y_1| = |x' - x_1|. \quad \ldots\ldots\ldots\ldots(2)$$

[1] The case of a zero value could be excluded from the rest of the discussion since the theorem is true if it occurs. It is included to simplify the proof.

Therefore $|y' - y_1| < \varepsilon$ if $|x' - x_1| < \varepsilon$, and the ε-condition is satisfied by taking $h = \varepsilon$.

24. Continuity of Functions which are Combinations of Continuous Functions.

We are now ready to prove the fundamental theorems concerning the continuity of combinations of continuous functions.

24. 1. THEOREM I. *The sum or the difference of two functions $\varphi(x)$, $\psi(x)$, which are both continuous at the value x_1 of x, is also continuous at x_1.*

Let $f(x) = \varphi(x) + \psi(x)$. Then we have

$$f(x_1) = \varphi(x_1) + \psi(x_1), \quad f(x') = \varphi(x') + \psi(x'), \quad \text{............(1)}$$

so that

$$f(x') - f(x_1) = \varphi(x') - \varphi(x_1) + \psi(x') - \psi(x_1) \quad \text{..............(2)}$$

and

$$|f(x') - f(x_1)| \leqslant |\varphi(x') - \varphi(x_1)| + |\psi(x') - \psi(x_1)|. \quad \text{........(3)}$$

Since $\varphi(x)$, $\psi(x)$ are continuous, it follows that, corresponding to any assigned ε, we can find positive numbers h_1, h_2 such that

$$|\varphi(x') - \varphi(x_1)| < \varepsilon \text{ if } |x' - x_1| < h_1, \quad \text{.................(4)}$$

and

$$|\psi(x') - \psi(x_1)| < \varepsilon \text{ if } |x' - x_1| < h_2. \quad \text{................(5)}$$

In these inequalities we can replace h_1, h_2 by a common value h which is not greater than either of them, and we then have

$$|f(x') - f(x_1)| < 2\varepsilon \text{ if } |x' - x_1| < h. \quad \text{.................(6)}$$

Since ε can be chosen as small as we please, 2ε can be chosen as small as we please. The ε-condition of continuity is therefore satisfied by $f(x)$ at the value x_1 of x, and the theorem is proved.

The difference of two continuous functions is proved continuous in the same way.

24. 2. THEOREM II. *The product of the two continuous functions $\varphi(x)$, $\psi(x)$ is also continuous at x_1.*

We now use the symbol $f(x)$ for the product of the two functions, so that $f(x) = \varphi(x)\psi(x)$, and we have

$$f(x') - f(x_1) = \varphi(x')\psi(x') - \varphi(x_1)\psi(x_1). \quad \text{.................(7)}$$

On adding and subtracting the term $\varphi(x_1)\psi(x')$ on the right-hand side, this may be written

$$f(x') - f(x_1) = \{\varphi(x') - \varphi(x_1)\}\psi(x') + \varphi(x_1)\{\psi(x') - \psi(x_1)\}, \quad \text{......(8)}$$

and therefore

$$|f(x')-f(x_1)| \leqslant |\varphi(x')-\varphi(x_1)| \,.\, |\psi(x')| + |\psi(x')-\psi(x_1)| \,.\, |\varphi(x_1)|. \quad (9)$$

We have only to consider values of x near to x_1. Let $|\varphi(x)|$, $|\psi(x)|$ have upper barriers [1] M_1, M_2 in a range (x_1-k, x_1+k) of x. Since $\varphi(x)$, $\psi(x)$ are continuous, we have, as before,

$$\left.\begin{array}{l} |\varphi(x')-\varphi(x_1)| < \varepsilon \text{ if } |x'-x_1| < h_1, \\ |\psi(x')-\psi(x_1)| < \varepsilon \text{ if } |x'-x_1| < h_2, \end{array}\right\} \quad (10)$$

where ε is arbitrary and h_1, h_2 are appropriate. We can now replace h_1, h_2 by h, a number not greater than h_1, h_2 or k, for if the inequalities hold in any range they hold in a smaller range. We then have

$$|f(x')-f(x_1)| < (M_1+M_2)\varepsilon \text{ if } |x'-x_1| < h. \quad (11)$$

Here ε can be chosen as small as we please and therefore $(M_1+M_2)\varepsilon$ can be so chosen. The condition of continuity is therefore satisfied and the theorem is proved.

24. 3. THEOREM III. *The reciprocal of the function $\varphi(x)$ is also continuous at x_1, provided $\varphi(x_1) \neq 0$.*

Let $f(x)$ be used for the reciprocal, so that $f(x)=1/\varphi(x)$. Then

$$\begin{aligned} f(x')-f(x_1) &= \frac{1}{\varphi(x')} - \frac{1}{\varphi(x_1)} \\ &= -\frac{\varphi(x')-\varphi(x_1)}{\varphi(x')\varphi(x_1)} \quad (12) \end{aligned}$$

and

$$|f(x')-f(x_1)| = \frac{|\varphi(x')-\varphi(x_1)|}{|\varphi(x')| \,.\, |\phi(x_1)|}. \quad (13)$$

If, now, $|\varphi(x_1)| = N \neq 0$, there is a range of x, say (x_1-l, x_1+l), in which $|\varphi(x)|$ is not less than $\frac{1}{2}N$. Also we have

$$|\varphi(x')-\varphi(x_1)| < \varepsilon \text{ (arbitrary) if } |x'-x_1| < h_1 \text{ (appropriate)}, \quad (14)$$

and we can replace h_1 by h, a number not greater than h_1 or l. We then have

$$|f(x')-f(x_1)| < 2\varepsilon/N^2 \text{ if } |x'-x_1| < h. \quad (15)$$

As before, $2\varepsilon/N^2$ can be chosen as small as we please. The condition of continuity is therefore satisfied and the theorem is proved.

[1] It is shewn in Art. 33 that a function continuous in a range has upper and lower barriers. It is, however, not necessary to use this general result here; according to the definition of a continuous function there is always a range (x_1-k, x_1+k) in which the magnitude of the variation from the value at x_1 is less than a chosen constant, K say, and we may therefore take M_1 to be the greater of the two numbers $|\varphi(x_1)+K|$, $|\varphi(x_1)-K|$, and similarly for M_2.

The student should carefully observe the proviso $\varphi(x_1) \neq 0$ in this theorem.

24. 4. THEOREM IV. *The quotient $\psi(x)/\varphi(x)$ of the two functions $\varphi(x)$, $\psi(x)$ is continuous at x_1, provided $\varphi(x_1) \neq 0$.*

Since we have proved that the reciprocal $1/\varphi(x)$ of $\varphi(x)$ is continuous under the stated proviso, Theorem II shews that the product of $\psi(x)$ and $1/\varphi(x)$ is continuous. The theorem in question is therefore proved. The student should again carefully observe the proviso $\varphi(x_1) \neq 0$.

24. 5. It at once follows from Theorems I and II that the sum, difference and product of two functions which are continuous in a range are also continuous in that range. The reciprocal $1/\varphi(x)$ and the quotient $\psi(x)/\varphi(x)$ are also continuous in the range, provided that $\varphi(x)$ does not vanish anywhere in the range.

24. 6. The Theorems I and II concerning the addition and multiplication of two functions can clearly be extended to the addition and multiplication of any finite number of continuous functions.

25. Continuity of Integral Rational Algebraic Functions (Polynomials).

These functions are obtained from the independent variable and constants by the operations of addition, subtraction and multiplication, (potentiation for positive integral indices being included in multiplication). It is clear that any polynomial in x can be built up by repetition of these processes. Thus from x and c we obtain cx by multiplication, and from this cx^2 by multiplication by x, and so on. All monomials of the form cx^n, (n a positive integer), are obtained in this way, and by addition of such monomials we obtain a polynomial in x. Any integral rational algebraic function is reducible to such a polynomial.[1]

Now from Theorem II of the preceding Article, cx is continuous in any range because c and x are so, (treating c and x as functions of x). By the same theorem, the product $cx \cdot x$, that is cx^2, is continuous, and so on. Thus we prove cx^n continuous.

By Theorem I, the sum of all such functions is continuous, that is, a polynomial in x is continuous.

Summarizing, we have the results that

(i) the monomial cx^n, and (ii) the polynomial $\sum_{r=0}^{n} a_r x^r$

are continuous for any finite range of x.

[1] Thus $(1+x)(2+x)$ is an integral rational algebraic function and $2+3x+x^2$ the equivalent polynomial.

26. Continuity of Rational Algebraic Functions.

These functions are obtained from the variable x and constants when the operation of division is allowed as well as those of addition, subtraction and multiplication. Any such function is reducible to the quotient of two polynomials in x. From Theorem IV above, such a quotient is continuous, because the polynomials are so, in any range which does not include a value of x for which the denominator polynomial vanishes.

Ex. 1. Prove directly by the use of the ε-condition that the functions x^2, x^3, x^4, x^n, $1/x$, $1/x^2$, $1/x^n$, (n a positive integer), are continuous for any value of x, with the exception of 0 in the case of the fractional forms.

27. Continuity of Fractional Powers of x.

The beginner will probably be content to base the discussion of the continuity of fractional powers[1] of x on the results concerning fractional indices which are established with more or less rigour in books on elementary Algebra and with which he is familiar. We therefore assume these results in the present Article. An alternative discussion, which is more fundamental, is given in Chap. XXI.

27. 1. *The power* $x^{1/q}$, *(q a positive integer).* If q is *even*, x must be positive as there is no real even root of a negative number. The positive root is supposed taken. The continuity at the value 0 of x cannot be dealt with here because there is no range having 0 as centre in which the function is defined. This is a case of one-sided continuity, which is treated later, Art. 32. If q is *odd*, x may be positive, negative or zero; it is convenient to treat the last case separately, (see below).

We consider the continuity at a value x_1 which is restricted as mentioned, y_1 being the value of the function there. If x', y' are another pair of values of independent variable and function, we have

$$\begin{aligned} y' - y_1 &= x'^{1/q} - x_1^{1/q} \\ &= \frac{(x'^{1/q} - x_1^{1/q})(x'^{(q-1)/q} + x'^{(q-2)/q}x_1^{1/q} + \dots + x_1^{(q-1)/q})}{x'^{(q-1)/q} + x'^{(q-2)/q}x_1^{1/q} + \dots + x_1^{(q-1)/q}} \\ &= \frac{x' - x_1}{x'^{(q-1)/q} + x'^{(q-2)/q}x_1^{1/q} + \dots + x_1^{(q-1)/q}}. \quad \dots\dots\dots\dots(1) \end{aligned}$$

Since $x_1 \neq 0$, we can confine our attention to a neighbourhood $(x_1 - k, x_1 + k)$ in which x' and x_1 have the same sign. The denominator of the last expression is then of magnitude not less than $qM^{\frac{q-1}{q}}$,

[1] The graphs of some fractional powers of x are shewn in Figs. 33, 34, pp. 139, 141.

where M is the smaller of the two numbers $|x_1-k|$, $|x_1+k|$. Thus we can write

$$|y'-y_1| < \frac{|x'-x_1|}{qM^{\frac{q-1}{q}}}. \qquad \text{.........................(2)}$$

It follows that we have

$$|y'-y_1| < \varepsilon, \ (\varepsilon \text{ arbitrary}), \text{ if } |x'-x_1| < \varepsilon q M^{\frac{q-1}{q}}, \qquad \text{.........(3)}$$

provided, of course, that $\varepsilon q M^{\frac{q-1}{q}} \leqslant k$. Now the condition of continuity plainly allows us to take ε less than any fixed number already introduced and therefore such that the proviso mentioned is satisfied. The function is therefore proved continuous for all values of x, other than 0, for which it is defined.

If q is odd, the function is also continuous at the value 0 of x. For if $x_1=0$ we have $y_1=0$, (the value of $0^{1/q}$ according to the ordinary convention), and $y'-y_1=x'^{1/q}$. Thus

$$|y'-y_1| < \varepsilon \quad \text{if} \quad |x'| < \varepsilon^q, \qquad \text{......................(4)}$$

and the condition of continuity is satisfied.

27. 2. *The power* $x^{p/q}$, *(p, q positive integers).* For this function p/q is supposed expressed in its lowest terms and x is positive if q is even. Since $x^{p/q}=(x^{1/q})^p$, the function is the product of p (equal) continuous functions and is therefore continuous for any range if q is odd, and for a range in which x is positive if q is even.

27. 3. *The power* $x^{-p/q}$, *(p, q positive integers).* This function is the reciprocal of $x^{p/q}$ and is therefore continuous for any range in which that function is defined, provided that $x^{p/q}$ (or x) does not vanish in it.

27. 4. Functions obtained by adding together constant multiples of fractional powers of x are continuous for values of x for which the continuity of the separate terms has been demonstrated.

27. 5. In like manner a quotient of two such functions is continuous for values of x for which the denominator does not vanish.

27. 6. *Irrational powers of x.* The continuity of powers of x with irrational indices, (x being now positive), will be assumed at this stage; a formal proof of this property is given in Chap. XXI. The statements in the above Sub-articles 4 and 5 are thus valid when the word *fractional* is replaced by *irrational*, subject to the proviso that x is positive.

28. Continuity of the Trigonometrical Functions.

28. 1. *Continuity of* $\sin x$. Let x_1 be any value of x. Denoting the function by y, we have

$$\begin{aligned} y' - y_1 &= \sin x' - \sin x_1 \\ &= 2\cos \tfrac{1}{2}(x' + x_1) \sin \tfrac{1}{2}(x' - x_1). \quad\quad (1) \end{aligned}$$

Now $|\cos \frac{1}{2}(x' + x_1)| \leqslant 1$ and $|\sin \frac{1}{2}(x' - x_1)| < |\frac{1}{2}(x' - x_1)|$.[1] Hence

$$|y' - y_1| < |x' - x_1|, \quad\quad (2)$$

and the ε-condition of continuity is satisfied by taking ε as the value of h. Therefore $\sin x$ is continuous for any value x_1 and hence in any range of x.

28. 2. *Continuity of* $\cos x$. Here we have

$$y' - y_1 = -2 \sin \tfrac{1}{2}(x' + x_1) \sin \tfrac{1}{2}(x' - x_1), \quad\quad (3)$$

leading, as in the preceding Sub-article, to the relation

$$|y' - y_1| < |x' - x_1|. \quad\quad (4)$$

The continuity of $\cos x$ thus follows as for $\sin x$.

28. 3. *Continuity of* $\tan x$, $\operatorname{cosec} x$, $\sec x$, $\cot x$. The continuity of these functions, apart from isolated values of x, follows from the fact that they are either reciprocals or quotients of $\sin x$ and $\cos x$. In the case of $\tan x$, (i.e. $\sin x/\cos x$), and $\sec x$, (i.e. $1/\cos x$), the values of x which make $\cos x$ zero must be excluded, that is, the functions mentioned are not continuous when $x = (n + \frac{1}{2})\pi$. For $\operatorname{cosec} x$, $\cot x$ we must exclude the values given by $x = n\pi$ from the ranges of continuity.

29. Function of a Function. Change of Independent Variable in a Function.

The class of functions which are immediately recognizable as continuous can be much extended by the use of the general theory of the formation of a *function of a function*. Simple cases of this process have already been used, requiring no special explanation.

Let y be a single-valued continuous function of x in a range (a, b) and let (α, β) be the corresponding range[2] of values of y, the values α, β of y not, in general, occurring at the terminals a, b of the range of x. Further, let z be a given single-valued continuous function of y in a range which includes the range (α, β). The function y of x,

[1] The student will recall that the latter result is demonstrated in elementary Trigonometry in the form that 'the sine of an angle is less than its circular measure.'

[2] That there is a definite closed range of y is formally demonstrated in Art. 33.

being continuous, will take each value between α and β at least once; it may take one of these values more than once, but the value of z is the same for the same value of y, however numerous the values of x for which it occurs.

Since with any value x_1 of x in the range (a, b) there is associated one value only of y, y_1 say, in the range (α, β), and with the value y_1 of y there is associated one value only of z, z_1 say, we can define a new single-valued function of x in the range (a, b) by associating with x_1 the value z_1 of z. This process, in which y plays the rôle of an intermediary variable, is said to define z as a function of x by treating it as a function of a function, (that is, as a function of y which is itself a function of x), and in doing so, the independent variable of the given function z is changed from y to x. If we write $y=f(x)$ and $z=\varphi(y)$, the relation between z and x obtained by this process is symbolized by the equation

$$z=\varphi\{f(x)\}. \qquad (1)$$

The particular case in which y is constant in the range (a, b) of x requires a word of explanation. The range (α, β) here shrinks to a single value α, and z, as a function of x, is also constant in the range (a, b). We suppose in this case that z is defined in a range of y which includes the value α, for otherwise there is no range of continuity of the function $\varphi(y)$.

The process described above can be extended to cases where a further change of independent variable is made. Let the range of values of z, as defined by the equation $z=\varphi\{f(x)\}$, be (α', β'), and suppose that w is a single-valued continuous function $\psi(z)$ of z in a range which includes the range (α', β'). Then the expression of w as a single-valued function of x in the range (a, b) is given by

$$w=\psi[\varphi\{f(x)\}], \qquad (2)$$

or the equivalent form

$$w=\psi(z) \text{ where } z=\varphi(y) \text{ and } y=f(x). \qquad (3)$$

Thus w is expressed as a function of a function of a function of x.

Ex. 1. Let $y=\dfrac{1+x}{1-x}$ in the range $(-1, 1]$ of x, so that the corresponding range of values of y is $(0, \infty]$, and write $z=\sqrt{y}$. Then the expression of z as a function of x in the range $(-1, 1]$ is

$$z=\left(\frac{1+x}{1-x}\right)^{1/2}. \qquad (4)$$

It is readily shewn that the independent variable y of the function z can only be changed to x in the above way when x is confined to the range of values indicated.

Ex. 2. Let $y=\sqrt{x}$ in the range $(0, 1]$ of x, so that y has the range $(0, 1]$, and write $z=\dfrac{1+y}{1-y}$, $w=\sqrt{z}$. The former of these relations shews that z has the range $(1, \infty]$ and the latter that w has the range $(1, \infty]$. The expression of w as a function of x in the range $(0, 1]$ is

$$w=\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)^{1/2}. \quad\text{......(5)}$$

It is readily verified that w is not defined as a function of x in this way for values of x outside the range stated.

30. Continuity of a Function of a Function.

We now shew that the function of x defined by the equation $z=\varphi\{f(x)\}$ is continuous in the range (a, b).

Consider the value x_1 in (a, b) and let y_1, z_1 be the corresponding values of y and z. Let x', y', z' be another set of corresponding values. Because z is a continuous function of y at y_1, corresponding to any arbitrarily chosen ε we can find a positive number k such that

$$|z'-z_1|<\varepsilon \quad \text{if} \quad |y'-y_1|<k, \quad\text{......(1)}$$

and because y is a continuous function of x at x_1, we can find a positive number h such that

$$|y'-y_1|<k \quad \text{if} \quad |x'-x_1|<h, \quad\text{......(2)}$$

(where k takes the place of the usual symbol ε). We thus have

$$|z'-z_1|<\varepsilon \quad \text{if} \quad |x'-x_1|<h. \quad\text{......(3)}$$

Therefore the condition that z may be a continuous function of x at x_1 is satisfied. Since x_1 is any value in the range (a, b), z is a continuous function of x in that range.

This investigation requires no modification when y is constant for the range (a, b) of x, provided that, as in the preceding Article, we suppose that z is defined in a range of y which includes the value y_1.

By an obvious extension of the above argument, we establish the continuity of a function of a function of a function of x, and so on.

Ex. 1. Shew that $\left(\dfrac{1+\sqrt{x}}{1-\sqrt{x}}\right)^{1/2}$ is continuous in the range[1] $[0, 1]$ of x.

31. Continuity of Fractional Powers of Polynomials. Ordinary Algebraic Functions.

Let y be a polynomial in x, say $y=\sum_{r=0}^{n} a_r x^r$, and therefore continu-

[1] For the method of extending the range of continuity to include the lower terminal 0, see Art. 32 below.

ous in any finite range. Further, let $z=y^{p/q}$, where p, q are positive integers. Then by the theorem just proved, the function z of x given by the equation

$$z=\left(\sum_{r=0}^{n} a_r x^r\right)^{p/q}, \quad \text{.........................(1)}$$

is a continuous function of x in any range if q is odd, and in any range for which y is positive if q is even.

We at once infer that the reciprocal function $1/z$, or $\left(\sum_{r=0}^{n} a_r x^r\right)^{-p/q}$, is continuous in any range in which z is defined and does not vanish.

It further follows from the general theorems of Art. 24 that all functions built up from the variable x and constants by means of the fundamental operations, inclusive of fractional radication, are continuous, subject to the restrictions already indicated. The functions obtained in this way are described as *ordinary algebraic functions* and are *surd* when fractional powers are involved.

Ex. 1. *The function* $\sqrt{(a^2-x^2)}$. Since a^2-x^2 is positive when x lies in the range $[-a, a]$, the given function is continuous for such values. The extension of the range of continuity to include the terminals $-a, a$ is made in Ex. 2 of the following Article.

Ex. 2. The function $(a^2-x^2)^{1/3}$ is continuous for all values of x.

Ex. 3. The function $(a^2-x^2)^{-1/3}$ is continuous except for the values $-a, a$ of x.

Ex. 4. The function $\sqrt{(x^2-a^2)}$ is continuous in the ranges $[a, \infty]$, $[-\infty, -a]$.

32. One-sided Continuity. Upper and Lower Continuity.[1]

If a function $f(x)$ is single-valued in a range (a, b), $(a<b)$, and is not defined for values of x less than a, the definition of continuity given in Art. 22. 3 does not apply at a because the function is not defined in any range such as $(a-h, a+h)$, of which a is the centre. In this case we define $f(x)$ to have *upper continuity* or to be *continuous from above* if the values in the range $(a, a+h)$, $(h>0)$, close up to equality when h is continually diminished, that is, if we can choose a value of h such that

$$|f(x')-f(a)|<\varepsilon, \ (\varepsilon \text{ arbitrary}), \text{ if } 0<x'-a<h. \quad \text{.........(1)}$$

If this condition is satisfied the continuity is said to be *one-sided*.

In like manner we define *lower continuity* or *continuity from below* at the value b of a range (a, b), $(a<b)$, when the function is not defined for values of x greater than b.

[1] Often the term *right-hand* (*left-hand*) *continuity* is used for *upper* (*lower*) *continuity*.

Further, we may consider a value x_1 of x *within* the range of definition of $f(x)$ and define in like manner lower and upper continuity at the value x_1 by considering the values of $f(x)$ on the lower and upper sides of x_1, respectively. From the definitions, it is clear that if the function has both upper and lower continuity at x_1, it is continuous in the ordinary sense at that value.

When a function is said to be continuous in a closed range (a, b), it is implied that the function has ordinary continuity at each point *within* the range and also has upper or lower continuity at the respective ends of the range. The function *may*, of course, be continuous beyond the ends of the range, unless the contrary is stated, so that there is ordinary continuity at the ends.

The theorems of Art. 24 concerning the continuity of combinations of continuous functions apply equally to one-sided continuity. The process of extending the range beyond the terminals, explained in the last paragraph of the next Article, can be conveniently used to justify this statement.

Ex. 1. *The function* $x^{p/q}$, (*q even*). This function is not defined for negative values of x. We consider the upper continuity at the point $x=0$, where, according to the usual convention, the function is equal to zero. Denoting the function by y, we have, for $x'>0$,

$$y'=x'^{p/q}, \quad y_1=0, \quad\text{.......(2)}$$

and so

$$|y'-y_1|=x'^{p/q}. \quad\text{.......(3)}$$

Thus

$$|y'-y_1|<\varepsilon, \ (\varepsilon \text{ arbitrary}), \text{ if } x'<h=\varepsilon^{q/p}. \quad\text{.......(4)}$$

The condition of upper continuity is therefore satisfied.

In particular, $\sqrt{x}$ is continuous in the range $(0, \infty]$. It follows, by change of origin to the point $x=-a$, that $\sqrt{(a+x)}$ is continuous in the range $(-a, \infty]$.

Ex. 2. *The function* $\sqrt{(a^2-x^2)}$, $(a>0)$. We examine the continuity at the ends of the range $(-a, a)$, outside which the function is not defined. In this range, $\sqrt{(a^2-x^2)}=\sqrt{(a+x)}\sqrt{(a-x)}$. Now $\sqrt{(a-x)}$ has ordinary continuity at $x=-a$, (and therefore also upper continuity), while $\sqrt{(a+x)}$ has upper continuity at the same point. Hence the product of these two functions has upper continuity at $x=-a$. Similarly, the product has lower continuity at $x=a$. Employing the result of Ex. 1 of the preceding Article, it follows that $\sqrt{(a^2-x^2)}$ is continuous in the range $(-a, a)$.

Ex. 3. *The function* $\sqrt{(x^2-a^2)}$, $(a>0)$. Here $\sqrt{(x^2-a^2)}=\sqrt{(x-a)}\sqrt{(x+a)}$ in the range $(a, \infty]$ and $\sqrt{(x-a)}$, $\sqrt{(x+a)}$ are both upper continuous at $x=a$. Thus the product of these functions is upper continuous at $x=a$. Similarly, the product $\sqrt{(a-x)}\sqrt{(-a-x)}$ is lower continuous at $x=-a$. Employing the result of Ex. 4 of the preceding Article, it follows that $\sqrt{(x^2-a^2)}$ is continuous in the ranges $(a, \infty]$, $[-\infty, -a)$ of definition.

33. Uniform Continuity of a Function in a Range.

We have called a function of x continuous in a range (a, b) when it is continuous for each value of x in the range,[1] that is, when for each value such as x_1 we can find, corresponding to any arbitrary positive value of ε, a value of h for which

$$|f(x') - f(x_1)| < \varepsilon \quad \text{if} \quad |x' - x_1| < h. \qquad \text{.................(1)}$$

The introduction of this definition raises the question whether, for any given ε, a *common* value of h, other than zero, can always be found to satisfy the condition (1) for *all* values of x_1 in the range, that is, whether we can secure a given degree of closing up to equality of the values of the function by an appropriate choice of sub-range $(x_1 - h, x_1 + h)$, *of the same magnitude throughout.* It is not difficult to shew that the answer is in the affirmative, and this we proceed to do by proving that a negative answer is erroneous.

Suppose that for some value of ε it is not true that a common value of h can be found for the whole range. Employ the method of successive decimal subdivisions of the range. The range being at first divided into ten equal parts, the assumption requires that in one at least of these parts the same failure occurs for that value of ε. Select the first part, reckoning from the lower terminal, for which this occurs. Subdivide this part similarly and make a corresponding selection, and so on indefinitely. The aggregate of lower ends of the selected parts defines a number, x_1 say, which lies in each of the selected parts.[2] We can therefore find a neighbourhood of x_1 as small as we please in which the same failure must occur. Now since the function is continuous at x_1, there exists a range, $(x_1 - k, x_1 + k)$ say, within which the variation[3] of the function is less than ε. The contradiction is evident and the initial assumption is therefore erroneous. Thus a common value of h can be found for any given ε.

The existence of a common h for each ε is described by saying that the function is *uniformly continuous* in the range. We have thus

[1] If the function is not defined beyond the terminals of the range, the corresponding conditions for one-sided continuity are supposed to apply at the terminals. It is, however, convenient to extend the range beyond the terminals in the manner suggested at the end of the present Article. Special consideration of the ends of the original range is thus avoided.

[2] The range $(x - h, x + h)$ corresponding to a point x in one of the parts is, of course, not required to lie wholly in it.

[3] By the 'variation' is here meant the magnitude of the difference of the values of the function at two points of the sub-range. It cannot be greater than twice the greatest variation from the value at a fixed point of the sub-range.

proved the theorem that *a function which is continuous in a closed range is also uniformly continuous in that range.*

It follows, as a corollary, that the aggregate of the values of the continuous function $f(x)$ in a closed range has upper and lower barriers, and hence has definite upper and lower bounds. The function is therefore said to have upper and lower bounds in the range.

In proof, observe that, by the theorem, we can divide the whole range into a finite number, n say, of parts in each of which the variation of the function is less than an assigned ε. The magnitude of the difference of the values of the function at any two values of x in the range is therefore less than the finite number $n\varepsilon$.

The above theorem applies to a closed range (a, b). A function continuous in the closed range can always be supposed defined (or re-defined) so that the continuity extends beyond the range above and below. Thus if $a < b$, the value at a point $a - x$ below a can be taken equal to that at $a + x$ above a, and similarly for a range going beyond b. The proof of the theorem then applies to (a, b), regarded as part of an extended range of continuity. If, however, the range $\{a, b\}$ is unclosed at one end, a say, a range $(a, a+k)$, which may be arbitrarily small, must be excluded in the application of the theorem. For example, consider the function $1/x$, which is defined in the range $[0, 1)$ but not in the closed range $(0, 1)$. This function is not bounded in the former range, for we can take a value of x in that range sufficiently small to make $1/x$ larger than any assigned number.

34. Existence of Greatest and Least Values of a Continuous Function.

We have seen that the aggregate of values of a continuous function in a range has definite bounds. We now shew that there is a value of x for which the function is equal to the upper bound, $B^{(u)}$ say, and a value for which it is equal to the lower bound, $B^{(l)}$ say, so that these bounds are actually the greatest and least values of the function.

Consider, at first, the upper bound and subdivide the range into ten equal parts. It is clear that the upper bound for the whole range can neither be greater nor less than the greatest of the upper bounds of the separate parts. It is therefore equal to the upper bound of at least one of the parts. Select the first part, counting from the lower end of the range, for which this occurs, subdivide it and select a part, and so on. The aggregate of the lower ends of these parts defines a number, x_1 say, which lies in them, and the process shews that we can choose a part as small as we please such that x_1 lies in it and such that the upper bound of that part is equal to the upper bound of the function for the whole range.

We now shew that the value $f(x_1)$ of the function for the value x_1 of x is the upper bound of the function for that part, and therefore for the whole range. Suppose that the value $f(x_1)$ at x_1 differs from the upper bound $B^{(u)}$. On account of the continuity of the function, the part can be taken so small that the variation of the function in it is less than the difference between $f(x_1)$ and $B^{(u)}$. But this is impossible since it would make all the values of the function in the part definitely less than $B^{(u)}$, whereas by the nature of a bound we must be able to find values as near to $B^{(u)}$ as we please. We conclude that $f(x_1)=B^{(u)}$, and the theorem is demonstrated.

In the same way we shew that the lower bound $B^{(l)}$ is a value of the function at a point of the range.

The results obtained are commonly expressed by saying that *a continuous function attains its bounds.*

A *discontinuous* function may increase or decrease towards a bound which it does not attain. A simple example is the function defined by $y=x^2$ when $x \neq 0$ and $y=1$ when $x=0$. Here the function decreases towards the bound 0 as $|x|$ decreases but it does not attain that value. A continuous function, on the other hand, which is defined by $y=x^2$ when $x \neq 0$ must have the value 0 when $x=0$, on account of the continuity, and it attains the bound 0. A more complicated case is the function $(1-x^2)\cos(1/x)$ in the range $(-1, 1)$. The formula leaves the function undefined at $x=0$, but whatever value lying between -1 and $+1$ we assign to it at that point, the upper and lower bounds $+1, -1$ are not attained since $|\cos(1/x)| \leqslant 1$ and $1-x^2<1$ when $x \neq 0$.

The importance of the demonstrations of this and the preceding Articles is obscured by the instinctive appeal to graphical representation. It must be remembered, however, that the definition given of a continuous function is an algebraic one and it includes functions for which graphical representations displaying their characters in detail are impossible. The theorems demonstrated shew that the algebraic definition involves certain properties which are associated with continuous graphs but we must guard against assuming that it involves all such properties. The student will gradually extend his knowledge of formulae representing functions which are not graph-like.

35. One-way Continuous Functions.

Let y, or $f(x)$, be a one-way continuous function of x and let it have the values α, β at the ends a, b of the range (a, b) of x considered.

Since the function is continuous, it has any (every) value between α and β for at least one value of x in (a, b), and since the function is one-way it has no value outside the range (α, β). Further, if the function is up-way it has the greater of two values at the greater of two values of x. Suppose now that the function is *strictly* up-way, so that it cannot have the same value for two values of x. The values of y are then definitely paired off with the values of x, or there is a one-one correspondence, so that to each value, x_1, of x there corresponds only one value, y_1, of y, and to the value y_1 of y there corresponds only the value x_1 of x. In the same way it is proved that there is a one-one correspondence if y is strictly down-way. Figure 14 represents a strictly up-way function and Fig. 15 a strictly down-way function in the range (a, b).

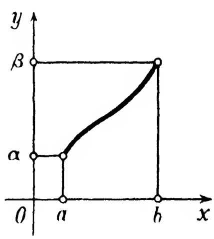


FIG. 14.

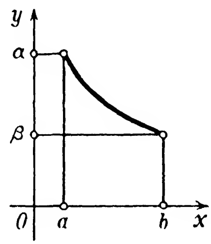


FIG. 15.

It is easily shewn, conversely, that if y is a continuous function of x in the range (a, b) and if there is a one-one correspondence between the values of x and y, the function y is strictly one-way in the range. For, suppose that the values y_1, y_2, y_3 correspond to values x_1, x_2, x_3 which are in ascending order, so that $x_1 < x_2 < x_3$. Then if y_1, y_2, y_3 are not in ascending or descending order, suppose that $y_1 < y_2$ and that $y_2 > y_3$, as represented in Fig. 16. If, now, we take a number η lying between y_1 and y_2 and also between y_2 and y_3, there is a value of x between x_1 and x_2 for which $y = \eta$ and also one between x_2 and x_3 for which $y = \eta$ (by the theorem of intermediate values, Art. 22. 6). Thus there are two values of x corresponding to one value, η, of y, and the supposed one-one correspondence does not exist.

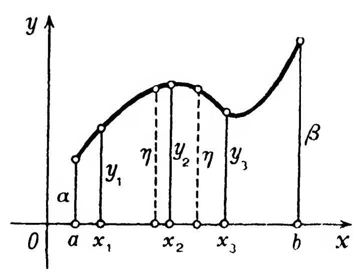


FIG. 16.

If $y_1 = y_2$ or $y_2 = y_3$, there are again two values of x corresponding to one value of y and again the one-one correspondence breaks down.

Thus for any (every) pair of values x_1, x_2 of x, if $x_2 > x_1$, then either y_2 is always greater than y_1 or always less than y_1, that is, the function y is either strictly up-way or strictly down-way.

36. Inversion of a Functional Relation.

We are now prepared to enunciate a simple but fundamental theorem concerning the interchange of the rôles of the independent and dependent variables in a functional relation.

If y, or $f(x)$, is a strictly one-way continuous function of x in a range (a, b), we can regard x as a single-valued function of y in the range (α, β), where α, β are the values of y at the terminal values a, b of x. This statement is justified as a consequence of the one-one correspondence of the values of y and x, which associates one value of x (and one only) with each value of y in the range (α, β). The function of y so defined is clearly strictly one-way. The process of passing from y as a function of x to x as a function of y is called the *inversion of the functional relation* $y=f(x)$, and the function of y so obtained is called the *inverse of the function* $f(x)$.

The functional relation in its inverted form is written $x=\arg f(y)$, where '$\arg f$' is to be treated as a single functional symbol and y is the independent variable. The symbol arises from the practice of describing the independent variable as the *argument* of the function considered. In the present case x is the argument of the original function y, or $f(x)$.

Having established the existence of the function $\arg f(y)$ in the range (α, β), we can, of course, employ any other symbol for the independent variable y. If we employ x for it we may use y for the original independent variable; the process of inversion is then the passage from the relation $x=f(y)$ to the relation $y=\arg f(x)$.

When the function $f(x)$ can be adequately graphed, it is plain that the *same* curve represents the relation $y=f(x)$ and the inverse relation $x=\arg f(y)$, for, if in the direct function y_1 is plotted against x_1, in the inverse function the value x_1 is plotted against y_1. The only change in graphing is thus the interchange of the rôles of the axes Ox, Oy.

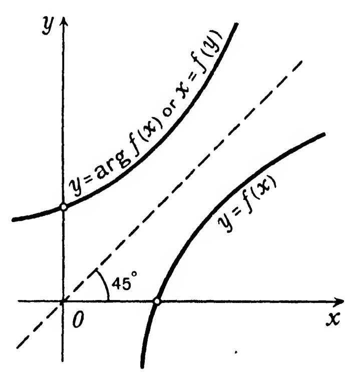


FIG. 17.

When in the inverse function the independent variable is denoted by x and we plot the relation $y=\arg f(x)$, the new graph has the same relation to the axes Ox, Oy as the old graph to the axes Oy, Ox, respectively; in other words, the new graph is the geometrical image of the old in the line bisecting the angle xOy; see Fig. 17.

When $f(x)$ is an established function having a special functional symbol, that symbol will be used instead of f in the symbol for the inverse function. Thus in the case of the Trigonometrical functions, the inverse function corresponding to $\sin x$ is $\operatorname{argsin} x$ (usually shortened to $\operatorname{arsin} x$), and so on.

The relation between a function and its inverse is reciprocal, and it depends on the history of the subject which of the two is treated as the direct function (with its special symbol).

In the case of the mutually inverse Exponential and Logarithmic functions, and also in the case of the Elliptic functions, when the inverses of the original functions were introduced they were assigned special symbols, on account of their importance, and the original functions have retained their special symbols, so that the usual notation for inverse functions is not required.

In the discussion of this Article we suppose that the function $f(x)$ is strictly one-way. The preceding Article shews that this assumption is a necessary one if there is to be one-one correspondence between x and y, that is, if x is to have only a single value corresponding to each value of y. Thus y, or $f(x)$, must be a strictly one-way function of x in the range considered if it is to invert into a one-way function of y in the corresponding range of that variable.

37. Inversion of Single-Valued Functions which are not One-way.

If a function y is not strictly one-way throughout the range of x considered, its inversion presents no theoretical difficulties so long as the range of x can be divided into sub-ranges in such a way that the function is strictly one-way in each, and this can be done in almost all practical cases. After such a subdivision has been effected, inversion for each of the sub-ranges separately defines a function x in a range of y whose terminal values correspond to those of x at the ends of the sub-range in question. We then regard the set of functions obtained by this process as together forming the complete inverse of the original function. It is immediately seen that the functions of this set will have ranges of definition which more or less overlap, as illustrated in Fig. 18, so that the complete inverse function will be multiple-

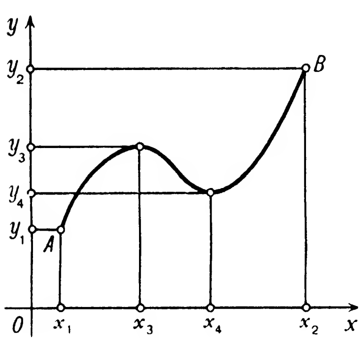


FIG. 18.

valued in parts, at any rate, of the whole range of y. In the theoretical discussions of functions, each function of the set would be treated separately as a single-valued function.

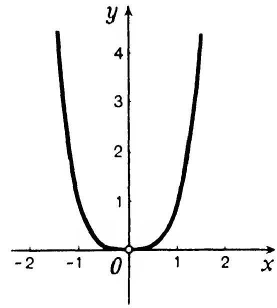


FIG. 19.
$y=x^4$.

Ex. 1. *The functional relation* $y=x^n$, (*n an even integer*). The inversion here leads to the two[1] nth roots of y. We have two sub-ranges $[-\infty, 0)$, $(0, \infty]$, in each of which x^n is strictly one-way, as shewn in Fig. 19, which represents the relation $y=x^4$. In the range $(0, \infty]$ of x, the value of x associated with a (positive) value of y is denoted by $y^{1/n}$, and in the range $[-\infty, 0)$ of x, the value of x associated with y is $-y^{1/n}$.

Ex. 2. *The relation* $y=$*constant*. Suppose that y, or $f(x)$, has the constant value c in the range (x_1, x_2), included in the range (a, b), but is otherwise strictly up-way in (a, b), (Fig. 20). On proceeding to invert the relation $y=f(x)$ we find that corresponding to the value c of y we have a continuous range of values of x extending from x_1 to x_2, that is, when y is taken as the independent variable, there is an indeterminacy in the value of the dependent variable x, of range (x_1, x_2), at the value c of y.

37. 1. *Inversion of the trigonometrical functions.* (i) $y=\sin x$. We consider a succession of sub-ranges of x in each of which y is strictly one-way. These are defined by $(2n\pi-\frac{1}{2}\pi,\ 2n\pi+\frac{1}{2}\pi)$ and $(2n\pi+\frac{1}{2}\pi,\ 2n\pi+\frac{3}{2}\pi)$, where n is any integer. In the first set y is strictly up-way, ranging from -1 to 1, in the second strictly down-way, ranging from 1 to -1.

According to the usual convention, the inverse function for $\sin x$, which is denoted by $\operatorname{arsin} y$, is defined *only* for the range $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ of x, (corresponding to the value 0 of n for the first set of sub-ranges), so that the relation $x=\operatorname{arsin} y$ gives x as an up-way function of y, increasing from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$ as y increases from -1 to 1.

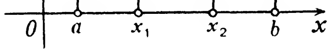


FIG. 20.

Suppose now that y_1 is any value in the range $(-1, 1)$, x_1 being the corresponding value of the function $\operatorname{arsin} y$. On inverting the relation $y=\sin x$ in an up-way sub-range of x, for a given value of n, the value of x to be associated with y_1 is $x_1+2n\pi$, while for inversion in a down-way sub-range, the value is $(2n+1)\pi-x_1$. These are the values of x for which $\sin x$ has the same value y_1. They are included in the formula $n\pi+(-1)^n x_1$, (as is known from elementary Trigonometry), and therefore in the general formula

$$n\pi+(-1)^n \operatorname{arsin} y. \quad\text{..............................(1)}$$

This formula represents the 'complete inverse function' for $\sin x$. The functions for values of n other than 0 are described as the *adjoined functions*

[1] That is, the two 'real' roots.

to arsin y. The up-way functions are obtained by taking n even, the down-way by taking n odd.

(ii) $y = \cos x$. This function is strictly down-way in the sub-ranges of x defined by $(2n\pi, 2n\pi + \pi)$, and strictly up-way in the sub-ranges defined by $(2n\pi + \pi, 2n\pi + 2\pi)$. According to the usual convention, the function arcos y is defined *only* for the down-way sub-range $(0, \pi)$, (corresponding to $n = 0$), so that the relation $x = \text{arcos}\, y$ gives x as a down-way function of y, decreasing from π to 0 as y increases from -1 to 1.

If x_1, y_1 are corresponding values in these ranges, the inversion of the given relation in a down-way sub-range of x, specified by n, leads to $2n\pi + x_1$, and in an up-way sub-range to $2n\pi - x_1$, as the values of x corresponding to y_1. It follows that the 'complete inverse function' for $\cos x$ is given by the general formula $2n\pi \pm x$, that is, by

$$2n\pi \pm \text{arcos}\, y, \qquad \text{.......(2)}$$

which therefore gives the adjoined functions to arcos y.

(iii) $y = \tan x$. This is strictly up-way in the sub-ranges of x given by $[n\pi - \frac{1}{2}\pi, n\pi + \frac{1}{2}\pi]$. According to the usual convention, the inverse function artan y is defined for the sub-range $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ of x, so that the relation $x = \text{artan}\, y$ gives x as a strictly up-way function of y, having the above range, that of y being $[-\infty, \infty]$. The adjoined functions are here easily seen to be given by the formula

$$n\pi + \text{artan}\, y. \qquad \text{.......(3)}$$

(iv) $y = \text{cosec}\, x$, $y = \sec x$, $y = \cot x$. The sub-ranges of x for which the inverse functions arcosec y, arsec y, arcot y are defined are respectively the same as those for arsin y, arcos y, artan y, and the adjoined functions are given respectively by the formulae

$$n\pi + (-1)^n \text{arcosec}\, y, \quad 2n\pi \pm \text{arsec}\, y, \quad n\pi + \text{arcot}\, y. \qquad \text{.......(4)}$$

It should be observed that if x_1 is the value of arcosec y when $y = y_1$, so that $\text{cosec}\, x_1 = y_1$, x_1 is also the value of arsin y when $y = 1/y_1$, for then $\sin x_1 = 1/y_1$. Thus we have the relation

$$\text{arcosec}\, y = \text{arsin}\, \frac{1}{y}. \qquad \text{.......(5)}$$

Similarly we conclude that

$$\text{arsec}\, y = \text{arcos}\, \frac{1}{y}, \quad \text{arcot}\, y = \text{artan}\, \frac{1}{y}. \qquad \text{.......(6)}$$

The inverse function arcosec y is not defined for values of y in the range $[-1, 1]$. It is a strictly down-way function, having the ranges $[0, -\frac{1}{2}\pi)$ $(\frac{1}{2}\pi, 0]$ corresponding to the ranges $[-\infty, -1)$, $(1, \infty]$ of y. The function arsec y is strictly up-way, having the ranges $(0, \frac{1}{2}\pi]$, $[\frac{1}{2}\pi, \pi)$ corresponding to the ranges $(1, \infty]$, $[-\infty, -1)$ of y. The function arcot y is strictly down-way, having the ranges $[-\frac{1}{2}\pi, 0]$, $[0, \frac{1}{2}\pi)$ corresponding to the ranges $[0, -\infty]$, $[\infty, 0)$ of y.

Reference to the inversion of the trigonometrical functions is made again in Chap. VI, Art. 150, where further properties with the graphs of the inverse functions are given.

38. Continuity of Inverse Functions.

We now resume the general discussion of the inversion of a continuous function y or $f(x)$ which is strictly one-way in the range of x considered, and prove that the inverse function is continuous in the corresponding range.

For definiteness, suppose that y is strictly up-way in the range (a, b) of x, so that the inverse function is a strictly up-way function of y in the corresponding range (α, β). Take now a fixed value x_1 of x in (a, b) and let y_1 be the corresponding value of y. Then if (x', y') is another pair of corresponding values, the difference $x' - x_1$ continually increases with $y' - y_1$. Suppose, at first, that $x' > x_1$ and consequently that $y' > y_1$. Because of the one-one correspondence of x and y there corresponds to a positive value h (however small) of $x' - x_1$ a positive value k of $y' - y_1$, and if $y' - y_1 < k$ then also $x' - x_1 < h$. In the same way if $x' < x_1$ then $y' < y_1$, and corresponding to the value h of $x_1 - x'$ there is a value k' of $y_1 - y'$. Also if $y_1 - y' < k'$, then $x_1 - x' < h$. Combining these two results we conclude that for a given h, however small,

$$|x' - x_1| < h \text{ provided } |y' - y_1| < k^*, \quad \text{................(1)}$$

where k^* is the smaller of k and k'. The condition for the continuity of x as a function of y is therefore satisfied at the value y_1. Since y_1 may be any value in the range (α, β), the inverse function is continuous in that range.

In the same way we can prove the theorem when y is down-way in the range (a, b) of x.

To avoid over-elaboration we have supposed that the condition of ordinary continuity applies at the ends of the stated ranges. If it does not, one-sided continuity is substituted for it.

Ex. 1. Because x^n is continuous and one-way in the range $(0, \infty]$, the inverse function $x^{1/n}$ is also continuous in that range.

Ex. 2. The inverse trigonometrical functions are continuous in their respective ranges.

39. Function Defined by Means of an Intermediary Variable (Parameter).

If y, z are two functions of x, whether single- or multiple-valued, which have a range of definition in common, one or more values of y and also one or more values of z will be associated with each value of x in that range. Thus the common value of x associates a certain number of values of y with a certain number of values of z, and, in

a general sense, this association creates a functional relation between y and z. In the following discussion we introduce restrictions which enable us to define z, say, as a *single-valued* function of y in a certain range of that variable.

Suppose that y, z are single-valued, continuous functions of x in a certain range (a, b), and write

$$y=f(x), \quad z=\varphi(x). \quad \ldots\ldots\ldots\ldots(1)$$

Suppose, further, that $f(x)$ is strictly one-way in the range (a, b) and that (α, β) is the corresponding range of its values. Then the relation $y=f(x)$ defines x as a single-valued (one-way) continuous function of y in the range (α, β). Thus z is defined as a single-valued function of y in the range (α, β) by the equations

$$z=\varphi(x), \quad x=\arg f(y), \quad \ldots\ldots\ldots\ldots(2)$$

which give

$$z=\varphi\{\arg f(y)\}. \quad \ldots\ldots\ldots\ldots(3)$$

Further, since z is a continuous function of x and x is a continuous function of y, it follows from Art. 30 that the relation (3) defines z as a continuous function of y in the range (α, β).

The variable x, which here plays an intermediary rôle in determining the relation between z and y, is called a *parameter*, and the functional relation between z and y is said to be *given parametrically* by means of the equations (1).

The above process corresponds, in the theory of single-valued functions, to the elimination of a variable (here x) between two equations containing it, as given in treatises on Algebra. Algebraic elimination usually leads in such a case to a multiple-valued function of y, as it does not introduce any restriction of ranges. For example, the algebraic elimination of x between the equations

$$y=ax^2+bx+c, \quad z=\alpha x^2+\beta x+\gamma \quad \ldots\ldots\ldots\ldots(4)$$

leads to the equation [1]

$$\{a(\gamma-z)-\alpha(c-y)\}^2=(a\beta-\alpha b)\{b(\gamma-z)-\beta(c-y)\}. \quad \ldots\ldots\ldots\ldots(5)$$

This is a quadratic equation for the expression of z in terms of y, so that it involves two single-valued functions of y.

From the geometrical standpoint, if y, z are taken as rectangular coordinates in a plane, the equation (3) defines a curve which is said

[1] Thus, multiplying the second equation by a and the first by α and subtracting, we get

$$x=\{a(\gamma-z)-\alpha(c-y)\}/(\alpha b-a\beta);$$

on multiplying the second by b and the first by β and subtracting, we get

$$x^2=\{b(\gamma-z)-\beta(c-y)\}/(a\beta-\alpha b).$$

to be given parametrically by the equations (1). For any value of the parameter x in the range (a, b), the equations (1) give definite values of y and z and hence a definite plot-point in the plane. This plot-point is usually described as 'the point x.' The aggregate of such plot-points defines the curve or graph given by equation (3). In this connection it is usual to employ another letter, often t, for the parameter, so that x, y can be taken as the rectangular coordinates, and we then write

$$x = f(t), \quad y = \varphi(t). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Ex. 1. *The equations* $x = a\cos\theta$, $y = a\sin\theta$, $(a > 0)$. ....(7)

The letter θ is here used for the parameter and x, y for rectangular coordinates. The elimination of θ leads at once to the relation

$$x^2 + y^2 = a^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

and the equations (7) are thus parametric equations of this circle.

To express y as a single-valued function of x we may proceed as in the general case above, noticing that if θ is restricted to the range $(0, \pi)$ we have $\theta = \operatorname{arcos}(x/a)$, so that

$$y = a\sin\{\operatorname{arcos}(x/a)\} = a\surd[1 - \cos^2\{\operatorname{arcos}(x/a)\}]$$
$$= a\surd(1 - x^2/a^2) = \surd(a^2 - x^2), \quad \ldots\ldots\ldots\ldots(9)$$

y being positive for the given range of θ. The same relation between x and y is obtained if θ is restricted to any of the ranges given by $(2n\pi, 2n\pi + \pi)$.

Similarly, if θ lies in a range given by $(2n\pi + \pi, 2n\pi + 2\pi)$. we find

$$y = -\surd(a^2 - x^2). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

40. Functions of Two Independent Variables; Dimetric Functions.

For the next step in the theory of functions of one variable it is convenient to introduce the language and notation employed for functions of two independent variables. The description of such functions, to which the term *dimetric* [1] is applied, is very similar to that of functions of one variable, (*monometric* functions).

Let x, y be two variables, each continuous *per se*. Either of these variables, x say, may have any value within a certain range, and for any particular value of x, y may have any value within a certain range, (which will, in general, depend on the value of x).

Suppose, now, that with each possible pair of values of x and y there is associated a third definite number. The aggregate of such

[1] This term is formed from Greek *metron* (measure) and the prefix *di* (two), and indicates that the value of the function depends on the measures of two variables. Similarly, *trimetric* indicates dependence on the measures of three variables and *polymetric* on the measures of several.

numbers forms a third variable which we denote by z. This dependent variable z is said to be a single-valued function of x and y, which are called the independent variables.

The general symbol for a function of the two variables x, y is $f(x, y)$, where the comma between x and y *must* be inserted, for otherwise the symbol would represent a function of the product xy. Other letters may be used instead of f, as in the case of a monometric function, and special letters are employed for established functions.

The simplest geometrical representation of a dimetric function is obtained by means of a spatial (or 'solid') figure, in which a plot-point is defined for each triad of values of the three variables x, y, z. Having chosen a plane of representation for the variables x, y and a pair of rectangular axes Ox, Oy in it, we can represent any pair of values, (x_1, y_1) say, by a plot-point N_1 in that plane. We now take a third axis Oz perpendicular to the plane of the axes Ox, Oy, and therefore perpendicular to each of them. Any triad of values (x_1, y_1, z_1) is represented by a plot-point P_1, which is defined by (drawing) a line N_1P_1 of length $|z_1|$ from N_1 parallel to Oz, and in the same or the opposite sense according as z_1 is positive or negative. This mode of representation is illustrated in Fig. 21.

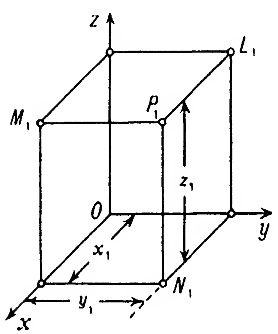


FIG. 21.

The point P_1 is said to have coordinates (x_1, y_1, z_1) relative to the rectangular axes Ox, Oy, Oz, and it is usual to refer to the 'point' (x_1, y_1, z_1) instead of to the triad of values. Since, by definition, x_1, y_1 are the algebraic distances of N_1 from O measured in the directions Ox, Oy, respectively, x_1, y_1, z_1 are the algebraic distances of P_1 from O measured in the directions Ox, Oy, Oz, respectively; thus the representation is symmetrical with respect to x, y and z, and the lengths of the perpendiculars P_1L_1, P_1M_1 to the planes yOz, zOx are respectively $|x_1|$, $|y_1|$, P_1 being on the same side of yOz (or zOx) as Ox (or Oy) when x_1 (or y_1) is positive. As in the cases of one and two variables, we assume that there is a one-one correspondence between the aggregate of triads (x, y, z) and the points of the spatial figure.

It is sometimes convenient to use oblique instead of rectangular axes. In the above description of spatial representation, the plot-point N_1 has then coordinates (x_1, y_1) with respect to oblique axes Ox, Oy. The axis Oz is

oblique to the plane xOy and the plot-point P_1 is defined by drawing N_1P_1 of length $| z_1 |$ parallel to Oz. The lines P_1L_1, P_1M_1 are now drawn parallel to Ox, Oy, respectively. Of course, an angle here described as oblique may be a right-angle as a special case.

If, now, we suppose the value z_1 of the third variable to be that of a single-valued function for the values x_1, y_1 of the independent variables, there is only one point P_1 corresponding to the plot-point N_1 of the number pair (x_1, y_1). We may at this stage drop the suffixes of the variables and consider generally the plot-point N, (x, y), and the corresponding plot-point P, (x, y, z).

For a fixed value of x, the plot-points N lie on a straight line parallel to Oy, and as N moves along this line, P remains in a plane parallel to yOz and describes a curve (in the most general sense) in that plane. The whole set of such curves, corresponding to the set of values of x, defines a surface (in the most general sense), and this surface is a geometrical representation of the function z. This representation is illustrated in Fig. 22.

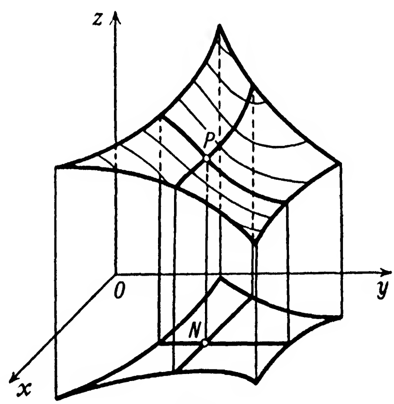


FIG. 22.

It is clear that if the surface is given and P is taken anywhere on it, we have only to draw the perpendicular PN to the plane xOy to determine the value of z which is associated with the values x and y corresponding to the point N.

The limitation of the ranges of the variables x, y will be geometrically defined by a curve or curves bounding the parts of the plane xOy in which the plot-point N can lie. Thus if N is confined to a *circle* of centre O and radius r, the values of x, y are subject to the inequation[1] $x^2 + y^2 \leqslant r^2$. The variable x must therefore lie in the range $(-r, r)$, and for a given value of x, say x_1, in this range, y has the range defined by $y^2 \leqslant r^2 - x_1^2$. Similarly, if N is confined to a

[1] An *inequation* is a symbolic expression of inequality which involves one or more variables. It defines, in general, one or more ranges or domains in which the variable or set of variables must lie. Thus the inequation $h^2 - x^2 > 0$ requires that x should lie in the range $[-h, h]$; the inequation $(x-a)^2 + (y-b)^2 < r^2$ requires that the point (x, y) should lie *inside* the circle of radius r around (a, b) as centre. The term is used whether the range or domain is open or closed. Thus $h^2 - x^2 \geqslant 0$ is an inequation which requires that x should lie in the closed range $(-h, h)$. An *inequality*, in the restricted sense, involves constants only.

rectangle of sides $2a$, $2b$ parallel respectively to Ox, Oy and having its centre at O, the ranges of x, y are respectively $(-a, a)$, $(-b, b)$, each being independent of the value of the other variable.

The parts of the plane in which N can lie are commonly called the *domain* of N and the same word is used for the corresponding aggregate of pairs of values of x, y which correspond to N; thus we speak of the *domain of the variables* x, y, whether it is defined geometrically or analytically.

41. Construction of Spatial Diagrams.

The diagrams corresponding to the spatial mode of representation are perspective (projective) figures. In the above Figures the lines labelled Oy, Oz may be looked on as the actual axes so named of the spatial figure. The line labelled Ox represents the axis Ox which is actually perpendicular to the plane of the paper and directed towards the reader. We describe as the *standard case* that in which the line Ox of the diagram is *equally* inclined to Oy and Oz.

A line in the spatial figure which is parallel to one of the axes is represented in the diagram by a line which is parallel to its representation and whose length is proportional to the actual line. The scale of representation is taken to be the same for lines along Oy and Oz, so that configurations in or parallel to the plane yOz are undistorted in the diagram. In the standard case the scale along Ox is usually taken as one-half that along Oy and Oz.

The diagram may be regarded as a perspective view of the spatial figure, with the view-point at a great distance from O and the plane of the picture parallel to the plane yOz. The standard case, with half-scale along Ox, corresponds to taking the line from O to the view-point at equal (acute) angles with Oy and Oz and at an angle artan $\frac{1}{2}$, (about 27°), with the axis of x.

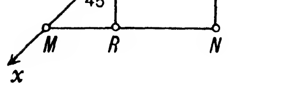


FIG. 23.

It is hardly necessary to say that when letters which appear in the diagram are mentioned in discussions, the reference is to the corresponding points in the spatial figure.

Ex. 1. Prove that in the method of projection described, the lines of projection make equal angles of about 71° 30′ with Oy, Oz.[1]

Ex. 2. Shew that if the lines of projection make an angle of 45° with Ox, and if yOz is taken as the picture-plane, the scales along the axes are unchanged.

Ex. 3. *Isometric projection.* Here the picture-plane is taken perpendicular to the lines of projection, which are at equal angles to the coordinate axes. Shew that this angle is about 54° 44′.

[1] The results of Examples 1, 3 follow from the known result of Solid Geometry that the inclinations α, β, γ of a line OP to the axes Ox, Oy, Oz satisfy the relation $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

41. 1. *Specification of a spatial plot-point by rectangular coordinates in the plane of representation.* Let the plane yOz be the plane of representation and let P, the plot-point corresponding to the number-triad (x, y, z), have rectangular coordinates (ξ, η) relative to axes $O\xi$, $O\eta$ drawn along Oy, Oz (Fig. 23). Then if PN (in space) is perpendicular to the plane xOy, and NM (in space) is perpendicular to Ox, the lengths of the lines OM, MN, NP in the plane of representation are $\frac{1}{2}x$, y, z, respectively, and hence

$$\left.\begin{aligned} \xi &= MN - MR = MN - OM \sin \tfrac{1}{4}\pi = y - x/(2\sqrt{2}), \\ \eta &= NP - NS = z - x/(2\sqrt{2}). \end{aligned}\right\} \quad \text{.............(1)}$$

In particular, a point whose coordinates in space are in the ratio

$$x : y : z = 2\sqrt{2} : 1 : 1, \quad \text{..............................(2)}$$

projects into the origin O.

Suppose, now, that P moves in space along the curve of intersection of the surfaces $z = f(x, y)$, $z = \varphi(x, y)$. Along this curve we have

$$f(x, y) = \varphi(x, y), \quad \text{..................................(3)}$$

and if we suppose this equation equivalent to $y = \psi(x)$, we get from (1),

$$\xi = \psi(x) - x/(2\sqrt{2}), \quad \eta = f\{x, \psi(x)\} - x/(2\sqrt{2}). \quad \text{.................(4)}$$

These equations express the relation between ξ and η in terms of the intermediary variable (parameter) x. By means of them the projection of the curve of intersection on the picture-plane is obtained.

42. Continuity of a Dimetric Function.

The general notion of continuity of a function of one variable is applicable to a function of two variables, but instead of considering a range of one variable we have now to consider a *domain* of the two variables. Let z_1 be the value of the function when x, y have the particular values x_1, y_1. The condition of continuity at (x_1, y_1) requires that the values of the function, for values of x, y in a domain containing (x_1, y_1), should close up to equality as the dimensions of the domain are continually diminished, (its shape, for simplicity, being supposed unaltered).

The simplest way to put this condition into strict form is to suppose the domain always a circle with the point (x_1, y_1) as centre. Let r be its radius at any stage of diminution. Then corresponding to any assigned positive number ε, (however small), we must be able to find a value of r (other than zero) such that the magnitude of the difference between z_1 and the value z' of z at any (every) point inside the circle is less than ε.

Now the square of the distance of the point (x, y) from the point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2$, and the inequation which expresses

that the point (x', y') lies inside this circle is $(x'-x_1)^2+(y'-y_1)^2<r^2$. Hence the symbolic statement of the condition of continuity is

$$|z'-z_1|<\varepsilon \text{ if } (x'-x_1)^2+(y'-y_1)^2<r^2, \quad \text{.............}(1)$$

where r is an appropriately chosen number.

If a square domain of side $2a$ and with its centre at (x_1, y_1) is used instead of the circle, the condition of continuity takes the form

$$|z'-z_1|<\varepsilon \text{ if } |x'-x_1|<a, \; |y'-y_1|<a. \quad \text{.............}(2)$$

We often require to consider the continuity of a function $f(x, y)$ at a point (x_1, y_1) with respect to the variables x, y separately. Suppose that y has the fixed value y_1; then there is a range of x for each value of which $f(x, y_1)$ has a definite value, that is, $f(x, y_1)$ is a function of the single variable x. This function is continuous for the value x_1 of x provided that

$$|f(x', y_1)-f(x_1, y_1)|<\varepsilon \text{ if } |x'-x_1|<h, \quad \text{.............}(3)$$

where h is a suitably chosen number.

In the same way we may consider the continuity of the function $f(x_1, y)$ of the single variable y. It is continuous for the value y_1 of y provided that

$$|f(x_1, y')-f(x_1, y_1)|<\varepsilon \text{ if } |y'-y_1|<k, \quad \text{.............}(4)$$

where k is suitably chosen.

Now the condition of continuity for the function $f(x, y)$ of two variables expressed in the form (2) shews that the two conditions (3), (4) are satisfied provided that (2) is satisfied, that is, the function is continuous at (x_1, y_1) with respect to x and y separately if it is continuous with respect to both variables considered together.

It is easily seen that the converse of this result is not true; the conditions (3), (4), in fact, have regard only to the closing up of values of z corresponding to curves on the representative surface lying in planes through the point (x_1, y_1, z_1) and parallel to yOz, zOx.

A dimetric function is said to be *continuous in a domain* of (x, y) if it is continuous for all points (x_1, y_1) in the domain.

The notion of *uniform continuity* is applicable to dimetric functions, and we may state without proof that continuity in a domain implies uniform continuity in the domain.

We have, for simplicity, supposed in these statements that the domain considered lies within a larger domain of continuity, so that there is no question as to the nature of the continuity at the boun-

dary; this is a matter which we need not discuss at the present stage.

The continuity of the sum, product or quotient of two continuous dimetric functions is proved as in the case of monometric functions, with the proviso that the denominator of the quotient does not vanish at the point, or in the region, considered.

Ex. 1. *Continuity of a polynomial in x and y.* It is sufficient to demonstrate the continuity at the origin, for if we shift the origin to any other point at which the continuity is to be examined, the polynomial transforms to another polynomial.

Consider the typical term Cx^py^q of a polynomial. In a square domain of side $2h$, having its centre at the origin, we have $|x^p| \leqslant h^p$, $|y|^q \leqslant h^q$. Thus

$$|Cx^py^q| \leqslant |C| h^{p+q}$$
$$< \varepsilon, \ (\varepsilon \text{ arbitrary}), \text{ if } h < \{\varepsilon/|C|\}^{\frac{1}{p+q}}. \qquad \text{.....................(5)}$$

A square domain is therefore found in which the variation of the term considered, from its value 0 at the origin, is less than an arbitrarily assigned positive number ε. This term is therefore continuous at the origin.

The continuity at the origin of the sum of any number of such terms, that is, of a polynomial expression, at once follows. Since the origin can be transferred to any other point we conclude that a polynomial function is continuous at any (every) point.

43. Functions of Three or More Variables; Polymetric Functions.

The definition of a dimetric function can evidently be extended to apply to *polymetric* functions. Consider, for simplicity, the case of three independent variables x, y, z. We can represent each triad of values (x, y, z) by a point P in space having those values as coordinates. The aggregate of points P may be supposed to be those of a spatial domain (volume), for example, a cuboid specified by the inequations $|x - x_0| \leqslant h$, $|y - y_0| \leqslant k$, $|z - z_0| \leqslant l$. If, now, we introduce a new variable, u say, and associate with each triad or point a definite value of u, the aggregate of such values of u is described as a (single-valued) *trimetric* function of the three independent variables (x, y, z) in the domain of definition. The functional symbol f is used as before and we may write $u = f(x, y, z)$.

The definition of a continuous trimetric function corresponds exactly to that for a dimetric function, a spatial domain taking the place of a plane domain in the latter case.

The further definitions and results of the previous Article apply equally to the case of three variables.

The language of Geometry is employed also in the case of polymetric functions of more than three variables. If there are n variables $x_1, x_2, \ldots, x_n$, each set of values $(x_1, x_2, \ldots, x_n)$ is spoken of as a *point* in a *domain* of an n-metric space.

Ex. 1. Shew that a polynomial in x, y and z is continuous at any (every) point.

44. Function of a Single Variable defined Implicitly. Implicit Functions.

The theory of functions of a single variable which are defined implicitly is a generalization of the theory of inversion of functions, Art. 36, in which x is defined as a function of y by means of the functional relation $y=f(x)$. This relation gives y directly as a function of x. In the more general theory we start with a relation between x and y which gives neither of the variables directly as a function of the other. Such a relation is most conveniently described in the language of dimetric functions.

Let z, or $f(x, y)$, be a single-valued function of x and y in a certain domain. If there is an aggregate of pairs of values of x and y for which the function $f(x, y)$ has the value 0, that aggregate is said to be defined by the relation $f(x, y)=0$. The aggregate may consist of only one pair of values. As illustrative examples, take the three cases

$$\text{(i) } z=x^2+y^2, \quad \text{(ii) } z=x^2+y^2+c^2, \quad \text{(iii) } z=x^2+y^2-c^2, \quad (c>0).$$

(i) In the first, when z vanishes, we have the relation $x^2+y^2=0$, which is satisfied only by giving both x and y the value 0.

(ii) In the second, the corresponding relation is $x^2+y^2+c^2=0$, and this is purely formal since there is no pair of values of real variables x, y which satisfy it.

(iii) In the third, the corresponding relation is $x^2+y^2-c^2=0$, and it is satisfied by the aggregate of pairs of values defined by $-c \leqslant x \leqslant c$, $y=\pm\sqrt{(c^2-x^2)}$.

The theory of implicit functions discusses the possibility of defining y as a function of x, (or x as a function of y), by means of (or in accordance with) the relation $f(x, y)=0$. In other words, a function $\varphi(x)$ of x is to be found, if possible, such that, if in the function $f(x, y)$ we give y the value $\varphi(x)$ for all values of x in a certain range, the function $f\{x, \varphi(x)\}$ of x has the value zero throughout that range.

Spatial representation of the function z by means of a surface is here very useful to prepare the way for the analytical discussion. Finding the aggregate of pairs of values (x, y) for which z vanishes corresponds to finding the points P on the representative surface for which the ordinates PN vanish, that is, to finding the intersection of the surface with the plane xOy. For a continuous surface, the intersection, if there is any, will, in general, consist of a curve (or curves), but the surface may be such as to cut or touch that plane

only in isolated points. The consideration of the curve of intersection, when it exists, at once suggests the possibility of defining the ordinate y of the curve as a function of the abscissa x, and this corresponds to the possibility of finding a function $y=\varphi(x)$ to satisfy the relation $f(x, y)=0$.

We now proceed to state and prove a simple theorem concerning the existence of a single-valued function defined by the relation $f(x, y)=0$.

44. 1. THEOREM I. *Let the single-valued function* z, *or* $f(x, y)$, *be defined in the (rectangular) domain specified by* $|x-x_0|\leqslant h$, $|y-y_0|\leqslant k$, *where* (x_0, y_0) *is the middle point of the domain and* (x, y) *is another point of it. Suppose further*

(i) *that* $f(x_1,y)$ *is a strictly up-way continuous function of* y *in the range* (y_0-k, y_0+k) *for any (every) fixed value* x_1 *of* x *(including* x_0*) in the range* (x_0-h, x_0+h) ;

(ii) *that* $f(x, k)$ *is positive and* $f(x, -k)$ *is negative throughout the range* (x_0-h, x_0+h) *of* x.

Then the relation $f(x, y)=0$ *defines a single-valued function,* $\varphi(x)$ *say, of* x *in the range* (x_0-h, x_0+h), *such that for each value of* x *in this range,* $\varphi(x)$ *lies in the range* (y_0-k, y_0+k) *and* $f\{x, \varphi(x)\}=0$.

To prove this theorem, consider the function $f(x_1, y)$ of y. This is, by hypothesis, negative when $y=-k$ and positive when $y=k$. Since it is continuous in the range $(-k, k)$ it must take the value 0 at least once in that range. Further, since it is strictly up-way in the same range it can take the value 0 once only. Thus for each value x_1 of x the relation $f(x, y)=0$ defines one and only one value of y lying in the range (y_0-k, y_0+k), that is, it defines a unique[1] single-valued function y, or $\varphi(x)$, of x in the range (x_0-h, x_0+h), y lying always in the range (y_0-k, y_0+k). The theorem is therefore proved.

The existence of such a function when $f(x_1, y)$ is strictly down-way and the signs of $f(x, k)$, $f(x, -k)$ are interchanged, is proved in the same way.

[1] The theorem does not exclude the existence of other functions of x defined implicitly by the relation $f(x, y)=0$ in the range (x_0-h, x_0+h) of x; but the values of any such functions must lie outside the range (y_0-k, y_0+k). For example, if the relation is $2ay-x^2-y^2=0$ and we take $x_0=y_0=0$, $h<a$, $k=a$, the function of which the theorem proves the existence is simply $a-\sqrt{(a^2-x^2)}$. There is, in this case, a second function defined in the range $(-h, h)$ by the same relation, namely $a+\sqrt{(a^2-x^2)}$, but its values lie outside the range $(-a, a)$. The graphs of the two functions are portions of a circle of radius a, the centre being at the point $(0, a)$.

In the applications of the theorem it is sufficient to suppose that (x_0, y_0) is a point at which $f(x, y)$ vanishes so that $f(x_0, y_0)=0$ and $y_0=\varphi(x_0)$, and this we shall do for the remainder of the discussion.

We proceed to prove a second theorem which concerns the continuity of the function $\varphi(x)$.

44.2. THEOREM II. *If the condition, additional to* (i) *and* (ii) *in Theorem I, is imposed that for any (every) fixed value y_2 of y in the range (y_0-k, y_0+k), $f(x, y_2)$ is a continuous function of x in the range (x_0-h, x_0+h), then the function $\varphi(x)$ is continuous in that range.*

Consider, at first, the continuity of $\varphi(x)$ at x_0, where $\varphi(x_0)=y_0$. Since $f(x_0, y)$ is strictly up-way and vanishes when $y=y_0$, it is positive when $y=y_0+k'$ and negative when $y=y_0-k'$, where k' has any positive value, however small, less than k. Further, since $f(x, y_0+k')$ and $f(x, y_0-k')$ are continuous in the range (x_0-h, x_0+h) of x, there is a positive number h' such that the former is positive and the latter negative in the range (x_0-h', x_0+h'). The argument for the existence of the function $\varphi(x)$ in the case of Theorem I can now be applied to the rectangular domain defined by $|x-x_0|\leqslant h'$, $|y-y_0|\leqslant k'$, and it shews that, within this domain, $\varphi(x)$ lies in the range (y_0-k', y_0+k'). Thus, however small k' may be, there is a range (x_0-h', x_0+h') of x such that $\varphi(x)$ lies in the range (y_0-k', y_0+k'), that is, $\varphi(x)$ is continuous at the value x_0 of x.

The same argument can be applied for any other point (x_1, y_1) which is in the original domain and for which $f(x_1, y_1)=0$. Thus $\varphi(x)$ is a continuous function in the range (x_0-h, x_0+h).

If $f(x_1, y)$ is down-way the same result of course follows.

We can now deduce from Theorems I and II a third theorem which is very convenient of application although it is a little less definite in one respect.

44.3. THEOREM III. *If $f(x_1, y)$ is a strictly up-way, continuous function of y in the range (y_0-k, y_0+k) for any (every) fixed value x_1 of x (including x_0) in the range (x_0-h, x_0+h), and if, moreover, $f(x, y_1)$ is continuous in the range (x_0-h, x_0+h) of x for any (every) fixed value y_1 of y in the range (y_0-k, y_0+k), then the relation $f(x, y)=0$ defines a continuous function $\varphi(x)$ of x in a range (x_0-h', x_0+h'), where $h'>0$. The condition* (ii) *of Theorem I is here not imposed.*

The theorem follows at once from the fact that if, as we have assumed, $f(x, k)$, $f(x, -k)$ are continuous in the range (x_0-h, x_0+h),

there is a range (x_0-h', x_0+h') in which they have the same signs as at x_0. Since $f(x_0, y_0)=0$ and the function $f(x_0, y)$ is up-way, these signs are respectively positive and negative. The condition (ii) is therefore satisfied for the range (x_0-h', x_0+h') and the theorem follows. The theorem clearly holds also if $f(x_1, y)$ is strictly down-way instead of up-way.

The student will note that in Theorem I we required the strictly-one-way condition to prove that the function $\varphi(x)$ was single-valued. Without this condition, the argument shews that one or more values of y are determined by the relation $f(x_1, y)=0$ for each value of x_1 in the range (x_0-h, x_0+h). The corresponding function $\varphi(x)$ may thus be multiple-valued.

44. 4. *The polynomial relation.* An important case of the general theorem, Theorem III, is that of the polynomial relation

$$f(x, y)=a_1x+b_1y+a_2x^2+b_2xy+c_2y^2+\dots$$
$$+a_nx^n+b_nx^{n-1}y+\dots+l_ny^n=0, \ (b_1\neq 0). \quad \text{.........(1)}$$

If we put $x=0$ we get

$$f(0, y)=b_1y+c_2y^2+\dots+l_ny^n, \quad \text{.........(2)}$$

and we know, by Art. 20, that this function of y is, in a range $(-k_0, k_0)$, say, of y, strictly up-way or down-way according as b_1 is positive or negative. We shall suppose b_1 positive. If we give x any other value, x_1 say, we find

$$f(x_1, y)=a_1x_1+a_2x_1^2+\dots+a_nx_1^n+(b_1+b_2x_1+\dots+b_nx_1^{n-1})y$$
$$+(c_2+c_3x_1+\dots+c_nx_1^{n-2})y^2+\dots+l_ny^n, \quad \text{.........(3)}$$

and this function is also up-way in a range $(-k_1, k_1)$ of y provided that $b_1+b_2x_1+\dots+b_nx_1^{n-1}$ is of the sign of b_1, which we know to be true, Art. 20. 1, for a range $(-h, h)$ of x_1. We can, for example, take h so that

$$|b_2x_1+\dots+b_nx_1^{n-1}|\leqslant \tfrac{1}{2}|b_1| \quad \text{.........(4)}$$

for all values of x_1 in the range $(-h, h)$, and $f(x_1, y)$ is then strictly up-way for all such values of x_1 in a range $(-k, k)$ of y, defined by

$$k=\frac{\frac{1}{2}|b_1|}{M^*+\frac{3}{2}|b_1|}, \quad \text{.........(5)}$$

where M^* is an upper barrier to the magnitudes of all the coefficients of the powers of y after the first each multiplied by the index of the corresponding power, Art. 20. 3.

Since $f(x_1, y)$ is a polynomial in the variable y it is a continuous function of y, and similarly, for any fixed value y_2 of y, $f(x, y_2)$ is a continuous function of x.

The conditions required by Theorem III are therefore all satisfied and we can assert the existence of a unique continuous function defined by the polynomial relation (1) in the neighbourhood of the origin.

45. Definition of Functions classed as Algebraic.

The preceding Sub-article serves as an introduction to a general definition of *algebraic functions*. In Art. 31 we have given a definition of *ordinary* algebraic functions. These can be described as given *explicitly* by means of the elementary operations, inclusive of rational radication (but exclusive of logarithmation), applied to the independent variable. Now if $f(x)$ is a function obtained in this way, the relation $y=f(x)$ can be rationalized and leads to a polynomial relation of the form $\Sigma a_{rs}x^r y^s=0$, where r, s are positive integers.[1] On the other hand, in the preceding Sub-article we have started with a polynomial of this kind (subject to a certain restriction) and shewn that it defines a function of x. It does not follow, and in fact it is not true, that this function can always be expressed explicitly as an ordinary algebraic function, but it naturally suggests itself that we should put all the functions which can be defined by polynomial relations between x and y into a single class and call them all *algebraic functions*.

The functions which cannot be so defined are classed as *transcendental*; for example, an irrational power of x is transcendental. The trigonometrical functions are also transcendental. Other types of transcendental functions will be introduced subsequently.

Ex. 1. The function $x^{m/n}$, (m, n integers), leads to the polynomial relation $y^n=x^m$.

Ex. 2. The function $x+\sqrt{(x^2-a^2)}$ leads to $y^2-2xy+a^2=0$.

Ex. 3. The function $\dfrac{x+\sqrt{(x^2-a^2)}}{x-\sqrt{(x^2-a^2)}}$ leads to $a^2y^2-2y(2x^2-a^2)+a^2=0$.

Ex. 4. The function $a_0+a_1x^{1/3}+a_2x^{2/3}$ leads to the polynomial relation $(a_0-y)^3-3(a_0-y)xa_1a_2+xa_1^3+x^2a_2^3=0$.

46. Dimetric Function defined Implicitly.

The theorems proved above concerning a function of one variable defined implicitly can be readily extended to the case of two (or more) variables.

Let the single-valued function u, or $f(x, y, z)$, be defined in the (cuboidal) domain specified by the inequations $|x-x_0|\leqslant h$, $|y-y_0|\leqslant k$, $|z-z_0|\leqslant l$, and suppose (i) that u is a strictly up-way continuous function of z in the range (z_0-l, z_0+l) for any (every) fixed pair of values (x, y) in the (rectangular) domain specified by $|x-x_0|\leqslant h$, $|y-y_0|\leqslant k$, (ii) that $f(x, y, l)$ is positive and $f(x, y, -l)$ negative throughout the same domain of (x, y). Then the arguments used in the proof of Theorem I above suffice to shew that the relation $f(x, y, z)=0$ defines z as a single-valued function of (x, y) in the same

[1] See Chrystal, *Algebra*, Part 1, p. 197.

rectangular domain, and this function is unique under the restriction that z lies in the range $(z_0 - l, z_0 + l)$.

The extensions of the above Theorems II and III can be made in a similar manner and will present no difficulty to the student.

47. Abridged Aggregates. Focal Bounds of an Abridged Aggregate.

For our next steps in the study of the elementary properties of functions it is convenient to state and prove some further general theorems regarding aggregates. These theorems are applied subsequently to two special forms of aggregates, *sequences* and *function aggregates*.

We may illustrate the meaning of the expression 'an abridged aggregate' by considering the infinite aggregate of the reciprocals of the positive integers, namely $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$. By abridgment, generally, we mean the removal of any number of elements, finite or infinite, according to any assigned law. The most important process for the particular aggregate considered is that of removing elements in succession according to their magnitude. If the above aggregate of reciprocals is denoted by R, the successive abridged aggregates, which we denote by $R_1, R_2, \dots$, begin with the elements $\frac{1}{2}, \frac{1}{3}, \dots$, respectively. Thus R_n is the aggregate

$$\frac{1}{n+1}, \frac{1}{n+2}, \dots, \frac{1}{n+p}, \dots, \ (p \text{ a positive integer}). \quad \dots\dots\dots\dots(1)$$

A second kind of abridgment would be effected by removing the even elements in succession. In this case R_n would be the aggregate

$$1, \frac{1}{3}, \dots, \frac{1}{2n-1}, \frac{1}{2n+1}, \frac{1}{2n+2}, \frac{1}{2n+3}, \dots, \frac{1}{2n+p}, \dots. \quad \dots\dots\dots(2)$$

When two or more processes of abridgment of the same aggregate are contemplated, they may be distinguished by special symbols, such as $\mathcal{P}_1$, $\mathcal{P}_2$, and so on. The aggregate is then described as *subjected to the processes* $\mathcal{P}_1$, $\mathcal{P}_2$, and so on.

Consider, now, a general infinite aggregate A whose elements are barred above and below, so that it has upper and lower bounds $B^{(u)}$, $B^{(l)}$. Suppose that it is continually abridged by some non-terminating process $\mathcal{P}$ and that any stage of the process is specified by the value of a variable ν which increases indefinitely, either continuously or discontinuously, as the abridgment proceeds. We denote the abridged aggregate at stage ν by A_ν. The aggregate A_ν has an upper bound $B_\nu^{(u)}$, which is plainly not greater than $B^{(u)}$ nor less than $B^{(l)}$. The aggregate of the upper bounds $B_\nu^{(u)}$ therefore has a lower bound, which we call the *focal upper bound* of the aggregate A when abridged

under process $\mathcal{P}$, (or when ν increases indefinitely), and which we denote by $B_f^{(u)}$. In the same way it is shewn that there is a *focal lower bound* $B_f^{(l)}$, which is plainly not greater than $B_f^{(u)}$. The difference $B_f^{(u)} - B_f^{(l)}$ is called the *focal range of the aggregate A under process* $\mathcal{P}$.

We may illustrate this result by means of the above aggregate R. Here we have $B^{(u)} = 1$, $B^{(l)} = 0$. Under the first process of abridgment ν can be taken as n, and we have

$$B_n^{(u)} = \frac{1}{n+1}, \quad B_n^{(l)} = 0. \quad \text{..........................(3)}$$

The lower bound of the aggregate of the upper bounds $B_n^{(u)}$ is thus clearly 0, and this is therefore the value of the focal upper bound $B_f^{(u)}$ under the process in question. The value of $B_f^{(l)}$ is also plainly 0, and the focal range therefore vanishes. Under the second process of abridgment ν can again be taken as n. The bounds of the aggregate are now unaltered by the abridgment. The lower bound of the aggregate of upper bounds is 1 and the upper bound of the aggregate of lower bounds is 0. The focal range is therefore 1 in this case.

48. Focus or Limit of an Aggregate.

Suppose now that the two focal bounds of an aggregate A abridged under process $\mathcal{P}$ are equal, so that the focal range vanishes. The aggregate is then said to have a *focus* under process $\mathcal{P}$, and the common value of the focal bounds is denoted by F and called the *focus of the aggregate*. This focus is also called the *limit* of the aggregate as ν increases indefinitely, and the appropriate symbol for it is then

$$\lim_{\nu \to \infty} A_\nu. \quad \text{..................................(1)}$$

The symbolic statement

$$A_\nu \to F \text{ as } \nu \to \infty \quad \text{............................(2)}$$

is also employed in this connection. The aggregate A is said to *converge* to the focus or limit F under the process $\mathcal{P}$, or as ν increases indefinitely.

It is easy to find an ε-condition for the existence of a focus. According to the definition of a bound, all the elements of the aggregate of upper bounds $B_\nu^{(u)}$ will, for sufficiently large values of ν, differ from the lower bound $B_f^{(u)}$ of this aggregate by less than an arbitrary number, taken as small as we please, and similarly for the difference between the elements of the aggregate of lower

bounds $B_\nu^{(l)}$ and its upper bound $B_f^{(l)}$. Symbolically, we can express this by saying that, corresponding to a positive number ε, a positive number, N say, can be found such that

$$B_\nu^{(u)} - B_f^{(u)} < \varepsilon, \quad B_f^{(l)} - B_\nu^{(l)} < \varepsilon \text{ if } \nu > N. \qquad \text{(3)}$$

We thus have

$$B_\nu^{(u)} - B_\nu^{(l)} - (B_f^{(u)} - B_f^{(l)}) < 2\varepsilon \text{ if } \nu > N. \qquad \text{(4)}$$

If the focal range vanishes, so that $B_f^{(u)} = B_f^{(l)}$, this gives

$$B_\nu^{(u)} - B_\nu^{(l)} < 2\varepsilon \text{ if } \nu > N. \qquad \text{(5)}$$

Let now α_ν, α_ν' be two elements of the aggregate A_ν. Since they lie in the range $(B_\nu^{(l)}, B_\nu^{(u)})$, we have

$$|\alpha_\nu - \alpha_\nu'| < 2\varepsilon \text{ if } \nu > N, \qquad \text{(6)}$$

which is thus a necessary ε-condition. This is merely the formal statement of the obvious fact that under the supposed conditions the elements of the abridged aggregate A_ν must close up to equality as ν is indefinitely increased.

We can now shew, conversely, that if the condition (6) is satisfied, the focal bounds must be equal, because we must then have

$$B_\nu^{(u)} - B_\nu^{(l)} \leqslant 2\varepsilon \text{ if } \nu > N. \qquad \text{(7)}$$

For if $B_\nu^{(u)} - B_\nu^{(l)}$ were greater than 2ε, elements α_ν, α_ν' of the aggregate A_ν could be found so nearly equal to $B_\nu^{(u)}$, $B_\nu^{(l)}$, respectively, that the difference $|\alpha_\nu - \alpha_\nu'|$ would not be less than 2ε, contrary to the condition (6). Hence we have $B_f^{(u)} - B_f^{(l)} \leqslant 2\varepsilon$. Thus the two focal bounds cannot differ by any assignable number and are therefore equal.

It is further clear that the elements α_ν close up to the focus F as ν increases indefinitely. A formal proof is based on the condition (7), of which a consequence is

$$0 \leqslant B_\nu^{(u)} - \alpha_\nu \leqslant 2\varepsilon \text{ if } \nu > N. \qquad \text{(8)}$$

Now since

$$0 \leqslant B_\nu^{(u)} - B_f^{(u)} < 2\varepsilon \text{ if } \nu > N', \ (N' \text{ appropriate}), \qquad \text{(9)}$$

we have

$$|B_f^{(u)} - \alpha_\nu| < 2\varepsilon \text{ if } \nu > N^*, \qquad \text{(10)}$$

where N^* is the greater of the numbers N, N', or, since $B_f^{(u)}$ is now the focus F,

$$|F - \alpha_\nu| < 2\varepsilon \text{ if } \nu > N^*. \qquad \text{(11)}$$

This expresses the property stated.

Conversely, if by any means a number, G say, is found which possesses the property

$$|G-\alpha_\nu|<\varepsilon \text{ (arbitrary) if } \nu>N \text{ (appropriate)}, \quad \text{.......(12)}$$

the abridged aggregate has a focus, namely G. This condition is described as the *second form of condition* for a focus, the *first form of condition* being expressed by (6).

Returning to the aggregate R of the preceding Article, we observe that under the first process of abridgment there is a focus, or limit, 0 to which all the elements of the aggregate $\frac{1}{n+1}, \frac{1}{n+2}, \dots, \frac{1}{n+p}, \dots$, close up as n is indefinitely increased. Thus

$$F=\lim_{n\to\infty} R_n=0. \quad \text{..........................(13)}$$

Under the second process of abridgment the focal bounds are unequal and the focus does not exist.

The above theory is applied in Art. 54 to aggregates of values of a function $f(x)$ in a range of the variable x. The typical cases are the following. (i) The aggregate of values of $f(x)$ in a range $[x_1, x_1+h)$, $(h>0)$. The aggregate is here abridged by diminishing h indefinitely, x_1 being fixed, and we may suppose that the variable ν of the general theory is equal to $1/h$, so that it increases indefinitely. (ii) The aggregate of values of $f(x)$ in a double range $(x_1-h, x_1]$, $[x_1, x_1+h)$. The aggregate is again abridged by diminishing h, or increasing $1/h$, indefinitely.

We now proceed to consider the application of the above theory to *sequences*.

49. Sequences.

We have defined an aggregate as a set of elements regarded as a whole, Art. 9, and although the elements of a number-aggregate have a natural order, that of their algebraic magnitudes, it does not follow that they are to be considered in this or any other particular order. When an infinite aggregate of numbers is such that its elements can be put into one-one correspondence with the positive integers, it is said to be *enumerable*.[1] If, then, the element which corresponds to the integer n is denoted by a_n, the aggregate is represented by

$$a_1, a_2, \dots, a_n, \dots, \quad \text{..........................(1)}$$

and is said to form an infinite *sequence* whose nth element is a_n. The sequence as a whole is usually symbolized by $\{a_n\}$.

The following examples illustrate the possibility, or otherwise, of ordering an aggregate so as to form a sequence.

[1] The adjective in this connection clearly does not imply that the enumeration comes to an end, for the aggregate is infinite.

(i) When the elements of the aggregate of reciprocals of the positive integers are given their natural order, we obtain the sequence $\{1/n\}$.

(ii) Consider next the aggregate consisting of all the rational fractions in the range $[0, 1)$. When the elements are given their natural order there is evidently no definite element next before or after a given one, for between any two rational fractions in the range there are others. If, however, we order the fractions (supposed in their lowest terms) so that those of lower denominator precede those of higher denominator while those of lower numerator precede those of higher numerator and the same denominator, then the elements form a sequence of which the earlier terms are $1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots$.

(iii) Let the aggregate consist of all the real numbers in the range $(0, 1)$. When the elements are given their natural order there is again no definite element next before or after any chosen one. Further, in this case there is no means of ordering the elements so as to form a sequence.

The process of abridgment as applied to sequences is *always* understood to be that in which the elements are omitted in order, unless the contrary is stated. Let this process be applied to the sequence $\{a_n\}$. The integer n here takes the place of ν and the nth abridged sequence starts with the element a_{n+1}, that is, it is

$$a_{n+1},\ a_{n+2},\ \ldots,\ a_{n+p},\ \ldots, \qquad \text{.....................(2)}$$

or

$$\{a_{n+p}\},\ (p=1, 2, \ldots). \qquad \text{..........................(3)}$$

We suppose that the elements are barred above and below, so that there are definite focal bounds of the sequence. When the focal bounds are equal, so that there is a focus or limit, the sequence is said to *converge* or to be *convergent*. If the focal bounds are unequal, the sequence is said to be *divergent* and to have a finite oscillation. The sequence is also divergent if one or both focal bounds do not exist, that is, if the elements are not barred both above and below. The focus or limit of a convergent sequence is the value to which the elements of the indefinitely abridged sequence close up, and it is usually symbolized by[1] $\lim_{n\to\infty} a_n$, or, more concisely, by $[a_n]$. The conditions (6) and (11) of Art. 48 for the existence of a focus here take the respective forms

$$|\, a_n - a_{n'}\,| < \varepsilon \text{ (arbitrary) if } n,\ n' > N \text{ (appropriate)}, \quad \ldots(4)$$

$$|\, F - a_n\,| < \varepsilon \text{ (arbitrary) if } n > N \text{ (appropriate)}. \quad \ldots\ldots.(5)$$

A simple but important case is that of a *one-way* sequence, that is, a sequence whose elements either do not decrease (up-way sequence) or do not increase (down-way sequence) as n increases.

[1] The upper and lower focal bounds of a sequence are also called the *upper* and *lower limits* of the sequence as n increases indefinitely, and are then symbolized by $\overline{\lim}_{n\to\infty} a_n$, $\underline{\lim}_{n\to\infty} a_n$.

An up-way sequence is convergent if its elements are barred above, a down-way sequence if they are barred below. To see this, consider an up-way sequence which is barred above, so that it has an upper bound $B^{(u)}$ which is evidently also the focal upper bound since the elements do not decrease as n increases. Since an element, a_n say, of the sequence exists which differs from $B^{(u)}$ by less than any assigned ε, all the elements following a_n possess the same property, that is,

$$|B^{(u)} - a_{n+p}| < \varepsilon, \ (p \geqslant 0). \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Comparing this condition with (5), we see that $B^{(u)}$ satisfies the condition for the existence of a focus F.

In the same way, for a down-way sequence the lower bound is the limit.

An important special case is that of a sequence formed by the successive decimal approximations to a number. The sequence formed is an up-way sequence of which the focus is the number defined by the successive approximations.

50. Divergence of a Sequence to $+\infty$ or $-\infty$.

If the elements of a sequence $\{a_n\}$ have not both an upper and a lower barrier, the sequence is said to be *unbounded*. In this case $|a_n|$ is greater than an arbitrarily assigned positive number, M say, however large, for *some* value of n, although as n is increased $|a_n|$ may continue to take values less than a fixed number R.

Suppose, at first, that the elements of the sequence increase systematically with n in the sense that

$$a_n > M, \ (M \text{ arbitrary}), \text{ if } n > N, \ (N \text{ appropriate}). \ldots\ldots(1)$$

Then the sequence is said to diverge to (the limit [1]) $+\infty$ as $n \to \infty$, and we write

$$[a_n] = +\infty \quad \text{or} \quad \lim_{n\to\infty} a_n = +\infty. \quad \ldots\ldots\ldots\ldots\ldots(2)$$

In the same way, when

$$a_n < -M, \ (M \text{ arbitrary}), \text{ if } n > N, \ (N \text{ appropriate}), \quad \ldots\ldots(3)$$

the sequence is said to diverge to $-\infty$, and we write

$$[a_n] = -\infty \quad \text{or} \quad \lim_{n\to\infty} a_n = -\infty. \quad \ldots\ldots\ldots\ldots\ldots(4)$$

If the increase is not systematic, the sequence is said to have an *infinite oscillation* as $n \to \infty$.

We now consider examples of sequences, some of which are of frequent application in the sequel.

[1] The common use of the word 'limit' in this connection involves a rather strained application of the meaning of the term.

Ex. 1. *The sequence* $\frac{1}{2}, 1\frac{1}{2}, \frac{3}{4}, 1\frac{3}{4}, \frac{7}{8}, 1\frac{7}{8}, \frac{15}{16}, 1\frac{15}{16}, \ldots$, *where* $a_{2n}=2-\frac{1}{2^n}$, $a_{2n+1}=1-\frac{1}{2^{n+1}}$. The sequence is barred above and below, and we have $B^{(u)}=2$, $B^{(l)}=\frac{1}{2}$. On abridgment the successive upper bounds have the constant value 2; their lower bound (the upper focal bound) is therefore 2, thus $B_f^{(u)}=2$. The successive lower bounds form the sequence $\frac{3}{4}, \frac{3}{4}, \frac{7}{8}, \frac{7}{8}, \ldots$, of which the upper bound is 1, so that $B_f^{(l)}=1$. The focal range is therefore 1 and the sequence is not convergent.

Ex. 2. *The sequence* $\{k^n\}$. We shew that this is convergent if k lies in the range $[-1, 1)$ but is otherwise divergent. We have $a_{n+1}/a_n=k$, and the sequence is therefore one-way if k is positive, being up-way if $k>1$ and down-way if $k<1$.

(i) $k>1$. Writing $k=1+h$, $(h>0)$, we have

$$k^n=(1+h)^n=1+nh+\ldots+h^n>nh, \qquad \ldots\ldots\ldots\ldots(5)$$

and therefore

$$k^n>M, \ (M \text{ arbitrary}), \text{ if } n>M/h. \qquad \ldots\ldots\ldots\ldots(6)$$

The sequence is therefore unbounded above and diverges to $+\infty$. Thus

$$\lim_{n\to\infty} k^n=+\infty, \ (k>1). \qquad \ldots\ldots\ldots\ldots(7)$$

(ii) $0<k<1$. We now write $k=1/(1+h)$, so that

$$k^n=\frac{1}{(1+h)^n}<\frac{1}{nh}. \qquad \ldots\ldots\ldots\ldots(8)$$

Hence $k^n<\varepsilon$, (ε arbitrary), if $n>1/(\varepsilon h)$. The sequence thus converges to the value 0, and we have

$$F=\lim_{n\to\infty} k^n=0, \quad (0<k<1). \qquad \ldots\ldots\ldots\ldots(9)$$

(iii) $-1<k<0$. The elements are now alternately negative and positive. The sequence is barred above and below and there are focal bounds each equal to 0, so that again we have $F=0$.

(iv) $k<-1$. The sequence is now unbounded above and below and has an infinite oscillation as $n\to\infty$.

(v) $k=1$. The elements are all equal and $F=1$.

(vi) $k=-1$. The elements are alternately equal to $+1$ and -1 and there is a focal range of magnitude 2.

Ex. 3. *The sequence* $\left\{\left(1+\frac{1}{n}\right)^n\right\}$. For any positive integer n, the Binomial Theorem gives

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= 1+n\cdot\frac{1}{n}+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3}+\ldots+\frac{1}{n^n}\\ &= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots\\ &\qquad +\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{n-1}{n}\right), \qquad \ldots\ldots\ldots\ldots(10)\end{aligned}$$

and

$$\left(1+\frac{1}{n+1}\right)^{n+1}=1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)+\dots$$
$$+\frac{1}{n!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\dots\left(1-\frac{n-1}{n+1}\right)+\frac{1}{(n+1)^{n+1}}. \quad \text{....(11)}$$

Since $1-\frac{1}{n+1}$, $1-\frac{2}{n+1}$, ..., $1-\frac{n-1}{n+1}$ are respectively greater than $1-\frac{1}{n}$, $1-\frac{2}{n}$, ..., $1-\frac{n-1}{n}$, a comparison of the terms in the two expansions having the same number of factors of this type shews that

$$\left(1+\frac{1}{n+1}\right)^{n+1}>\left(1+\frac{1}{n}\right)^{n}. \quad \text{..........................(12)}$$

The sequence is therefore up-way.

Further, the terms after the first two of the expansion (10) are respectively less than those of the expression

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\dots+\frac{1}{2^{n-1}}; \quad \text{..........................(13)}$$

whose sum is[1] $1-\frac{1}{2^{n-1}}$ and therefore less than 1. Hence we have

$$\left(1+\frac{1}{n}\right)^{n}<3. \quad \text{..................................(14)}$$

The sequence is therefore barred above and has a focus or limit which is less than 3. This limit is actually an irrational number whose value is shewn later, Ex. 3, Art. 162, to be 2·7182818, correct to seven places of decimals. It is an important number, being the base of *natural logarithms*, and is denoted by the letter e. See also Art. 177.

Ex. 4. *The sequence* $\{k^n/n^p\}$, ($k>1$, p *a fixed positive integer*). Here we have

$$\frac{a_{n+1}}{a_n}=\frac{k^{n+1}}{(n+1)^p}\frac{n^p}{k^n}=\frac{k}{\left(1+\frac{1}{n}\right)^p}>\frac{k}{\left(1+\frac{1}{r}\right)^p} \text{ if } n>r. \quad \text{..............(15)}$$

Now $k\Big/\left(1+\frac{1}{r}\right)^p>1$ if $1+\frac{1}{r}<k^{1/p}$, that is, if $r>1/(k^{1/p}-1)$. Thus $a_{n+1}/a_n>1$ after a certain value, n_1 say, of n, and the sequence is then up-way with a ratio of two successive terms which is greater than a fixed number M, where $M>1$. Hence if $n>n_1$, we have $a_n>a_{n_1}M^{n-n_1}$. The elements thus increase indefinitely with n and the sequence is not barred above, or is divergent. We thus write

$$\lim_{n\to\infty} k^n/n^p=+\infty. \quad \text{..............................(16)}$$

Ex. 5. *The sequence* $\{k^{1/n}\}$, (k *positive*). Here we have

$$\frac{a_{n+1}}{a_n}=\frac{k^{\frac{1}{n+1}}}{k^{\frac{1}{n}}}=\frac{1}{k^{\frac{1}{n(n+1)}}}. \quad \text{..........................(17)}$$

[1] This follows from the result $\frac{1-a^n}{1-a}-1=a+a^2+\dots+a^{n-1}$ on putting $a=\frac{1}{2}$.

If $k > 1$ this gives $0 < a_{n+1}/a_n < 1$, while if $k < 1$, $a_{n+1}/a_n > 1$. The sequence is therefore down-way if $k > 1$ and up-way if $k < 1$.

Now if $k > 1$, $k^{1/n} > 1$ and we may write $k^{1/n} = 1 + h$, so that

$$k = (1+h)^n > 1 + nh. \quad \text{.......(18)}$$

Hence we have

$$0 < h < (k-1)/n. \quad \text{.......(19)}$$

Since k is fixed, $(k-1)/n$ (and hence also h) converges to zero as n is indefinitely increased. We therefore conclude that there is a focus given by

$$F = \lim_{n \to \infty} k^{1/n} = 1. \quad \text{.......(20)}$$

If, next, $k < 1$, we have $k^{1/n} < 1$ and we now write $k^{1/n} = 1/(1+h)$. Accordingly

$$k = \frac{1}{(1+h)^n} < \frac{1}{1+nh}; \quad \text{.......(21)}$$

whence we obtain

$$0 < h < \left(\frac{1}{k} - 1\right)\Big/ n. \quad \text{.......(22)}$$

Proceeding as before, we are again led to the result (20).

Ex. 6. *The sequence* $\{n^{1/n}\}$. We have

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{\frac{1}{n+1}}}{n^{\frac{1}{n}}}, \quad \text{.......(23)}$$

and hence

$$\left(\frac{a_{n+1}}{a_n}\right)^{n(n+1)} = \frac{(n+1)^n}{n^{n+1}} = \frac{\left(1+\frac{1}{n}\right)^n}{n} < \frac{3}{n}, \text{ (compare Ex. 3 above),}$$

$$< 1 \text{ if } n > 3. \quad \text{.......(24)}$$

The sequence is therefore down-way. Also we can write $n^{1/n} = 1 + h$, (where $h > 0$), and we have

$$n = (1+h)^n > 1 + nh + \tfrac{1}{2}n(n-1)h^2 > \tfrac{1}{2}n(n-1)h^2 > \tfrac{1}{4}n^2h^2 \text{ when } n > 3, \quad \text{...(25)}$$

since then $n - 1 > \frac{1}{2}n$. This gives

$$0 < h < 2/\sqrt{n}, \quad \text{.......(26)}$$

so that h converges to zero as n increases indefinitely. It follows that $n^{1/n}$ converges to the value 1, and we thus have the result

$$F = \lim_{n \to \infty} n^{1/n} = 1. \quad \text{.......(27)}$$

51. Difference-Sequence.

From a sequence $\{a_n\}$ we may form a new sequence $\{a_n - a_{n-1}\}$, whose elements are the differences of consecutive elements of $\{a_n\}$. We denote it by $\{d_n\}$, where $d_n = a_n - a_{n-1}$ and, in particular, $d_1 = a_1$. The new sequence may be called the (first) *difference-sequence* derived from $\{a_n\}$. Since

$$d_1 + d_2 + \dots + d_n = a_1 + (a_2 - a_1) + \dots + (a_n - a_{n-1}) = a_n, \quad \text{.......(1)}$$

the equation $a_n = \sum_{r=1}^{n} d_r$ defines the elements of the original sequence in terms of those of the difference-sequence, and we have

$$\{a_n\} = \{\sum_{r=1}^{n} d_r\}.$$

The original sequence may be called the *sum-sequence* derived from $\{d_n\}$.

So far, these sequences may be convergent or divergent. Suppose, now, that the sequence $\{a_n\}$ is convergent. Then the difference-sequence is also convergent and its focus or limit is 0, for the convergence of $\{a_n\}$ requires that we can find N such that

$$|a_{n'} - a_n| < \varepsilon \text{ (arbitrary) if } n, n' > N, \quad \ldots\ldots\ldots\ldots(2)$$

and hence, in particular, such that

$$|a_{n+1} - a_n| < \varepsilon \text{ if } n > N. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The latter condition is that for the convergence of the difference-sequence $\{a_n - a_{n-1}\}$ to zero. Thus we can state in symbolic form that if $[a_n]$ exists, then $[d_n] = 0$.

The converse result is not true; it is a necessary but not a sufficient condition for the existence of $[a_n]$ that $[d_n] = 0$. An example is given in the next Article.

52. Sum-Sequence. Infinite Series.

We now consider the sequences of the last Article in the reverse order. Starting with a sequence $\{a_n\}$, we form a new sequence $\{s_n\}$ whose nth element is the sum of the first n elements of $\{a_n\}$, so that $s_n = \sum_{r=1}^{n} a_r$ and $\{s_n\} = \{\sum_{r=1}^{n} a_r\}$. The new sequence $\{s_n\}$ is called, in accordance with the last Article, the sum-sequence derived from $\{a_n\}$. The latter sequence is plainly the difference-sequence derived from $\{s_n\}$, for $s_n - s_{n-1} = a_n$ and $s_1 = a_1$.

We have proved that for the convergence of $\{s_n\}$, $\{a_n\}$ must converge to zero. That this is not a sufficient condition for the convergence of $\{s_n\}$ is shewn by an example. Suppose that the sequence $\{a_n\}$ is $\{\frac{1}{n}\}$. Then $s_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$, and we have

$$s_{2n} - s_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \frac{n}{2n} = \frac{1}{2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The condition of convergence of $\{s_n\}$, namely

$$|s_{n'} - s_n| < \varepsilon \text{ (arbitrary) if } n, n' > N \text{ (appropriate)}, \quad \ldots\ldots(2)$$

is therefore not satisfied, since when $n' = 2n$, $|s_{n'} - s_n| > \frac{1}{2}$.

The sum-sequence $\{s_n\}$ is also called the *infinite series* whose terms form the sequence $\{a_n\}$, or more shortly, the infinite series of term-sequence $\{a_n\}$. The element s_n is called the nth *partial sum* of the infinite series. When the infinite series $\{s_n\}$ is convergent, its focus or limit $[s_n]$ is called the *sum of the infinite series* or the *value* of the series. We denote this sum by S. The condition of convergence of $\{s_n\}$ may now be stated in the alternative form, that there exists a number S such that, for an arbitrary ε, a number N can be found for which

$$|S - s_n| < \varepsilon \text{ if } n > N. \quad \text{......................(3)}$$

The term divergent is applied, as in the case of other sequences, to infinite series which are not convergent.

The above definitions of an infinite series and its sum, when convergent, are in accord with the modern theory of sequences. According to an older phraseology, which is that commonly employed, the form $a_1 + a_2 + \ldots + a_n + \ldots$, where the terms are supposed continued indefinitely, is called the infinite series whose nth term is a_n. If the sum $a_1 + a_2 + \ldots + a_n$ has a limit when n increases indefinitely, that limit is called the sum of the infinite series and is symbolized by $\sum_1^\infty a_n$. The two forms of definition are quite equivalent.

It should be noticed that if the term-sequence (and hence the series) is abridged by the omission of the first M terms, where M is any positive integer, the convergence of the series is not affected, as the first form of convergence (2) immediately shews. Thus in the tests of convergence below we can ignore the elements for which n is less than some integer M.

When a series is convergent, its terms may be looked on as an infinite set of ‘ corrections ’ which, when added in order, define a number by indefinitely close approximation.

In the Calculus, the terms of the infinite series which occur are usually functions of a variable, x say. The previous explanation of convergence (or divergence) then applies for any particular value, x_1 say, of x and the series is said to be convergent (or divergent) at x_1. There is usually a continuous range of values of x for each point of which the series is convergent. The series is then said to be convergent in a range, which may be open or closed and which is called a *range of convergence*. Since in such a range the series has a definite sum for each value of x in it, this sum is a function of x in the range. The infinite series then *defines* a function of x in the range. For example, the infinite series (geometric progression) of term-

sequence $\{x^n\}$ defines a function of x in the open range $[-1, 1]$ of convergence. In this case the function is known to be $1/(1-x)$, but such a series can also be applied to define *new* functions.[1]

The important class of series in which the term-sequence is of the form $\{c_n x^n\}$, where the coefficients c_n depend on n but not on x, have received a special name. They are called *power series*. Many examples of power series will be dealt with in the present Volume but their general theory is beyond its scope.

When either the first or the second form of condition of convergence is applied to a series for any value x_1 of x in a range of convergence, it may or may not be possible to choose the *same* N for all values of x_1 and for the *same* ε, such that

$$|s_{n'} - s_n| < \varepsilon \text{ if } n, n' > N, \quad \text{or} \quad |S - s_n| < \varepsilon \text{ if } n > N. \quad \text{......(4)}$$

If the same value of N serves for all values of x_1, the series is said to be *uniformly convergent*[2] in the range. Power series are always uniformly convergent within their range of convergence, (Art. 264).

Many secondary tests of convergence or divergence have been devised. These often decide the question of the convergence of a particular series very easily without recourse to the general criterion. Only the simplest of these tests will be investigated here and the application of them will only be incidental. The reason is that any series which we shall examine represents a function which is already known and from which we start to find the series. In carrying out the process, a formula is found for the difference between the value of the function and the value of any partial sum, s_n say, of the series. The proof of convergence of the series then consists in shewing directly that this difference converges to zero as n increases indefinitely, so that the value of the function is the sum (or value) of the series.

52. 1. *Secondary tests of convergence.* It is convenient to consider, at first, series whose terms are all positive. Series whose terms are all negative do not require separate treatment, as a change in the sign of all the terms does not affect the convergence or divergence and merely changes the sign of the sum of the series in the case of convergence.

1°. *Series of positive terms.* We denote the term-sequence by $\{a_n\}$ and the sum-sequence, or the series, by $\{s_n\}$.

[1] Compare the remarks of Arts. 146. 5 and 162. 1 with respect to the definitions of the trigonometrical and exponential functions by means of series.

[2] This definition may be compared with that of *uniform continuity*, Art. 33.

(i) Since $s_n - s_{n-1} = a_n$ and a_n is positive, $\{s_n\}$ is up-way. The necessary and sufficient condition for its convergence is therefore that it should be barred above. The upper bound, being its focus or limit, is then the sum of the series.

Corollary. If the term-sequence is expressed in the form $\{b_n - b_{n-1}\}$, the sum-sequence is convergent provided that $\{b_n\}$ is barred above. Here $\{b_n\}$ is up-way, since $b_n - b_{n-1} = a_n$. The result stated therefore follows.

Again, $s_n = b_n - b_0$, so that if B is the upper bound of $\{b_n\}$, s_n has the upper bound $B - b_0$, which is therefore the sum of the series.

(ii) If the series $\{s_n\}$ is convergent and if the sequence $\{c_n\}$ is barred above and below, the series of term-sequence $\{a_n c_n\}$ is convergent. For if B, B' are the bounds of $\{c_n\}$ and $|B| \geqslant |B'|$,

$$|a_{n+1}c_{n+1} + a_{n+2}c_{n+2} + \dots + a_{n'}c_{n'}| \leqslant |B| \,.\, |a_{n+1} + a_{n+2} + \dots + a_{n'}|. \quad \dots(5)$$

Now since $\{s_n\}$ is convergent we can find N such that

$$|a_{n+1} + a_{n+2} + \dots + a_{n'}| < \varepsilon / |B|, \text{ (ε arbitrary), if } n' > n > N, \quad \dots(6)$$

and therefore such that

$$|a_{n+1}c_{n+1} + a_{n+2}c_{n+2} + \dots + a_{n'}c_{n'}| < \varepsilon \text{ if } n' > n > N. \quad \dots\dots\dots(7)$$

The condition of convergence is therefore satisfied by the series of term-sequence $\{a_n c_n\}$.

Corollary. It follows that if the terms of a series are positive and respectively less than, or equal to, the terms of another series which is convergent, then the first series is convergent. Here we take c_n positive and less than, or equal to, 1 for all values of n. Then $\{c_n\}$ is barred above and below and the series of term-sequence $\{a_n c_n\}$ is convergent.

(iii) If $\{b_n\}$ is an up-way sequence which is barred above, and if, for each value [1] of n, (greater than M), we have $a_n \leqslant b_n - b_{n-1}$, the series $\{s_n\}$ is convergent. For the series of term-sequence $\{b_n - b_{n-1}\}$ is convergent, by the Corollary to paragraph (i) above, and if we take the sequence $\{c_n\}$ such that for each value of n, we have

$$a_n = c_n(b_n - b_{n-1}),$$

c_n is positive and not greater than 1, so that $\{c_n\}$ is barred above and below. The series of term-sequence $\{c_n(b_n - b_{n-1})\}$, or $\{a_n\}$, is therefore convergent, by (ii).

[1] It is explained above that any condition of convergence of this character need only be satisfied for values of n greater than some integer M.

Ex. 1. *The series of term-sequence* $\{a_n\}=\{1/n^p\}$, $(p>1)$. Here we can take $b_n=-k/n^{p-1}$, for an appropriate positive value of k. For if we choose a positive integer r such that $p>1+1/r$, we have, for $n>1$,

$$\left(1+\frac{1}{n}\right)^{p-1}-1>\left(1+\frac{1}{n}\right)^{1/r}-1=\frac{1+\frac{1}{n}-1}{\left(1+\frac{1}{n}\right)^{\frac{r-1}{r}}+\left(1+\frac{1}{n}\right)^{\frac{r-2}{r}}+\dots+1}$$

$$>\frac{1/n}{k},\text{ if we take } k=2^{\frac{r-1}{r}}+2^{\frac{r-2}{r}}+\dots+1. \quad \dots(8)$$

Then $$b_{n+1}-b_n=-k\left\{\frac{1}{(n+1)^{p-1}}-\frac{1}{n^{p-1}}\right\}=k\left\{\left(1+\frac{1}{n}\right)^{p-1}-1\right\}\frac{1}{(n+1)^{p-1}}$$

$$>\frac{n+1}{n}\frac{1}{(n+1)^p}>\frac{1}{(n+1)^p}. \quad \dots\dots(9)$$

Thus $1/n^p<b_n-b_{n-1}$, where b_n is barred above by 0. The convergence of $\{1/n^p\}$ therefore follows.

Corollary 1. *Kummer's Test.* If $\{c_n\}$ is a sequence of positive elements and if, for each value of n, (greater than M),

$$c_{n-1}a_{n-1}/a_n-c_n\geqslant 1,$$

the series $\{s_n\}$ is convergent. Here we may put $b_n=-a_nc_n$, so that 0 is an upper barrier to b_n. For we then have, on multiplying the inequality stated by a_n, $a_n\leqslant b_n-b_{n-1}$ and this makes $\{b_n\}$ up-way. It follows that $\{s_n\}$ is convergent.

Corollary 2. *Raabe's Test.* If there exists a positive number K such that for each value of n, (greater than M),

$$(n-1)(a_{n-1}/a_n-1)\geqslant K>1,$$

the series is convergent. Here we may put $b_n=-na_n/(K-1)$, so that 0 is an upper barrier to $\{b_n\}$. For the inequality stated gives $(n-1)a_{n-1}-na_n\geqslant(K-1)a_n$, and hence $a_n\leqslant b_n-b_{n-1}$, so that $\{b_n\}$ is up-way. It follows that $\{s_n\}$ is convergent.

Corollary 3. *D'Alembert's Test.* If for all values of n, (greater than M),

$$a_n/a_{n-1}\leqslant k<1,$$

$\{s_n\}$ is convergent. Here we may put $b_n=-ka_n/(1-k)$, for the stated inequality then makes $a_n\leqslant b_n-b_{n-1}$. If for all values of n, (greater than M), $a_n/a_{n-1}=k<1$, then $a_n=a_1k^{n-1}$ and the series is a geometrical progression of common ratio k. The convergence in this case is proved in elementary Algebra.

2°. *Series having positive and negative terms.* The term-sequence is again denoted by $\{a_n\}$ and the sum-sequence by $\{s_n\}$.

(i) If the series of term-sequence $\{|a_n|\}$ is convergent, so is the series of term-sequence $\{a_n\}$. For in 1°, (ii), starting with the former sequence, we may take the sequence $\{c_n\}$ to be $\{(\text{sig } a_n)1\}$, where [1] $(\text{sig } a_n)$ means the algebraic sign of a_n, and the sequence $\{|a_n| c_n\}$ is then simply $\{a_n\}$. The stated result then follows. In words, the result is that a series is convergent if the series of moduli of its terms is convergent.

When the series of term-sequence $\{|a_n|\}$ is convergent, the series of term-sequence $\{a_n\}$ is said to be *absolutely convergent.* Thus *absolute convergence* is a species of *convergence.*

(ii) If the elements of $\{a_n\}$ are alternately positive and negative and if, moreover, $\{|a_n|\}$ is one-way and converges to zero, then $\{s_n\}$ is convergent. For

$$|s_{n'} - s_n| = |a_{n+1} + a_{n+2} + \dots + a_{n'}| < |a_{n+1}|, \qquad \dots\dots\dots(10)$$

by Ex. 1, Art. 5, and since $\{|a_n|\}$ converges to zero, we can find a value of N such that $|a_{n+1}| < \varepsilon$ if $n > N$. The condition of convergence is therefore satisfied by $\{s_n\}$. Such a series is described as *alternating.*

It is convenient to prove here that the error made in stopping such a series at any term is less than the magnitude of the next following term. We have shewn in Ex. 1, Art. 5, that s_n, or

$$(a_1 + a_2 + \dots + a_{r-1}) + (a_r + a_{r+1} + \dots + a_n),$$

lies between s_r and s_{r+1} for all values of n (greater than $r+1$). We conclude, on increasing n indefinitely, that the sum S of the infinite series $\{s_n\}$ lies in the range (s_r, s_{r+1}). Further, the final result in part (iii) of the Example referred to shews that S lies *between* s_r and s_{r+1}, for it follows from the inequality $|s_n - s_r| > |a_{r+1} + a_{r+2}|$ that the difference $s_n - s_r$ does not converge to zero when n increases indefinitely, and in like manner it follows that the difference $s_n - s_{r+1}$ does not converge to zero.

We thus reach the important result that, in the case of alternating series here considered, the difference $|S - s_r|$ between the sum of the series and the partial sum to any number r of terms is less than $|s_{r+1} - s_r|$, that is, less than the magnitude of the $(r+1)$th term of the series. This result is of very frequent application.

53. Discontinuities. Monometric Functions.

The functions which we have examined in illustrating the condition of continuity have either been continuous throughout the

[1] The symbol 'sig' is an abbreviation of *signum* (or *sign*).

whole or a continuous portion of the range $[-\infty, \infty]$ of the independent variable, or have been continuous in two or more ranges separated either by isolated values for which the function was not continuous or by ranges of the independent variable in which the function was not defined.

Thus, the functions x^2, x^n, (n a positive integer), $\sin x$ are continuous in the range $[-\infty, \infty]$; $1/x$ is continuous in the ranges $[-\infty, 0]$, $[0, \infty]$; $\sqrt{x}$ is continuous in the range $(0, \infty]$; $\sqrt{(a^2 - x^2)}$ is continuous in the range $(-a, a)$; $\sqrt{(x^2 - a^2)}$ is continuous in the ranges $[-\infty, -|a|)$, $(|a|, \infty]$; $\tan x$ is continuous in the ranges given by $[n\pi - \frac{1}{2}\pi, n\pi + \frac{1}{2}\pi]$, where n is an integer; and so on.

In this connection the adjective *discontinuous* is commonly employed in the sense of *not-continuous*. Thus a function is said to be *discontinuous* at a point P if it is not continuous at that point, and the function is said to have a *discontinuity* there. The functions subsequently considered have only isolated *point-discontinuities*. When a function of this kind is discontinuous at a point P, it is continuous elsewhere in a neighbourhood of P, the neighbourhood being two-sided if the function is defined on both sides of P. It is not necessary for us to consider in a general manner the possible modes of variation of a function in the neighbourhood of a point of discontinuity, but it is essential for the sequel to distinguish the three cases of the following Articles 54-56.

54. Potential Continuity.

We consider the behaviour of a function $f(x)$ in a neighbourhood of the value x_1 of x. The function is supposed to be defined for all values of x in the neighbourhood other than x_1; it may or may not be defined at x_1. The function is said to be *potentially continuous* at x_1 if it is possible to define or redefine its value there so that it becomes continuous.[1] In other words, the function is potentially continuous at x_1 if we can find a *fixed* number N such that the difference $|f(x) - N|$ can be made less than an arbitrarily chosen ε by restricting x to a range $(x_1 - h, x_1 + h)$, $(x \neq x_1)$, where h is an appropriately chosen positive number. We may express this condition in the curter form

$$|f(x) - N| < \varepsilon, \text{ (ε arbitrary) if } |x - x_1| < h, \text{ (h appropriate)}, \quad \ldots(1)$$

where x is different from x_1. If this condition is satisfied the function becomes continuous at x_1 when we assign it the value N there.

[1] This definition is meant to include functions which are already continuous at x_1.

The condition for potential continuity can be stated without introducing the number N. In the first place, the condition (1) plainly involves the closing up to equality of the values of the function, other than that at x_1, in a neighbourhood $(x_1 - h', x_1 + h')$ of x_1 when h' is indefinitely diminished, that is, it involves the condition

$$|f(x) - f(x')| < \varepsilon' \text{ (arbitrary) if } \left\{ \begin{matrix} |x - x_1| \\ |x' - x_1| \end{matrix} \right\} < h' \text{ (appropriate)}, \quad \ldots(2)$$

where x, x' are different from x_1. In the second place, the condition (2) involves the condition (1), for it ensures that the abridged aggregate of values of $f(x)$ in the neighbourhood defined by h has a focus or limit when h converges to zero, and if we denote the value of this limit by N, the condition (1) is satisfied, by Art. 48.

A potentially continuous function is said to be *completed* at x_1 by the introduction of the value N. We denote the completed function by $f^*(x)$, so that $f^*(x) = f(x)$ when $x \neq x_1$ and $f^*(x_1) = N$; this function is continuous at x_1. The original function $f(x)$, when not continuous at x_1, is said to have a *removable discontinuity* at x_1.

As in Art. 22. 3, we conclude that if, in this case, the completing value $f^*(x_1)$ is different from zero, there is a neighbourhood of x_1 such that for all points in it, except x_1, $f(x)$ has the same sign as $f^*(x_1)$.

Again, a function which is not potentially continuous at x_1 in the above sense may have one-sided potential continuity, that is, it may be possible to assign it a value at x_1 such that it becomes one-sided continuous there. The condition for upper potential continuity, for example, at x_1 is expressed symbolically by

$$|f(x) - f(x')| < \varepsilon \text{ if } \left\{ \begin{matrix} 0 < x - x_1 < h \\ 0 < x' - x_1 < h \end{matrix} \right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

the values x, x' now lying in the range $[x_1, x_1 + h]$.

Finally, a function may be potentially continuous on both sides at x_1 and have different completing values on the two sides. In this case the function is said to have a *simple unremovable discontinuity* at x_1 which is measured by the difference of the completing values there. If the two completing values are equal, the function is potentially continuous in the original sense.

An important special case of potentially continuous functions is that of barred one-way functions; it is easy to shew that these are always potentially continuous from above and from below at every point within their range of definition. At an end of a range they are one-sided potentially continuous.

Consider an up-way function $f(x)$ at a value x_1. By assumption, the aggregate of values of $f(x)$ in any neighbourhood $(x_1 - h, x_1]$ is barred and therefore has an upper bound, which is plainly that of the abridged aggregate when h is indefinitely diminished, since the function is up-way. Further, h can be chosen so that at the value $x_1 - h$ the function differs from this upper bound by less than ε, where ε is as small as we please. The difference between any value of the function in the neighbourhood $(x_1 - h, x_1]$ is then less than ε, and the condition for potential continuity from below is satisfied. Other cases of one-way functions are dealt with in the same way.

The nomenclature which has been commonly employed for completing values was suggested by the language of limits. In the case of upper (lower) potential continuity the completing value is described as the *limit of $f(x)$ as x converges to x_1 from above (below)*. This description suggests the corresponding symbols

$$\lim_{h\to 0} f(x_1 + h), \quad \lim_{h\to 0} f(x_1 - h). \quad\ldots\ldots\ldots\ldots(4)$$

These symbols are again further abbreviated to $f(x_1 + 0)$, $f(x_1 - 0)$.

In the case of potential continuity in the two-sided sense, the completing value is denoted by

$$\lim_{x\to x_1} f(x) \text{ or by } f(x_1 \pm 0). \quad\ldots\ldots\ldots\ldots(5)$$

We now consider some simple illustrations[1] of the process of completion, adducing at first some cases which occur in elementary Algebra and which have become so familiar that they easily escape recognition. Take first the function defined by the formula x^2. This formula defines the function for all values of x other than 0. Does it define the function at $x = 0$? The answer at the present day is in the affirmative because the convention that 0^2 is 0 has been long introduced and is familiar. But this is not an arbitrary convention ; the value 0 at $x = 0$ is the completing value which makes x^2 continuous there. The completed function is defined by the formula x^2 *together with the convention* $0^2 = 0$. Similarly, the function defined by the formula $\sqrt{x}$ is defined for all positive values of x and is also defined at $x = 0$ when the convention $\sqrt{0} = 0$ is introduced. In this case the completed function is upper continuous only at $x = 0$. A third case is that of the function defined by the formula a^x, where a is a positive constant. This case will be considered at length in Chap. VII, but it may be noted now that the convention $a^0 = 1$ of elementary Algebra introduces the value at $x = 0$ which makes the completed function continuous. Again, when the function $\sin x$ is defined geometrically, its value 0 at $x = 0$ is assigned by convention to complete the function, for there is no right-angled triangle of zero angle by means of which we can define $\sin 0$.

In the cases cited we could not violate the established conventions for any

[1] The remarks concerning 0^2, $\sqrt{0}$ and $\sin 0$ have reference to the early history of Algebra and Trigonometry, not to modern treatments of these subjects, where the postulates and definitions provide for these special values of x^2, $\sqrt{x}$ and $\sin x$.

purpose without calling attention to the fact that we were doing so. In other cases of quite simple character, there is no corresponding established convention. Consider, for example, the function defined by the formula $x/|x|$. This function is equal to 1 when $x > 0$ and equal to -1 when $x < 0$. The lower and upper completing values at $x=0$ are here respectively 1 and -1. The function is potentially continuous from above and below separately at $x=0$ but it is not defined at $x=0$.

Ex. 1. *The function* $y = \dfrac{1-\sqrt{(1-x^2)}}{x}$. This function is defined for all values of x in the range $(-1, 1)$ except the value 0. The formula may be transformed by writing

$$y = \frac{1-\sqrt{(1-x^2)}}{x}\cdot\frac{1+\sqrt{(1-x^2)}}{1+\sqrt{(1-x^2)}} = \frac{x^2}{x\{1+\sqrt{(1-x^2)}\}} = \frac{x}{1+\sqrt{(1-x^2)}},\ (x\neq 0). \quad \text{.....(6)}$$

Now in a small neighbourhood of $x=0$ we may take $|x| < \frac{3}{5}$, say, so that $1+\sqrt{(1-x^2)} > 1+\sqrt{(1-\frac{9}{25})} = \frac{9}{5}$. Hence we have

$$\frac{|x|}{1+\sqrt{(1-x^2)}} < \varepsilon,\ (\varepsilon \text{ arbitrary}),\ \text{if } |x| < \tfrac{9}{5}\varepsilon. \quad \text{..................(7)}$$

The function is therefore potentially continuous at $x=0$ and becomes continuous there when we assign it the value 0. Thus the completed function $\left\{\dfrac{1-\sqrt{(1-x^2)}}{x}\right\}^*$, whose value is 0 when $x=1$, is continuous for this value of x.

54. 1. *Theorems on potentially continuous functions.* We now give some simple theorems which much facilitate the discussion of potentially continuous functions.

54. 2. THEOREM I. *The sum* (*or difference*) *and the product of two functions which are potentially continuous at a value* x_1 *are also potentially continuous functions at* x_1, *and their completing values are respectively the sum* (*or difference*) *and the product of those of the two functions.*

This follows from the fact that the two functions are continuous when completed and therefore also their sum (or difference) and their product. The result applies also to two functions which are potentially continuous from above or from below.

The theorem can be extended to the quotient of two functions, provided that the completing value of the denominator is not zero. It can also be generalized to cover the case of an ordinary algebraic function of potentially continuous functions, subject to the limitation stated for a quotient.

In terms of the language of limits, the simple form of the theorem is enunciated by saying that the limit of the sum, difference, or product of two functions is equal to the sum, difference or product of their limits. The general result is often quoted in the abbreviated

form that the limit of an algebraic function is equal to the same function of the limits of the separate functions composing it. But the necessary limitations must not be overlooked in applying the theorem.

54. 3. THEOREM II. Another simple theorem which is of constant application is that *if an algebraic transformation of a function which is valid in its range or ranges of continuity leads to a function which is continuous at a terminal x_1 of one of these ranges, then the original function is potentially continuous at x_1 and its completing value is the value there of the function to which it is transformed.*

This is evident, since by assumption the function so completed is continuous at x_1.

We proceed to give some simple examples of the completion of potentially continuous functions with the aid of the preceding theorems. Subsequently we shall apply the process of completion systematically in the discussion of rates.

Ex. 2. *The function* $\dfrac{(1+x)^2-1}{x}$. This is a rational function of x and, by Art. 26, it is continuous for all values except $x=0$, where it takes the form 0/0 and is indeterminate. Apart from this value we have

$$\frac{(1+x)^2-1}{x}=\frac{2x+x^2}{x}=2+x. \qquad (8)$$

Now the transform on the right is continuous at $x=0$, its value there being 2. The given function is therefore potentially continuous at $x=0$ and is completed by assigning it the value 2 there. We thus have, for all values of x,

$$\left\{\frac{(1+x)^2-1}{x}\right\}^*=2+x, \qquad (9)$$

and, in particular, $$\left\{\frac{(1+x)^2-1}{x}\right\}^*_0=\lim_{x\to 0}\frac{(1+x)^2-1}{x}=2. \qquad (10)$$

Ex. 3. *The function* x^2/x. This is equal to x provided $x\neq 0$, but if $x=0$ it is undefined. By Theorem II, the function is potentially continuous at $x=0$, and if we assign it the value 0 there it becomes continuous. Hence we have, for all values of x,

$$(x^2/x)^*=x, \qquad (11)$$

and, in particular, $$(x^2/x)^*_0=\lim_{x\to 0}(x^2/x)=0. \qquad (12)$$

Ex. 4. *The function* $\dfrac{1-\sqrt{(1-x^2)}}{x}$. This has already been proved potentially continuous when $x=0$ by means of the ε-condition. We now give an alternative treatment, employing the above theorems.

The function $\sqrt{(1-x^2)}$ is continuous in the range $(-1, 1)$ of x. The given function, being the quotient of two functions which are continuous in the range $(-1, 1)$, is therefore continuous, by Theorem IV, p. 39, except for the

value 0 of x where the denominator vanishes. For this value the formula leads to the expression $0/0$, which is indeterminate.

As in Ex. 1 above, the transformation

$$\frac{1-\sqrt{(1-x^2)}}{x}=\frac{x}{1+\sqrt{(1-x^2)}} \quad \text{.........................(13)}$$

is valid if $x \neq 0$. The transform on the right is, however, continuous at $x=0$, its value there being 0. Therefore the given function is potentially continuous at $x=0$ and is completed when we assign the value 0 to it there.

Ex. 5. *The function* $\dfrac{1-\cos x}{\sin x}$. This is undefined when $x=0$ but is otherwise continuous in the range $[-\pi, \pi]$ We have, if $x \neq 0$,

$$\frac{1-\cos x}{\sin x}=\frac{2\sin^2 \frac{1}{2}x}{2\sin\frac{1}{2}x\cos\frac{1}{2}x}=\tan\tfrac{1}{2}x. \quad \text{........................(14)}$$

Since $\tan \frac{1}{2}x$ is continuous at $x=0$, we conclude that the given function is potentially continuous there, its completing value being 0.

55. Functions Bounded but not Potentially Continuous at a Point of Discontinuity.

In this case the values of the function do not close up to equality as x approaches x_1, and the behaviour of the function may be described by saying that it has *finite oscillations* however closely we approach x_1.

Consider, for example, the function $\sin(1/x)$. It is undefined at $x=0$ and therefore not continuous there. For other values of x it is continuous, since if we write $1/x=u$, we get the continuous function $\sin u$ of u, and u is a continuous function of x if $x \neq 0$. The value of $\sin(1/x)$ oscillates between $+1$ and -1, and these values occur when $x=\dfrac{2}{(4n+1)\pi}$ and $x=\dfrac{2}{(4n+3)\pi}$, respectively. Now such values of x occur in any neighbourhood (however small) of the value 0 of x, since n can be taken as large as we please. The values of the function therefore do not close up to equality as x converges to zero.

A corresponding statement holds for the function $\sin^2(1/x)$, the oscillation being in this case between 0 and $+1$.

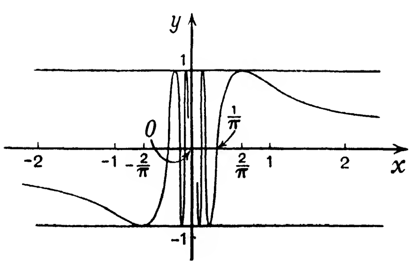


FIG. 24.

$y=\sin\dfrac{1}{x}.$

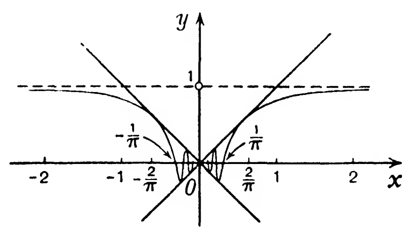


FIG. 25.

$y=x\sin\dfrac{1}{x}.$

On the other hand, the function $x\sin(1/x)$ is potentially continuous at $x=0$. For, since $|\sin(1/x)|\leqslant 1$, we have

$$|x\sin(1/x)|<\varepsilon \text{ (arbitrary) if } |x|<\varepsilon. \quad \text{.......................(1)}$$

The function therefore becomes continuous when completed by the value 0 at $x=0$.

Characteristic portions of the graphs of the functions $\sin(1/x)$, $x\sin(1/x)$ are shewn in Figs. 24, 25.

56. Functions Unbounded at a Discontinuity.

We now consider a case of discontinuity in which the function $f(x)$ is not barred above and below in a neighbourhood of the value x_1. The function $|f(x)|$ then takes values which exceed any assigned value as x approaches x_1, and the function $f(x)$ is said to have an *infinite discontinuity* at x_1. The simplest case is that in which the increase or decrease of $f(x)$ is systematic. Suppose that, given a positive number N, however large, a neighbourhood (x_1-h, x_1+h) of x_1 can be found such that $f(x)>N$ for all values of x other than x_1 in that neighbourhood. Then the function is said to diverge to $+\infty$ at x_1. Similarly if $f(x)<-N$, (N however large), in a neighbourhood (x_1-h, x_1+h) of x_1, the function is said to diverge to $-\infty$.

The symbolic statements in the two cases are

$$f(x_1\pm 0)=+\infty \quad\text{and}\quad f(x_1\pm 0)=-\infty. \quad \text{................(1)}$$

Sometimes the symbolic statements

$$\lim_{x\to x_1} f(x)=+\infty, \quad \lim_{x\to x_1} f(x)=-\infty \quad \text{.................(2)}$$

are employed; we have already referred to the fact that a rather strained application of the word *limit* is here involved.

Exactly similar considerations apply to one-sided neighbourhoods. If a neighbourhood $[x_1, x_1+h)$ exists in which $f(x)>N$, (N arbitrarily large), we say that $f(x)$ diverges to $+\infty$ from above when x approaches x_1, and so for other cases. The symbolic statement for the particular case mentioned is

$$f(x_1+0)=+\infty. \quad \text{.............................(3)}$$

If the limit-notation is used, the corresponding statement is

$$\lim_{h\to 0} f(x_1+h)=+\infty. \quad \text{..........................(4)}$$

It is a common occurrence that $f(x)$ diverges to $+\infty$ from above and to $-\infty$ from below (or vice versa) at a point of discontinuity.

Ex. 1. *The function* $1/x$. This is not continuous at $x=0$. From above the function diverges to $+\infty$, since $1/x>N$ when $0<x<h\leqslant 1/N$, and in like manner, from below it diverges to $-\infty$.

Ex. 2. *The function* $1/\sqrt{x}$. This diverges to $+\infty$ from above at $x=0$. It is undefined below $x=0$.

Ex. 3. *The function* $\dfrac{1}{\sqrt{(a^2-x^2)}}$, $(a>0)$. This is defined only in the range $[-a, a]$. It diverges to $+\infty$ from below at $x=a$ and from above at $x=-a$.

Ex. 4. *The function* $1/\sin(1/x)$. This has an unbounded oscillation at $x=0$.

57. Convergence or Divergence of a Function at Infinity.

When a function is defined in an infinite range it is convenient to introduce definitions of convergence and divergence 'at infinity' which correspond to those already given for an isolated point.

57.1. *Bounded functions.* Let the range of definition of $f(x)$ extend to $+\infty$, so that the function is defined for all values of x greater than some positive number M. We suppose that the function is bounded, so that $|f(x)|<R$, say, when $x>M$.

(i) Let us first suppose that the values of $f(x)$ close up to equality as x is indefinitely increased, so that, corresponding to an arbitrarily chosen ε, we can find a positive number N such that

$$|f(x')-f(x)|<\varepsilon \text{ if } x>N,\ x'>N. \quad \ldots\ldots\ldots\ldots\ldots(1)$$

In this case the function is said to converge at $x=+\infty$. According to the general theorem, Art. 48, this condition of convergence defines a fixed number, L say, such that

$$|f(x)-L|<\varepsilon',\ (\varepsilon' \text{ arbitrary}), \text{ if } x>N',\ (N' \text{ appropriate}). \quad \ldots(2)$$

The function is said to converge to the limit L when x increases indefinitely (or diverges to $+\infty$). The symbolic statement is

$$f(+\infty)=L \text{ or } \lim_{x\to+\infty} f(x)=L. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The statements

$$f(-\infty)=L',\ \lim_{x\to-\infty} f(x)=L' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

are interpreted in a similar manner.

(ii) If the values of the function do not close up to equality as x is indefinitely increased (decreased), the function does not converge at $x=+\infty$ $(-\infty)$ but is *finitely oscillatory* there.

Ex. 1. *The functions* $1/x$, $\sin(1/x)$. These converge to zero both when x diverges to $+\infty$ and when x diverges to $-\infty$; compare Fig. 5, p. 25, and Fig. 24, p. 89.

Ex. 2. *The functions* $\sin x$, $(\sin x)/x$. The former function has a finite oscillation -1 to $+1$ at $x=+\infty$ or $x=-\infty$. The latter function, however, converges to 0 for these values; see Fig. 110, p. 321.

57.2. *Unbounded functions.* A function $f(x)$ is said to be unbounded at $+\infty$ if $|f(x)|$ is greater than an arbitrarily assigned

positive number M for *some* values of x greater than an arbitrarily great number N. In this case the function may, of course, continue to take values less than a fixed number R for some values of x greater than N.

Suppose, at first, that the increase of values of the function is systematic, so that

$$f(x) > M, \ (M \text{ arbitrary}), \text{ if } x > N, \ (N \text{ appropriate}). \quad \text{......(5)}$$

The function is then said to diverge to $+\infty$ as x diverges to $+\infty$, and so in other cases. The symbolic statement in the case mentioned is

$$f(+\infty) = +\infty, \text{ or } \lim_{x \to +\infty} f(x) = +\infty. \quad \text{..................(6)}$$

If the increase is not systematic the function is said to have an *infinite oscillation* at $+\infty$ or $-\infty$, as the case may be.

Ex. 3. *The function* x^m, (m *a positive integer*). This diverges to $+\infty$ when $x \to +\infty$, and to $+\infty$ or to $-\infty$ when $x \to -\infty$, according as m is even or odd.

Ex. 4. *The functions* $x \sin x$, $1/\sin x$. These have infinite oscillations from $-\infty$ to $+\infty$ when x diverges to $+\infty$ or to $-\infty$.

Ex. 5. *The functions* $x^2 \sin^2 x$, $1/\sin^2 x$. These have infinite oscillations from 0 to $+\infty$ and from 1 to $+\infty$, respectively, when x diverges to $+\infty$ or to $-\infty$.

58. Foci (Limits) of Combinations of Sequences.

For some purposes it is convenient to associate with a sequence $\{a_n\}$ a function $f(x)$ of x which is potentially continuous when the sequence is convergent. One way of doing this is to take the value of the function to be a_n when x lies in the range $\left(\frac{1}{n}, \frac{1}{n+1}\right]$; thus

$$f(x) = a_1 \text{ when } x \text{ lies in the range } (1, \tfrac{1}{2}],$$
$$f(x) = a_2 \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad (\tfrac{1}{2}, \tfrac{1}{3}],$$

and so on. The function $f(x)$ is thus defined by means of the sequence in the range $(1, 0]$.

If the sequence is convergent, so that the elements of the indefinitely abridged sequence close up to equality, it is clear that the values of the function $f(x)$ in a range $[0, h)$, $(0 < h < 1)$, close up to equality as h is indefinitely diminished. Thus the function $f(x)$ is potentially continuous from above at the value 0 of x.

It is further clear that the foci of sequence and function are equal, that is, the completing value of the function at $x = 0$ is the focus (limit) of the sequence as $n \to \infty$. Otherwise said, we can regard the focus of the sequence as a completing element of it.

It is now plain that we can apply the theorems concerning the completing values of combinations of potentially continuous functions, Art. 54, to the functions associated with sequences in the above way, and hence to the sequences themselves. Let $\{a_n\}$, $\{b_n\}$ be two convergent sequences. Then the sequences $\{a_n+b_n\}$, $\{a_n-b_n\}$, $\{a_nb_n\}$, $\{b_n/a_n\}$ are convergent, and their foci (or limits) are respectively equal to the sum, difference, product and quotient of the foci of the two original sequences, provided only that in the case of the quotient the focus of $\{a_n\}$ is not zero. These theorems are usually quoted in the short form that *the limit of the sum, product or quotient of two convergent sequences is equal to the sum, product or quotient of their limits.* These results can evidently be generalized to cover further combinations of two or more sequences, as in the case of potentially continuous functions. The student will readily devise proofs of these theorems which do not introduce an associated function such as is employed here.

Ex. 1. *The sequence* $\left\{\dfrac{\alpha_1 n^2+\alpha_2 n+\alpha_3}{n^2}\right\}$. Here we have

$$a_n=\alpha_1+\alpha_2/n+\alpha_3/n^2. \qquad (1)$$

Now since $\lim\limits_{n\to\infty}\alpha_2/n=0$. $\lim\limits_{n\to\infty}\alpha_3/n^2=0$, therefore $\lim\limits_{n\to\infty}a_n=\alpha_1$, or, using the notation $[a_n]$ for the focus of the sequence,

$$[a_n]=\alpha_1. \qquad (2)$$

Ex. 2. *The sequence* $\left\{\dfrac{n^2}{\alpha_1 n^2+\alpha_2 n+\alpha_3}\right\}$. Denoting the typical element by b_n, we have $b_n=1/a_n$, where a_n is given by (1), and hence

$$[b_n]=1/[a_n]=1/\alpha_1. \qquad (3)$$

Ex. 3. *The sequence* $\left\{\left(1+\dfrac{1}{n}\right)^{kn}\right\}$, (*k a fixed integer*). Here we have

$$a_n=\left(1+\frac{1}{n}\right)^{kn}=\left(\left(1+\frac{1}{n}\right)^n\right)^k, \qquad (4)$$

and the focus is given by

$$[a_n]=\left[\left(1+\frac{1}{n}\right)^n\right]^k=e^k, \quad \text{(see Ex. 3, p. 75).} \qquad (5)$$

58. 1. *Application to infinite series.* The above results apply also to infinite series, since they are defined as sequences, Art. 52. Thus, if the series $\{s_n\}$, $\{s_n'\}$, whose term-sequences are $\{a_n\}$, $\{a_n'\}$, are both convergent, then the series whose term-sequence is $\{a_n+a_n'\}$ is also convergent, and if S, S' are the sums of the first two series, the sum of the third is $S+S'$.

The convergence of the third series remains and the sum is unaltered if in it each term a_n+a_n' is replaced by two consecutive terms a_n, a_n'; for corresponding to any partial sum of the modified series, there is a partial sum of

the unmodified one, which differs from it at most by the addition of one last term. Thus

$$(a_1 + a_1') + (a_2 + a_2') + \dots + (a_n + a_n') \text{ and } a_1 + a_1' + a_2 + a_2' + \dots + a_n + a_n'$$

are equal partial sums of the two series, while the partial sum

$$a_1 + a_1' + a_2 + a_2' + \dots + a_{n-1}' + a_n$$

differs from the first of those sums by the term a_n'. Since the terms a_n, a_n' converge to zero when n is increased indefinitely, the corresponding partial sums converge to the same limit, and thus the modified and unmodified series have the same sum.

59. Orders of Magnitude.

In ordinary language we say that one number (or quantity) is of *a higher order of magnitude* than a second when the ratio of the first to the second is a 'large' number, and the second is then of *a lower order of magnitude* than the first. Two numbers (or quantities) are of the same order of magnitude when their ratio is neither a large number nor a small fraction. The latitude allowed in the magnitude of the ratio will depend on the question that is being considered.

It is often convenient to make a comparison of a number with a power of 10. A number is said to be *of the order* 10^n if its ratio to 10^n is neither large nor small. If n is large, a ratio of 10 would usually not be looked on as large nor $\frac{1}{10}$ as small.

The usefulness of a consideration of the orders of magnitude of quantities, both in practical life and in science as a first guide to action, will be familiar to students in various connections and need not be insisted on here.

59. 1. *Orders of infinitesimals.* In the Calculus, orders of magnitude are most often considered with respect to variables which are simultaneously converging to zero, and the notions involved are constantly appealed to in the current treatment of rates. Let $f(x)$ be a function which converges to zero with x. The order of magnitude of $f(x)$ with respect to x is decided, in the cases with which we are concerned, by a comparison of the value of the function with that of a power of x. In the simplest cases, positive integral powers of x only need be considered, so that the standards of comparison are $x, x^2, \dots, x^n, \dots$. The function $f(x)$ is said to be *of the order of x^n as x converges to zero*, or to be *of the nth order with respect to x*, if the ratio $f(x)/x^n$ has two focal bounds of the same sign. The standards of comparison $x, x^2, \dots, x^n, \dots$ and the functions of the corresponding orders are spoken of as *infinitesimals* of the *first, second, ..., nth, ... orders.* The usual case is that in which the focal

bounds of $f(x)/x^n$ coincide, so that there is a focus or limit *which is not zero.*

The notation $f(x)=O(x^n)$ is sometimes used to express that the function $f(x)$ is of the order of x^n, that is, of the nth order, when x converges to zero.

Since the ratio of each power of x to the preceding power converges to zero with x, an infinitesimal of the nth order is ultimately indefinitely small[1] compared with one of the $(n-1)$th or lower order.

When there can be no doubt as to the standards of comparison, we say simply that $f(x)$ is *of the nth* (*or other*) *order of smallness.*

It is plain that fractional and negative orders can be introduced in the same manner, but an 'infinitesimal' of negative order is actually an 'infinital,' (that is, ultimately infinite).

If x converges to x_1 instead of to 0 the powers $x-x_1$, $(x-x_1)^2$, ... , $(x-x_1)^n$, ... , of course take the place of powers of x.

59.2. *Simple theorems concerning infinitesimals.* The following results concerning the orders of magnitude of combinations of functions are immediate consequences of the definitions.

I. *The order of a sum of two or more functions is that of the function* (*or functions*) *of least order.* Thus the function (or functions) of least order need alone be considered in finding the order of the sum.[2]

II. *The order of a product of two or more functions is the sum of the orders of the separate functions.* In particular, the order of the mth power of a function of order n is mn.

III. *The order of a quotient of two functions is the difference of the orders of the numerator and the denominator.* If, however, the two functions are of the same order, the quotient is *not* an infinitesimal; it may be described as of zero order. If the denominator is of higher order than the numerator, the quotient is an infinital.

Ex. 1. *The function* $a_2x^2+a_3x^3+\ldots+a_nx^n$. This is of the second order, if $a_2\neq 0$, for the ratio of it to x^2 is the polynomial $a_2+a_3x+\ldots+a_nx^{n-2}$, which has the focus a_2 as $x\to 0$.

Ex. 2. *The function* $(a_2x^2+a_3x^3+\ldots+a_nx^n)(b_4x^4+b_5x^5+\ldots+b_nx^n)$. This is of the sixth order, if $a_2b_4\neq 0$, for the ratio of it to x^6 is equal to

$$(a_2+a_3x+\ldots+a_nx^{n-2})(b_4+b_5x+\ldots+b_nx^{n-4}),$$

which has the focus a_2b_4 as $x\to 0$.

[1] Thus *high order* connotes *smallness* in the case of small quantities but *largeness* in the case of large quantities.

[2] It is customary to say in this connection that an infinitesimal can be rejected when of a higher order than the ones retained. It is ultimately indefinitely small compared with them.

Ex. 3. *The function* $\dfrac{b_4x^4+b_5x^5+\ldots+b_nx^n}{a_2x^2+a_3x^3+\ldots+a_nx^n}$. This is [redacted] order, if $a_2b_4 \neq 0$, for the ratio of it to x^2 is the rational function

$$\frac{b_4+b_5x+\ldots+b_nx^{n-4}}{a_2+a_3x+\ldots+a_nx^{n-2}},$$

which has the focus $\dfrac{b_4}{a_2}$ as $x \to 0$.

60. Symbolism for Differences. The Δ-Notation.

The discussion of the continuity of a function in this Chapter has involved repeated consideration of corresponding increments of the independent variable and the function. No special symbolism has been introduced for such increments as it was thought desirable to avoid distraction of the attention from the fundamental notions by the use of an unfamiliar notation. We now proceed to introduce the customary increment-notation and explain the elementary operations connected with it, in preparation for the following Chapters where it will be systematically employed.

60. 1. *The symbol* Δ. If the independent variable changes from a value x to another value x', or $x+h$, the algebraic increment $x'-x$, or h, is denoted by[1] Δx, (delta x), where Δx is a compound symbol for the algebraic number h and Δ is an *operative symbol*, which for the present is not to be separated from the variable operated on. The corresponding increment of the function $f(x)$ is denoted by $\Delta f(x)$, and we write

$$\Delta f(x)=f(x')-f(x)=f(x+h)-f(x)=f(x+\Delta x)-f(x). \quad \ldots\ldots.(1)$$

The Δ-notation may, of course, be used for two or more increments of x starting from the same or different values of that variable, appropriate affixes being employed to distinguish between them. Thus a single increment from x_1 to any value x, or x_1+h, may be denoted by Δx, (or by Δx_1, when attention must be called to the starting point x_1 of the increment). A second increment, from x_1 to another value x', or x_1+h', may then be denoted by $\Delta' x$, (or by $\Delta' x_1$), and so on. The corresponding increments of $f(x)$ are $\Delta f(x)$, $\Delta' f(x)$, {or $\Delta f(x_1)$, $\Delta' f(x_1)$ when the starting point x_1 is to be indicated}. If y is used for the dependent variable $f(x)$, these increments are denoted by Δy, $\Delta' y$, (or by Δy_1, $\Delta' y_1$), where $y_1=f(x_1)$. We thus have

$$\Delta y=y-y_1, \quad \Delta' y=y'-y_1. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$$

[1] Δ (the Greek capital *D*) may be regarded as an abbreviation of *difference*, the equivalent of algebraic increment. The small Greek delta, δ, is also employed in place of Δ to denote differences, but in this Book we shall only employ δx to denote a *small* increment of x.

The condition for the continuity of $f(x)$ at any value x can now be written in the more compact form

$$|\Delta f(x)| < \varepsilon, \ (\varepsilon \text{ arbitrary}), \text{ for } |\Delta x| < h, \ (h \text{ appropriate}). \quad \text{...(3)}$$

If a second function $g(x)$ is considered, $\Delta g(x)$ will denote its increment due to the increment Δx of x, that is,

$$\Delta g(x) = g(x+\Delta x) - g(x). \quad \text{......(4)}$$

Consider now the increment of the sum $f(x)+g(x)$ of the two functions; we have, by definition,

$$\begin{aligned}\Delta\{f(x)+g(x)\} &= \{f(x+\Delta x)+g(x+\Delta x)\} - \{f(x)+g(x)\}\\ &= \{f(x+\Delta x) - f(x)\} + \{g(x+\Delta x) - g(x)\}\\ &= \Delta f(x) + \Delta g(x). \quad \text{......(5)}\end{aligned}$$

The increment of the product $f(x)g(x)$ is found from

$$\begin{aligned}\Delta\{f(x)g(x)\} &= f(x+\Delta x)g(x+\Delta x) - f(x)g(x)\\ &= \{f(x)+\Delta f(x)\}\{g(x)+\Delta g(x)\} - f(x)g(x)\\ &= \Delta f(x)\,.\,g(x) + f(x)\,.\,\Delta g(x) + \Delta f(x)\,.\,\Delta g(x)\\ &= \Delta f(x)\{g(x)+\Delta g(x)\} + f(x)\,.\,\Delta g(x). \quad \text{......(6)}\end{aligned}$$

The increment of the quotient $f(x)/g(x)$ is found from

$$\begin{aligned}\Delta\left\{\frac{f(x)}{g(x)}\right\} &= \frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)} = \frac{f(x)+\Delta f(x)}{g(x)+\Delta g(x)} - \frac{f(x)}{g(x)}\\ &= \frac{\Delta f(x)\,.\,g(x) - f(x)\,.\,\Delta g(x)}{g(x)\{g(x)+\Delta g(x)\}}. \quad \text{......(7)}\end{aligned}$$

Ex. 1. Shew that $\Delta\{f(x)\}^2 = 2f(x)\Delta f(x) + \{\Delta f(x)\}^2$.

Ex. 2. Shew that $\Delta\{1/f(x)\} = -\Delta f(x)/[f(x)\{f(x)+\Delta f(x)\}]$.

The symbol Δ here introduced is the characteristic symbol of the theory of Finite Differences, which at many points has a close connection with the Differential Calculus. To realize or adequately to utilize the connection requires a consideration of the repetition of the operation represented by the symbol Δ.

60. 2. *Repetition of the operation* Δ. If we suppose Δx or h to be a fixed constant, we may proceed to find a formula for the increment of $\Delta f(x)$, or of $f(x+h) - f(x)$, when x is increased by h. This is denoted by $\Delta\{\Delta f(x)\}$. We have

$$\begin{aligned}\Delta\{\Delta f(x)\} = \Delta\{f(x+h) - f(x)\} &= \Delta f(x+h) - \Delta f(x)\\ &= \{f(x+2h) - f(x+h)\} - \{f(x+h) - f(x)\}\\ &= f(x+2h) - 2f(x+h) + f(x). \quad \text{......(8)}\end{aligned}$$

This difference of a difference is called the *second difference of* $f(x)$ due to the fixed increment h of x. The difference $\Delta f(x)$ is called the *first difference* when necessary to distinguish it. The symbol for the second difference is always abbreviated to the form $\Delta^2 f(x)$, where Δ^2 is, of course, not a square in the numerical sense, but represents a once-repeated operator.

The second difference is a function of x and we may take its difference due to the same increment h of x, as before. This gives the *third difference* of $f(x)$, which is denoted by $\Delta^3 f(x)$. We have

$$\begin{aligned}\Delta^3 f(x) &= \Delta\{\Delta^2 f(x)\}\\ &= \Delta f(x+2h) - 2\Delta f(x+h) + \Delta f(x)\\ &= \{f(x+3h) - f(x+2h)\} - 2\{f(x+2h) - f(x+h)\} + f(x+h) - f(x)\\ &= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x). \quad \text{......................}(9)\end{aligned}$$

This process may be repeated any number of times. The formulae obtained for the second and third differences have the same sets of numerical multipliers of $f(x), f(x+h), \ldots$ as occur in the expansion of $(x-1)^2$ and $(x-1)^3$, respectively, and on trial the corresponding statement will be found true for the fourth, fifth, ... differences as compared with $(x-1)^4$, $(x-1)^5, \ldots$. This suggests the general formula[1]

$$\begin{aligned}\Delta^n f(x) = f(x+nh) - nf\{x+(n-1)h\} + \frac{n(n-1)}{2!}f\{x+(n-2)h\}&\\ + \ldots + (-1)^r \tbinom{n}{r} f\{x+(n-r)h\} + \ldots + (-1)^n f(x),& \quad \text{......}(10)\end{aligned}$$

the validity of which can be demonstrated by Mathematical Induction. If we suppose the formula correct for some particular value of n, we deduce from it,

$$\begin{aligned}\Delta^{n+1} f(x) &= \Delta\{\Delta^n f(x)\}\\ &= \Delta f(x+nh) - n\Delta f\{x+(n-1)h\} + \frac{n(n-1)}{2!}\Delta f\{x+(n-2)h\}\\ &\qquad + \ldots + (-1)^n \Delta f(x)\\ &= f\{x+(n+1)h\} - f(x+nh) - n[f(x+nh) - f\{x+(n-1)h\}]\\ &\qquad + \ldots + (-1)^n\{f(x+h) - f(x)\}\\ &= f\{x+(n+1)h\} - (n+1)f(x+nh) + \frac{(n+1)n}{2!}f\{x+(n-1)h\}\\ &\qquad + \ldots + (-1)^{n+1} f(x), \quad \text{.............................}(11)\end{aligned}$$

[1] In this formula, $\binom{n}{r}$ is the symbol, now usually employed, for the Binomial Coefficient $\dfrac{n(n-1)\ldots(n-r+1)}{r!}$.

where the coefficient of $f\{x+(n-r)h\}$ is $-(-1)^r\binom{n}{r}+(-1)^{r+1}\binom{n}{r+1}$ that is,

$$-(-1)^r\frac{n(n-1)\dots(n-r+1)}{r!}+(-1)^{r+1}\frac{n(n-1)\dots(n-r)}{(r+1)!}, \quad \dots(12)$$

or
$$(-1)^{r+1}\frac{(n+1)n(n-1)\dots(n-r+1)}{(r+1)!}, \quad \dots\dots(13)$$

that is
$$(-1)^{r+1}\binom{n+1}{r+1}. \quad \dots\dots(14)$$

The suggested formula therefore reappears for the $(n+1)$th difference. Thus if the formula is correct for one value of n it is correct for the next greater value. Since the formula is true for the first, second and third differences, Induction shews that it is true generally. If the student compares in detail the process of forming the successive differences with the process of raising $x-1$ to successive powers by repeated multiplication by $x-1$, he will see that the powers $x^0, x^1, x^2, \dots, x^n$ and their coefficients appear in exactly the same way as $f(x), f(x+h), f(x+2h), \dots, f(x+nh)$ and their coefficients, so that the law of formation of the nth difference is exactly the same as that for the nth power of $x-1$.

We now shew how to express $f(x+nh)$ in terms of the successive differences of $f(x)$. From (8) and (9) we find, in turn,

$$f(x+2h)=f(x)+2\Delta f(x)+\Delta^2 f(x), \quad \dots\dots(15)$$
$$f(x+3h)=f(x)+3\Delta f(x)+3\Delta^2 f(x)+\Delta^3 f(x), \quad \dots\dots(16)$$

and the form of these results suggests that the general result is

$$f(x+nh)=f(x)+n\Delta f(x)+\binom{n}{2}\Delta^2 f(x)+\dots+\binom{n}{r}\Delta^r f(x)+\dots+\Delta^n f(x). \quad (17)$$

That this is the correct form is easily verified by the method of Induction. Assuming it to be true for any value of n, we have

$$\begin{aligned} f\{x+(n+1)h\} &= f(x+nh)+\Delta f(x+nh)\\ &= f(x)+n\Delta f(x)+\dots+\binom{n}{r}\Delta^r f(x)+\dots+\Delta^n f(x)\\ &\quad +\Delta f(x)+n\Delta^2 f(x)+\dots+\binom{n}{r-1}\Delta^r f(x)+\dots+\Delta^{n+1} f(x)\\ &= f(x)+(n+1)\Delta f(x)+\dots+\left\{\binom{n}{r}+\binom{n}{r-1}\right\}\Delta^r f(x)+\dots+\Delta^{n+1} f(x)\\ &= f(x)+(n+1)\Delta f(x)+\dots+\binom{n+1}{r}\Delta^r f(x)+\dots+\Delta^{n+1} f(x), \quad \dots\dots(18)\end{aligned}$$

which is obtainable from (17) by writing $n+1$ for n. Since the proposed formula is true when n has the values 1, 2 and 3, we infer it to be true generally.

61. The Operator E.

In the theory of Finite Differences a special symbol E is used for the operation of changing x into $x+\Delta x$, or $x+h$. We write Ex for $x+\Delta x$, or $x+h$, and $Ef(x)$ for $f(x+\Delta x)$, or $f(x+h)$.

It is desirable to deduce a few formulae connecting the two operators Δ and E. The fundamental formulae are

$$Ef(x)=f(x+h)=f(x+h)-f(x)+f(x)$$
$$=\Delta f(x)+f(x), \quad \text{.......(1)}$$

and
$$\Delta f(x)=Ef(x)-f(x). \quad \text{.......(2)}$$

The meaning of the repeated operator is evident. We have

$$E(Ex)=E(x+h)=x+2h, \quad \text{.......(3)}$$

or, with the same abbreviation of symbolism as in the case of Δ, $E^2x=x+2h$. So $E^3x=x+3h$, and generally,

$$E^nx=x+nh. \quad \text{.......(4)}$$

Again, $E\{Ef(x)\}=Ef(x+h)=f(x+2h)$ or $E^2f(x)=f(x+2h)$, and generally,

$$E^nf(x)=f(x+nh). \quad \text{.......(5)}$$

Let us apply these formulae to the results of the preceding Sub-article. We have

$$\Delta^2f(x)=f(x+2h)-2f(x+h)+f(x)=E^2f(x)-2Ef(x)+f(x), \quad \text{.......(6)}$$

and generally,

$$\Delta^nf(x)=E^nf(x)-nE^{n-1}f(x)+\frac{n(n-1)}{2!}E^{n-2}f(x)+\ldots$$
$$+(-1)^r\binom{n}{r}E^{n-r}f(x)+\ldots+(-1)^nf(x). \quad \text{.......(7)}$$

Again, on account of (5), equation (17) above takes the form

$$E^nf(x)=\Delta^nf(x)+n\Delta^{n-1}f(x)+\ldots+\binom{n}{r}\Delta^{n-r}f(x)+\ldots+f(x). \quad \text{.......(8)}$$

62. Separation of Operators.

It is very convenient, in connection with theorems of the types just investigated, to introduce a convention by which the operative symbols can be separated from the functions operated on. Beginning with the simplest formula $\Delta f(x)+f(x)$, it is written in the form

$$(\Delta+1)f(x), \quad \text{.......(1)}$$

on the understanding that this form is to be interpreted by the use of the distributive rule, that is, each of the terms Δ, 1 of the sum in the parentheses is to be associated with $f(x)$ and the results added. So $\Delta^2f(x)+2\Delta f(x)+f(x)$ is written in the form

$$(\Delta^2+2\Delta+1)f(x), \quad \text{.......(2)}$$

and $\Delta^nf(x)+n\Delta^{n-1}f(x)+\dfrac{n(n-1)}{2!}\Delta^{n-2}f(x)+\ldots+f(x)$ in the form

$$\left\{\Delta^n+n\Delta^{n-1}+\frac{n(n-1)}{2!}\Delta^{n-2}+\ldots+1\right\}f(x). \quad \text{.......(3)}$$

In exactly the same way, $Ef(x)-f(x)$ is written in the form

$$(E-1)f(x), \quad \text{.......(4)}$$

and $E^nf(x)-nE^{n-1}f(x)+\dfrac{n(n-1)}{2!}E^{n-2}f(x)+\ldots+(-1)^nf(x)$ in the form

$$\left\{E^n-nE^{n-1}+\frac{n(n-1)}{2!}E^{n-2}+\ldots+(-1)^n\right\}f(x). \quad \text{.......(5)}$$

The formulae (3), (5) immediately suggest the introduction of the forms $(\Delta+1)^n f(x)$ and $(E-1)^n f(x)$ as equivalents of them, on the understanding that $(\Delta+1)^n$ and $(E-1)^n$ are to be expanded by the Binomial Theorem as if Δ, E were numbers, in order to get back to the forms (3), (5).

The fundamental results (7), (8) of the preceding Article now take the respective forms

$$\Delta^n f(x)=(E-1)^n f(x), \quad E^n f(x)=(\Delta+1)^n f(x). \qquad \text{.................(6)}$$

Thus, as operators on $f(x)$, Δ^n is equivalent to $(E-1)^n$ and E^n to $(\Delta+1)^n$. This is expressed by writing the operator-equations

$$\Delta^n=(E-1)^n, \quad E^n=(\Delta+1)^n, \qquad \text{........................(7)}$$

which we look on as derived by repetition from the fundamental equations

$$\Delta=E-1, \quad E=\Delta+1. \qquad \text{...............................(8)}$$

The operators E, Δ can occur together. Thus $\Delta^p E^q f(x)$ means $\Delta^p\{E^q f(x)\}$, or $\Delta^p f(x+qh)$, and $E^q\Delta^p f(x)$ means $E^q\{\Delta^p f(x)\}$, which is again easily seen to be $\Delta^p f(x+qh)$. The operative factors Δ^p, E^q are therefore commutative.

In the same way a polynomial operator of the form $a\Delta^p E^q+b\Delta^r E^s+\dots$, (where a, b, ... are any numbers), when applied to $f(x)$ gives

$$a\Delta^p f(x+qh)+b\Delta^r f(x+sh)+\dots. \qquad \text{......................(9)}$$

It is readily shewn that the effect of any polynomial operator on $f(x)$ is not altered by the substitution of $E-1$ for Δ or of $\Delta+1$ for E, so that we can write

$$\Delta^p E^q=\Delta^p(\Delta+1)^q=\Delta^{p+q}+q\Delta^{p+q-1}+\frac{q(q-1)}{2!}\Delta^{p+q-2}+\dots+\Delta^p, \qquad \text{..... (10)}$$

and

$$\Delta^p E^q=(E-1)^p E^q=E^{p+q}-pE^{p+q-1}+\frac{p(p-1)}{2!}E^{p+q-2}+\dots+(-1)^p E^q. \qquad \text{...(11)}$$

63. The Binomial Difference Theorem.

The Binomial Theorem for the expansion of $(1+x)^n$ can be written [1] in the 'Remainder form'

$$\begin{aligned}(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\dots+\binom{n}{r}x^r\\
+\frac{1}{r!}\Big\{1.2.3\dots r(1+x)^{n-r-1}+2.3.4\dots(r+1)(1+x)^{n-r-2}+\dots\\
+(n-r)(n-r+1)\dots(n-1)\Big\}x^{r+1}, \ (r<n). \qquad \text{.............(1)}\end{aligned}$$

According to the preceding Article, we can replace x by Δ, that is, we can write

$$\begin{aligned}(1+\Delta)^n=1+n\Delta+\frac{n(n-1)}{2!}\Delta^2+\dots+\binom{n}{r}\Delta^r\\
+\frac{1}{r!}\Big\{1.2.3\dots r(1+\Delta)^{n-r-1}+2.3.4\dots(r+1)(1+\Delta)^{n-r-2}+\dots\\
+(n-r)(n-r+1)\dots(n-1)\Big\}\Delta^{r+1}, \qquad \text{.......................(2)}\end{aligned}$$

[1] This is quite an old form of the theorem. It is readily proved by Induction. See *Notes on the Mathematical Syllabus of the Schools Board*, Melbourne University Press (1925), p. 18.

and further, we can replace $1+\Delta$ by E. Let us apply each of the equivalent operators on the two sides of the equation (2) to $f(x)$. Then, since

$$(1+\Delta)^n f(x) = E^n f(x) = f(x+nh), \quad\ldots\ldots(3)$$

$$(1+\Delta)^{n-r-1}\Delta^{r+1}f(x) = E^{n-r-1}\Delta^{r+1}f(x) = \Delta^{r+1}f\{x+(n-r-1)h\}, \quad\ldots\ldots(4)$$

and so on, we get the equation

$$f(x+nh) = f(x) + n\Delta f(x) + \frac{n(n-1)}{2!}\Delta^2 f(x) + \ldots + \binom{n}{r}\Delta^r f(x)$$
$$+\frac{1}{r!}\Big[1.2.3\ldots r\Delta^{r+1}f\{x+(n-r-1)h\} + 2.3.4\ldots(r+1)\Delta^{r+1}f\{x+(n-r-2)h\} + \ldots$$
$$+(n-r)(n-r+1)\ldots(n-1)\Delta^{r+1}f(x)\Big]. \quad\ldots(5)$$

This is the expression of the Binomial Difference Theorem (with Remainder) of the theory of Finite Differences; the theorem in its various forms is of fundamental importance in that subject.

64. Miscellaneous Notes.

64. 1. *Note on the nature of a proof and on necessary and sufficient conditions.* In ordinary language, the word *proof* (or *prove*) is sometimes employed in the broad sense of the word *test*, as, for example, in the saying that 'the exception proves the rule.' In mathematics, the meaning of *proof* is that of *conclusive test*. A proof of a theorem (if satisfactory) shews that the theorem is a logical consequence of the axioms of the subject. A theorem in mathematics is, generally speaking, stated or proved to be true only subject to certain restrictions or conditions being imposed on the elements, or relations between the elements, with which it is concerned. A set of conditions is *necessary* when the theorem is not true unless those conditions are satisfied. A set of conditions is *sufficient* when the theorem is true if those conditions are satisfied. A set of conditions is *necessary and sufficient* when the theorem is true if those conditions are satisfied and otherwise is not true.

A necessary set of conditions may not be sufficient, that is, other conditions may also be necessary. A sufficient set of conditions may not be necessary, that is, the theorem may be true under conditions which are less restrictive or which are incompatible with the first.

Conditions which are sufficient but not necessary are often introduced to facilitate the proof of a theorem. The treatment of a theorem cannot be regarded as final unless the conditions imposed are both necessary and sufficient.

64. 2. *Note on dimensions and homogeneity.* The language of *dimensions* is used in Analysis for the characterization of certain types of formulae. Suppose that $F(x, y, z, \ldots)$ is a formula involving the literal numbers $x, y, z, \ldots$ (and possibly others with which we are not here concerned). We consider the effect on the formula of replacing $x, y, z, \ldots$ by multiples of these numbers, of which we will in the sequel write three only. If the same multiple, k say, is taken for all of them, so that they are replaced by kx, ky, kz, we say that they are treated as of the same dimensions.[1] If x, y, z are replaced by $k^m x, k^n y, k^r z$,

[1] The language here corresponds to that employed in Physics. If x is the measure of a physical quantity X and the unit of measurement for it is changed to $1/k$ of

where m, n, r are not necessarily integers, they are said to be treated as of m, n, r dimensions, respectively. One of the indices, m say, can always be taken to be 1, and y, z are then said to be treated as of dimensions n, r relative to x, which is of one dimension.

The type of formula with which we are here concerned in that in which the effect of the replacement indicated, for suitable values of m, n, r, is to multiply the formula by a power of k; that is, taking $m=1$, we are to have

$$F(kx, k^n y, k^r z) = k^p F(x, y, z). \qquad \text{.......................(1)}$$

The formula is then said to be *homogeneous* and of dimensions p in x, y, z (jointly) when these are treated as of 1, n, r dimensions, respectively.[1] Unless the contrary is stated, x, y, z are always supposed treated as of the *same* dimensions, so that $n=r=1$.

If $p=0$, the formula is said to be of zero dimensions or of no dimensions.

If we put $\dfrac{y}{x^n}x^n$ for y and $\dfrac{z}{x^r}x^r$ for z, the formula $F(x, y, z)$ can be treated as one in x, y/x^n, z/x^r, say $G(x, y/x^n, z/x^r)$. Now here y/x^n, z/x^r are unaltered when the replacements of x, y, z are made, so that the condition of homogeneity can be expressed by

$$G(kx, y/x^n, z/x^r) = k^p G(x, y/x^n, z/x^r), \qquad \text{.......................(2)}$$

and thus
$$\frac{1}{(kx)^p} G(kx, y/x^n, z/x^r) = \frac{1}{x^p} G(x, y/x^n, z/x^r). \qquad \text{.......................(3)}$$

Since k can be given any value, it appears that the right-hand side of this equation involves x only in the ratios y/x^n, z/x^r and the formula $F(x, y, z)$ can be put into the form $x^p H(y/x^n, z/x^r)$, or $x^p H(y/x, z/x)$, if $n=r=1$.

It is plain that the homogeneity and dimensions of the formula $F(x, y, z)$ are not affected by any algebraic operations which merely alter its form. This property is regularly employed in (partially) checking the correctness of algebraic operations, as exemplified below.

Ex. 1. The formula $x^\lambda y^\mu z^\nu$ is homogeneous and of $\lambda+\mu+\nu$ dimensions, if x, y, z are one-dimensional. It is of dimensions $\lambda+\mu n+\nu r$ if x, y, z are of dimensions 1, n, r, respectively.

Ex. 2. The polynomial $\Sigma A_{\lambda\mu\nu}x^\lambda y^\mu z^\nu$ is homogeneous and of dimensions $\lambda+\mu+\nu$ if that sum is the same for all the terms, (x, y, z being supposed one-dimensional).

Ex. 3. The sum of two homogeneous expressions of the same dimensions is homogeneous and of the same dimensions.

itself, the measure x is changed to kx. If y, z are measures of quantities Y, Z of the same kind as X, they are also changed in the same ratio and become ky, kz. The quantities are here said to be all of one dimension in the unit. If the units of Y, Z are changed to $1/k^n$, $1/k^r$ of themselves when the unit of X is changed to $1/k$ of itself, the measures y, z are multiplied by k^n, k^r, respectively. The quantities Y, Z are then said to be of n, r dimensions, respectively, in the unit of X.

[1] If the condition of homogeneity is satisfied, we can regard the formula as the measure of a physical quantity of dimensions p with respect to the unit of the quantity X of which x is the measure.

Ex. 4. The product of two homogeneous expressions of dimensions p, q is of dimensions $p+q$. The quotient of the first expression by the second is homogeneous and of dimensions $p-q$.

Ex. 5. In the expansion of $(a+x)^m$, (m a positive integer), each term must be of dimensions m in a, x jointly.

Ex. 6. Shew from considerations of dimensions that the equation $\dfrac{1}{x+h}=\dfrac{1}{x}-\dfrac{h}{x^2}+\dfrac{h^2}{x^2}\dfrac{1}{1+h/x}$ is incorrect.

Ex. 7. If the variables x, y are treated as one-dimensional, $\Delta y/\Delta x$ is of zero dimensions, $\Delta^2 y/(\Delta x)^2$ of dimensions -1, and generally, $\Delta^n y/(\Delta x)^n$ of dimensions $-(n-1)$.

Here we may write $\Delta x=x_2-x_1$, $\Delta y=y_2-y_1$ and x_1, x_2 are to be changed to kx_1, kx_2 and y_1, y_2 to ky_1, ky_2. Thus Δx changes to $k(x_2-x_1)$ or $k\Delta x$, and Δy to $k(y_2-y_1)$ or $k\Delta y$. Therefore $\Delta y/\Delta x$ is unaltered and hence is of zero dimensions.

Similarly $\Delta^2 y$ can be written $(y_3-y_2)-(y_2-y_1)$ and changes to $k\Delta^2 y$ while $(\Delta x)^2$ changes to $k^2(\Delta x)^2$. Thus $\Delta^2 y/(\Delta x)^2$ changes to $\dfrac{1}{k}\Delta^2 y/(\Delta x)^2$ and is of dimensions -1.

Ex. 8. If the variables x, y are treated as of dimensions 1, m, respectively, $\Delta y/\Delta x$ is of dimensions $m-1$, $\Delta^2 y/(\Delta x)^2$ of dimensions $m-2$ and $\Delta^n y/(\Delta x)^n$ of dimensions $m-n$.

EXAMPLES I

SIMPLE GRAPHS[1]

1. Sketch characteristic portions of the graphs of the following functions:
x^3, $\sqrt{x}$, $x^{3/2}$, $-1/x$, $1+1/x$, $2+1/x^2$, $2-1/x^2$, $1/\sqrt{x}$, $\sqrt{(-x)}$, $|x|$, $1/|x|$, $x+|x|$, $1/(1+x)$, $1/(x-1)^2$, $2x+x^2$, $x+1/x$, $x-1/x$, x^2-1/x, x^3+2x, $x^3-1/(x+1)$, $\sqrt{(2-x^2)}$, $1/\sqrt{(1-x^2)}$, $1/(x^2+1)$, $1/(x^2-1)$, $1\Big/\left(x+\dfrac{1}{x}\right)$, $\dfrac{1}{x-1}+\dfrac{1}{x+2}$, $\dfrac{1}{x^2-2x+3}$, $\sqrt{(x-x^2)}$, $x(x-1)^2$, x^3-3x^2+2x, x^4-x^2, $\dfrac{x+1}{x-1}$, $\left(\dfrac{x+1}{x-1}\right)^2$, $x-I(x)$, $(x^2-a^2)^{1/3}$.

2. Sketch characteristic portions of the graphs of the following functions:
$\sin 2x$, $|\sin x|$, $\sin^2 x$, $\cos^2 x$, $2\sin x+3\cos x$, $\frac{3}{5}\sin x+\frac{4}{5}\cos x$, $\cot x+\tan x$, $\sec x+\tan x$, $\sin x+\frac{1}{2}\sin 2x$, $\sin^2 x\cos x$, $\tan^2 x$, $\sec^2 x$, $\sin(x^2)$, $\cos\sqrt{x}$, $\operatorname{cosec}(1/x)$, $\sin(1/x^2)$, $x+\sin x$, $\sin|x|$, $\cos x-\frac{1}{2}x$, $\tan x-x$, $\tan x+1/x$, $\tan x-2\sin x$, $x\sin x$, $x^2\sin(1/x)$, $(1/x)\sin(1/x)$.

[1] The details of any but the simplest graphs are more easily determined with the aid of the Calculus; see Chapter IV.

3. Sketch the curves whose coordinates are expressed parametrically by the equations :

(i) $x=2t+3, y=3t-2$; (ii) $x=\dfrac{6t}{1+t^2}, y=\dfrac{1-t^2}{1+t^2}$; (iii) $x=\dfrac{1-t^2}{1+t^2}, y=\dfrac{t(1-t^2)}{1+t^2}$;

(iv) $x=\dfrac{t+t^3}{1+t^4}, y=\dfrac{t-t^3}{1+t^4}$; (v) $x=t-t^3, y=1-t^4$; (vi) $x=a\cos^3\theta, y=a\sin^3\theta$.

4. Sketch the curves given by the following equations :

$y^2=1/x$, $y^2=x^2/(2-x)$, $y^2=x^3$, $y^2=(x-1)(x^2+1)$, $y^2=(x-1)x(x+1)$, $(x^2+y^2)x-y^2=0$, $y^2=x^2(2+x)/(1-x)$.

EXAMPLES II

ELEMENTARY PROPERTIES OF FUNCTIONS

1. Investigate the ranges in which the following functions are one-way :

k^2/x, $\cos x$, $\tan x$, $\sqrt{(a^2-x^2)}$, x^2-x, $1+x^2$, $a+a^2/x$, $1/(1+x^2)$.

2. State which of the following functions are even and which are odd :

$1/(1+x^2)$, $x/(1+x^2)$, $x/(1+x)$, $(\sin x)/x$, $(1-\cos x)/x$, $(1-\cos x)/x^2$, $\sin x\cos^2 x$, $\sin^2 x\cos x$.

[A function $f(x)$ is *even* in a range $(-a, a)$ if, in this range, $f(-x)=f(x)$, and *odd* if $f(-x)=-f(x)$.]

3. Prove that any function $f(x)$ which is defined in a range $(-a, a)$ can be expressed as the sum of an even and an odd function in the range.

[Employ the identity $f(x)=\frac{1}{2}\{f(x)+f(-x)\}+\frac{1}{2}\{f(x)-f(-x)\}$.]

4. Find the homogeneous linear relation between the distances $x-a$, $x-b$, $x-c$ of the variable point P, (x), from the fixed points A, (a), B, (b), C, (c), on Ox.

Find also the linear relation between the squares of the same distances.

5. Find the constants a, b, c of the function ax^2+bx+c so that its graph may pass through the points $(-1, 4)$, $(0, 1)$, $(1, 2)$. Find the nearest point of the graph to Ox. [2, -1, 1 ; $(\frac{1}{4}, \frac{7}{8})$.]

6. A cable hangs between two points which are 100 feet apart horizontally and at heights 20, 25 feet above the ground. Assuming that the form of the cable is a parabola with vertical axis and that the minimum height of the cable between the supports is 18 feet, find where this occurs and also the height midway between the supports.

[34·84 feet horizontally from the lower support ; 18·38 feet.]

7. Find the shortest distance of the point $(0, c)$ from the parabola $y=x^2$.

[$\sqrt{(c-\frac{1}{4})}$, $\frac{1}{2}$, or c according as $c>$, $=$, or $<\frac{1}{2}$.]

8. Prove that the expression $(a+c)(ax^2+2bx+c)+2(b^2-ac)(x^2+1)$ has the same sign for all values of x provided $b^2>ac$.

9. Establish the identities

$$x+\frac{a^2}{x}=\left(\sqrt{x}\mp\frac{a}{\sqrt{x}}\right)^2\pm 2a,\ (x>0),\quad x+\frac{a^2}{x}=-\left\{\sqrt{(-x)}\mp\frac{a}{\sqrt{(-x)}}\right\}^2\mp 2a,\ (x<0).$$

Deduce rules for the determination of the extreme values of the functions $x+a^2/x$, $(ax^2+bx+c)/x$.

10. Find the order in magnitude of the numbers $\sqrt{10}$, 3, $\frac{19}{6}$, $\frac{117}{37}$, $\frac{721}{228}$. [Compare the squares of the numbers.]

11. Establish the identity $\left(\sum_{r=1}^{n} a_r b_r\right)^2 = \sum_{r=1}^{n} a_r^2 \sum_{s=1}^{n} b_s^2 - \sum_{r=1}^{n} \sum_{s=1}^{n} (a_r b_s - a_s b_r)^2.$

12. A polynomial of the fourth degree in x has the values 47, 8, 1, 2, 11 when x has the respective values $-2, -1, 0, 1, 2$. Find its value when $x = \frac{1}{2}$. [$\frac{17}{16}$.]

13. Prove by the method of Mathematical Induction that

$$\sum_{r=1}^{n} \frac{(x-2r)x(x-1)\dots(x-r+1)}{r!} = \frac{x(x-1)\dots(x-n)}{n!} - x.$$

14. Shew that if $u_r = (r+a)(r+1+a)\dots(r+m-1+a)$,

$$\sum_{r=1}^{n} u_r = \frac{(n+a)(n+1+a)\dots(n+m+a)}{m+1} + C,$$

and find the constant C by considering the case of $n = 1$.

15. Shew that if $u_r = 1/\{(r+a)(r+1+a)\dots(r+m-1+a)\}$,

$$\sum_{r=1}^{n} u_r = -1/\{(m-1)(n+1+a)(n+2+a)\dots(n+m-1+a)\} + C,$$

and find the constant C by considering the case of $n = 1$.

EXAMPLES III

CONTINUITY OF MONOMETRIC FUNCTIONS

1. Prove that if m is a positive integer,

$$(x+h)^m - x^m = h\{(x+h)^{m-1} + x(x+h)^{m-2} + \dots + x^{m-1}\}$$
$$= mhx^{m-1} + h^2\{(x+h)^{m-2} + 2x(x+h)^{m-3} + \dots + (m-1)x^{m-2}\}.$$

Taking $m = 4$, shew that if $|h| < |x|$,

$$|(x+h)^4 - x^4| < \varepsilon \text{ if } |h| < \varepsilon/(15|x|^3),$$
$$\left|\frac{(x+h)^4 - x^4}{h} - 4x^3\right| < \varepsilon \text{ if } 0 < |h| < \varepsilon/(11|x|^2).$$

2. Establish the identities

$$\sqrt{(x+h)} - \sqrt{x} = \frac{h}{\sqrt{x}\,.\,\{1 + \sqrt{(1+h/x)}\}}$$
$$= \frac{h}{2\sqrt{x}} - \frac{h^2}{8x^{3/2}}\left\{\frac{2}{1+\sqrt{(1+h/x)}}\right\}^2.$$

Hence shew that

$$|\sqrt{(x+h)} - \sqrt{x}| < \varepsilon \text{ if } |h| < \varepsilon\sqrt{x},$$
$$\left|\frac{\sqrt{(x+h)} - \sqrt{x}}{h} - \frac{1}{2\sqrt{x}}\right| < \varepsilon \text{ if } 0 < |h| < 2\varepsilon x^{3/2}.$$

3. Prove directly that the ε-condition for the continuity of x^2 at x_1 is satisfied by taking $|h| < -|x_1| + \sqrt{(x_1^2 + \varepsilon)}$.

4. Investigate the continuity of the functions $\sqrt{(a-x)}$, $x^{2/3}$ directly by means of the ε-condition.

5. Examine the ranges in which the following functions are continuous :

(i) $\sqrt{\{(x-a)(b-x)\}}$; (ii) $(1-x)^m(1+x)^n$; (iii) $\sqrt{(1+2\sin x)}$;
(iv) $1/\sqrt{(2+\sin x)}$; (v) $1/(a\cos^2 x+b\sin^2 x)$; (vi) $\sin(\pi\tan x)$;
(vii) $\dfrac{\sin \pi x}{x(1-x)}$; (viii) $\dfrac{\sec x}{x}$; (ix) $\sin\sqrt{(1-x)}$.

Sketch the graphs of the functions.

6. Shew that the function $f(x)=I(x)+\sqrt{\{x-I(x)\}}$ is continuous for all values of x, and that $f(x)\leqslant x+\frac{1}{4}$.

7. Find the most general continuous function $f(x)$ which satisfies the relation $f(x_1+x_2)=f(x_1)+f(x_2)$, where x_1, x_2 are any two values of the variable.

[Prove in turn that $f(nx)=nf(x)$, (n a positive integer), and $f(m/n)=am/n$, where $a=f(1)$, (m, n integers). The condition of continuity is now employed to prove that $f(\alpha)=a\alpha$, (α any real number), and the required function is thus given by $f(x)=ax$.]

8. Shew that $\sqrt{(1-x^2)}$ is uniformly continuous in the range $(-1, 1)$ by finding a common value of h corresponding to the given value 10^{-m} of ε, (m a positive integer).

9. Prove directly from the definitions that if the function $f(x)$ is such that, in a range (a, b), $\left|\dfrac{f(x+h)-f(x)}{h^p}\right|\leqslant K$, ($K$ a fixed positive number, p positive), for all starting points x and increments h, the function is uniformly continuous in the range.

Apply this to shew that the functions $\sin x$, $\sqrt{(a^2+x^2)}$, $x/(x^2+1)$, $\sqrt{(x^2-a^2)}$, x^n, $(x^2-a^2)^{1/3}$ are uniformly continuous within their ranges of definition.

EXAMPLES IV

SPATIAL DIAGRAMS. CONTINUITY OF DIMETRIC FUNCTIONS

1. Find the inclination (in space) to Oy of the line joining the origin O to the point P, $(1, 2, 3)$. [Project OP on Oy.]

2. In the standard method of projection, Art. 41, draw the representations of (i) a unit radius OP which is inclined at 45° to Oz and makes equal angles with Ox, Oy ; (ii) the perpendicular from P on the axis Oz.

3. Shew that the equation $ax+by+cz=1$ represents a plane making intercepts $1/a$, $1/b$, $1/c$ on the axes Ox, Oy, Oz, respectively.

[Shew that if P, Q are any two points on the surface, any point R on their join also lies on the surface.]

4. Sketch, with the use of representative sections, the forms of the following surfaces :

(i) $x^2+y^2=r^2$; (ii) $x^2/a^2+y^2/b^2=1$; (iii) $y^2=2ax$;
(iv) $x^2/a^2+y^2/b^2+z^2/c^2=1$; (v) $x^2+y^2=2z$; (vi) $x^2-y^2=2z$.

5. Shew that, in the standard representation, the plot of the curve of intersection of the sphere $x^2+y^2+z^2=4a^2$ and the circular cylinder $(x-a)^2+y^2=a^2$ is given by

$$\xi=a\sin\theta-\frac{a(1+\cos\theta)}{2\surd 2}, \quad \eta=\pm 2a\sin\tfrac{1}{2}\theta-\frac{a(1+\cos\theta)}{2\surd 2}.$$

6. Find the equations of the plots of the curves of intersection of the plane $lx+my+nz=0$ with the cylinders $x^2+y^2=r^2$, $x^2/a^2+y^2/b^2=1$.

7. Prove that if the functions $f(x, y)$, $\varphi(x, y)$ are continuous in a certain domain, so is their product $f(x, y)\varphi(x, y)$.

8. Shew that it is sufficient for the uniform continuity of a dimetric function $f(x, y)$ in a domain that $\left|\dfrac{f(x+h, y)-f(x, y)}{h^p}\right| \leqslant K$ and $\left|\dfrac{f(x, y+k)-f(x, y)}{h^p}\right| \leqslant K$, ($K$ a fixed positive number, p positive), for all pairs of points such as $(x+h, y)$, (x, y) and $(x, y+k)$, (x, y) in the domain.

9. Prove that the functions $1/(x^2+n^2y^2+nxy+1)$, $\sin x\sin y$, $\sin(mx+ny)$ are uniformly continuous in any finite domain.

10. Shew that the function $1/\surd(a^2-x^2-y^2)$ is uniformly continuous in a domain defined by $x^2+y^2<a^2$.

11. Shew that the conditions of Theorem I, Art. 44. 1, are satisfied by the function $f(x, y)=4x^2+8xy+6y^2-2x-5y$ when the origin is taken as the point (x_0, y_0), and verify that the equation $f(x, y)=0$ leads, in this case, to two single-valued functions of x provided that x lies in the range

$$(-\tfrac{1}{2}-\tfrac{1}{2}\surd\tfrac{33}{8},\ -\tfrac{1}{2}+\tfrac{1}{2}\surd\tfrac{33}{8}).$$

EXAMPLES V

SEQUENCES

1. Shew that the sequence $\frac{1}{2}$, $1\frac{1}{2}$, $\frac{3}{4}$, $1\frac{1}{4}$, $\frac{7}{8}$, $1\frac{7}{8}$, ..., where $a_{2n}=1+1/2^n$, $a_{2n+1}=1-1/2^{n+1}$, converges to the focus 1.

2. If $P(n)$ is a polynomial in n of degree r at most, shew that the sequence $\{P(n)/n^r\}$ is convergent and that its focus is that of the sequence $\{a/n^{r-s}\}$, where an^s is the term of highest degree in $P(n)$.

State the corresponding result when $P(n)$ is of degree higher than r.

3. Investigate the convergence of the sequences whose general elements are the following :

(i) $(-1)^n$; (ii) $n/(n+1)$; (iii) $(-1)^n n/(n+1)$; (iv) $n/(n^2+1)$;
(v) $(-1)^n/n$; (vi) $n+(-1)^n n$; (vii) n^p/k^n, (p a positive integer, $k>1$) ;
(viii) $\dfrac{n^2+1}{2n^2+3}$; (ix) $\dfrac{a_1n^2+a_2n+a_3}{n+b}$; (x) $\dfrac{m(m-1)\dots(m-n+1)}{n!}k^n$, ($|k|<1$) ;
(xi) $1+k+k^2+\dots+k^{n-1}$; (xii) n^k, ($k>0$) ; (xiii) $\surd(n+1)-\surd n$;
(xiv) $n!/n^n$; (xv) $n^{1/\surd n}$; (xvi) n^{1/n^2} ; (xvii) $(1+1/n^2)^n$;
(xviii) $\surd(n+a)\surd(n+b)-n$; (xix) $\dfrac{1^k+2^k+\dots+n^k}{n^{k+1}}$, ($k>-1$) ;

(xx) $\left(a+b\dfrac{\sqrt{1}}{n}\right)^2+\left(a^2+b\dfrac{\sqrt{2}}{n}\right)^2+\ldots+\left(a^n+b\dfrac{\sqrt{n}}{n}\right)^2$, $(0<a<1)$;

(xxi) $\sin(\pi/n)$; (xxii) $\sin\frac{1}{2}n\pi$; (xxiii) $\sin nk\pi$, (k any real number); (xxiv) $(\sin n)/n$; (xxv) $n\sin(k/n)$; (xxvi) $(\sin\frac{1}{2}n\pi)/\sqrt{n}$.

4. Employ the results of Art. 58 to establish the following results:

$$\left[\frac{n+1}{n}\right]=1;\ \left[\frac{n}{n+1}\right]=1;\ \left[\frac{n+1}{n^2+1}\right]=0;\ \left[\frac{n^2+1}{2n^2+3}\right]=\tfrac{1}{2};\ \left[\frac{a_1n^2+a_2n+a_3}{b_1n^2+b_2n+b_3}\right]=\frac{a_1}{b_1}.$$

5. If m, n are positive integers, m and nh $(=k)$ being fixed, shew that when n increases indefinitely, $n(n-1)\ldots(n-m)h^{m+1}$ converges to k^{m+1}.

Shew also that under the same conditions the expression

$$\big(1.2\ldots m+2.3\ldots(m+1)+\ldots+(n-m)(n-m+1)\ldots(n-1)\big)h^{m+1}$$

converges to $k^m/(m+1)$.

[According to Ex. 14, Set II, the sum of the co-factor of h^{m+1} is equal to $(n-m)(n-m+1)\ldots(n-1)n/(m+1)$.]

6. Shew that the sequence $\{a_n\}$ defined by

$$a_n=x\left(\frac{1}{n}+\frac{1}{n+x}+\frac{1}{n+2x}+\ldots+\frac{1}{n+nx}\right)$$

is convergent provided that $x+1$ is positive.

[Shew that the condition $|a_{n+1}-a_n|<\varepsilon$ if $n>N$ is satisfied. The limit defines the logarithmic function $\log_e(1+x)$, Chap. VII.]

7. Shew that the sequence $\{a_n\}$ defined by $a_n=\dfrac{n!\,n^{x-1}}{x(x+1)\ldots(x+n-1)}$ is convergent provided that x is not a negative integer.

[Use the result of Art. 348. The limit defines the Gamma-function $\Gamma(x)$, Chap. X.]

8. Shew that the sequence $\{a_n\}$ defined by

$$a_n=x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\ldots\left(1-\frac{x^2}{n^2\pi^2}\right)$$

is convergent.

[Use the result of the preceding Example. The limit defines the function $\sin x$.]

9. If $a_n=a_{n-1}+a_{n-2}$ and $a_1=0$, $a_2=1$, shew that

$$a_{n-1}a_{n+1}-a_n^2=(-1)^{n-1},$$

and find the focus of the sequence $\{a_{n+1}/a_n\}$.

[The first result follows at once by Induction. The focus is given by the limiting form of the equation $a_n/a_{n-1}=1+a_{n-2}/a_{n-1}$.]

10. Prove that if a_1, b_1 are positive and $a_{n+1}=\frac{1}{2}(a_n+b_n)$, $b_{n+1}=\sqrt{(a_nb_n)}$, the sequences $\{a_n\}$, $\{b_n\}$ are one-way and have a common focus.

[The focus is called the arithmo-geometric mean of a_1, b_1.]

11. Shew that if the sequence $\{a_n\}$ is convergent, $[a_n]=[s_n/n]$, where $s_n=a_1+a_2+\ldots+a_n$.

[Let $[a_n]=L$ and write $a_n=L+\alpha_n$, $s_n/n-L=\sigma_n/n$. If $|\alpha_n|<\varepsilon$ for $n>m$, shew that $\dfrac{|\sigma_n|}{n}<\dfrac{|\sigma_m|}{n}+\varepsilon\left(1-\dfrac{m}{n}\right)$.]

12. A sequence $\{a_n\}$ is defined by $a_n = \dfrac{3a_{n-1}-2}{a_{n-1}}$. Prove that if $a_1 > 2$, $[a_n] = 2$. Discuss the convergence in other cases.

13. Shew that if the function $f(x)$ is such that $\left|\dfrac{f(x+h)-f(x)}{h}\right| < k < 1$ for any (every) increment h, the sequence $\{a_n\}$ for which $a_{n+1} = f(a_n)$, (a_1 assigned), is convergent and that its focus is the only real root of the equation $x = f(x)$. Interpret this result graphically.

[Shew that $|a_{n+1} - a_n| < k\,|a_n - a_{n-1}|$.]

14. Shew that the function $\frac{1}{2}\cos x$ satisfies the condition for $f(x)$ in the preceding Example, and find to six decimal places the root of the equation $x = \frac{1}{2}\cos x$ by employing the sequence defined by $a_{n+1} = \frac{1}{2}\cos a_n$, taking $a_1 = 0{\cdot}45$.

[$x = 0{\cdot}450184$. The value $0{\cdot}45$ of a_1 is found to be near the root by comparing the values of x and $\frac{1}{2}\cos x$ in a table of $\cos x$. See, for example, the tables mentioned on p. 344.]

15. Shew that the positive root of the equation $x = 2\sin x$ is $1{\cdot}89549$, correct to five places of decimals.

[Plainly the root is converged to if the condition of Ex. 13 is satisfied in a neighbourhood of the root, the first approximation a_1 being taken to lie within it.]

16. Shew that the positive root of the equation $x^2 = x + k$, $(k > 0)$, is the focus of the sequence $\{a_n\}$ for which $a_{n+1} = \sqrt{(a_n + k)}$, where a_1 is positive. Illustrate the convergence graphically.

17. Employing the results

$$\cos(k/2^n) = \sqrt{\left\{\frac{1+\cos(k/2^{n-1})}{2}\right\}},\quad \sin(k/2^n) = \sqrt{\left\{\frac{1-\cos(k/2^{n-1})}{2}\right\}},$$

shew that $[\cos(k/2^n)] = 1$, $[\sin(k/2^n)] = 0$.

18. Employing the result

$$\cos(k/2^n) = \sqrt{\left\{\frac{1+\cos(k/2^{n-1})}{2}\right\}} > \sqrt{\{\cos(k/2^{n-1})\}},$$

shew that $\dfrac{\sin k}{2^n\cos^2 k} > \sin(k/2^n) > \dfrac{\sin k}{2^n}$.

EXAMPLES VI

INFINITE SERIES

1. Examine the convergence of the series whose term-sequences are the following:

$$\text{(i)}\ \left\{\frac{1}{2n-1}\right\};\quad \text{(ii)}\ \left\{\frac{1}{(2n+1)^2}\right\};\quad \text{(iii)}\ \left\{\frac{1}{(2n+1)^p}\right\};\quad \text{(iv)}\ \left\{\frac{1}{1+\sqrt{n}}\right\};$$

$$\text{(v)}\ \left\{\frac{n-1}{n^k}\right\};\quad \text{(vi)}\ \left\{\frac{n^p}{(n-1)^q}\right\},\ (p, q > 0);\quad \text{(vii)}\ \left\{\frac{1}{(n+a)^k}\right\};$$

$$\text{(viii)}\ \left\{\frac{1}{n^2+an+b}\right\};\quad \text{(ix)}\ \left\{\frac{1}{n^{(n+1)/n}}\right\};\quad \text{(x)}\ \left\{\operatorname{artan}\frac{k}{1+(n^2+n)k^2}\right\}.$$

2. Shew that the series of term sequence $\left\{\frac{n}{np+q}\right\}$ is divergent.

[Here the condition $a_n \to 0$ as $n \to \infty$ is not satisfied.]

3. Shew that the series $1 - \frac{1}{2} + \frac{1}{3} - \ldots + (-1)^{n+1}\frac{1}{n} + \ldots$ is convergent.

[Shew that $|s_{n+p} - s_n| < \varepsilon$ if $n+1 \geqslant 1/\varepsilon$, or treat as an alternating series. The sum is $\log_e 2$, Chap. VII.]

4. Shew that the series of term-sequence $\left\{\frac{1}{n(n+1)}\right\}$ is convergent and has the sum 1.

[Write $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ and sum from 1 to n.]

5. Shew that the series of term-sequence $\left\{\frac{n}{(n+1)(n+2)}\right\}$ is divergent but that of term-sequence $\left\{(-1)^{n+1}\frac{n}{(n+1)(n+2)}\right\}$ is convergent.

6. Shew that if the series of term-sequences $\{a_n\}$, $\{b_n\}$, $(a_n, b_n > 0)$, are both convergent, so is the series of term-sequence $\{\sqrt{(a_n b_n)}\}$.

Hence prove that if the series of term-sequence $\{a_n\}$ is convergent, so is that of term-sequence $\{\sqrt{(a_n a_{n+1})}\}$.

[For the first part use the result $\sqrt{(a_n b_n)} < \frac{1}{2}(a_n + b_n)$.]

7. If the sequence $\{a_n\}$ is one-way, shew that the series whose term-sequence is $\{a_n\}$ is convergent if the series whose term-sequence is $\{\sqrt{(a_n a_{n+1})}\}$ is convergent.

8. If the series of term-sequence $\{a_n{}^2\}$ is convergent, shew that the series of term-sequence $\{a_n/n\}$ is also convergent.

[The series of term-sequence $\{1/n^2\}$ is convergent and therefore also that of term-sequence $\{\sqrt{(a_n{}^2/n^2)}\}$.]

9. Shew that the sum of the series of term-sequence $\left\{\frac{1}{n^2+2n}\right\}$ is $\frac{3}{4}$.

10. Prove that the series of term-sequence $\{a_n\}$, $(a_n > 0)$, is convergent (divergent) if the upper focal bound of the sequence $\{(a_n)^{1/n}\}$ is less (greater) than 1.

[Use the D'Alembert test. The criterion is known as the Cauchy test.]

11. Shew that the series $1 + \alpha + \beta^2 + \alpha^3 + \beta^4 + \ldots + \alpha^{2n-1} + \beta^{2n} + \ldots$ is convergent if $0 < \alpha < \beta < 1$.

[Compare with the geometric series $1 + \beta + \beta^2 + \ldots + \beta^{2n} + \ldots$. What information is obtained from the application of the Cauchy and D'Alembert tests?]

12. Investigate the ranges of convergence of the series whose term-sequences are the following:

(i) $\{x^n/n!\}$; (ii) $\{(-1)^n x^{2n-1}/(2n-1)!\}$; (iii) $\{(-1)^n x^{2n}/(2n)!\}$;

(iv) $\{(-1)^{n-1}x^n/n\}$; (v) $\left\{(-1)^n \frac{x^{2n+1}}{2n+1}\right\}$; (vi) $\left\{\frac{(2n)!\,x^{2n+1}}{2^{2n}(n!)^2(2n+1)}\right\}$;

(vii) $\{\sin(x/n)\}$; (viii) $\left\{\frac{1}{n} - \frac{1}{n+x}\right\}$;

(ix) $\left\{\frac{\alpha(\alpha+1)\dots(\alpha+n-1)}{n!}\frac{\beta(\beta+1)\dots(\beta+n-1)}{\gamma(\gamma+1)\dots(\gamma+n-1)}x^n\right\}$, ($\alpha$, β, γ not zero nor a negative integer); (x) $\{n^k x^{n-1}\}$; (xi) $\left\{\frac{x^{n+1}}{n(n+1)}\right\}$; (xii) $\left\{\frac{(n!)^2 x^n}{(2n)!}\right\}$; (xiii) $\{nx^{n^2}\}$; (xiv) $\{x^n/n^n\}$; (xv) $\left\{(-1)^{n-1}\frac{x^{2n}}{2n(2n-1)}\right\}$; (xvi) $\{(nx)^n\}$;

(xvii) $\left\{\left(\frac{2.4\dots 2n}{1.3\dots(2n-1)}\right)^p x^{n-1}\right\}$; (xviii) $\left\{\frac{3.5\dots(2n+1)}{2.4\dots 2n}\frac{n^n}{3+2n}\right\}$.

13. Shew that the sequence $\{a_n\}$ defined by $a_n = \left(1+\frac{x^2}{1^2}\right)\left(1+\frac{x^2}{2^2}\right)\dots\left(1+\frac{x^2}{n^2}\right)$ is convergent.

[The sequence $\{(1+k/n)^n\}$, $(k>0)$, is proved to have an upper barrier in a manner similar to that employed for the sequence $\{(1+1/n)^n\}$, Art. 50, Ex. 3. In the present case we thus have

$$1+\frac{x^2}{n^2} < \left(1+\frac{x^2}{n^4}\right)^{n^2} = \left(1+\frac{x^2}{n^4}\right)^{n^4/n^2} < K^{1/n^2}, \; (K \text{ constant}),$$

and therefore $a_n < K^{s_n}$, where $s_n = 1+\frac{1}{2^2}+\dots+\frac{1}{n^2}$. Since $\{s_n\}$ converges, $\{a_n\}$ is barred above and therefore converges. The focus defines the function $(\sinh \pi x)/(\pi x)$, Chap. VII.]

14. If $\{a_n\}$ is a sequence of positive terms and $\{p_n\}$, $\{q_n\}$ are sequences defined by

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2},$$

shew that the sequence $\{p_n/q_n\}$ converges if the series of term-sequence $\{a_n\}$ diverges.

[Shew that $p_n q_{n-1} - p_{n-1} q_n = -(p_{n-1}q_{n-2} - p_{n-2}q_{n-1}) = (-1)^n K$, say, and hence that $\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = (-1)^n \frac{K}{q_n q_{n-1}}$. Also $\left\{\frac{1}{q_{n-1}q_n}\right\}$ is down-way and converges to zero.]

15. Shew that the series of term-sequence $\left\{\frac{nx}{n^2x^2+1} - \frac{(n-1)x}{(n-1)^2x^2+1}\right\}$ is convergent for any value of x but not uniformly convergent in the neighbourhood of $x=0$.

[However large n may be, there is always a value of x which makes $nx + 1/(nx) = 2$, namely $x = 1/n$. Thus it is impossible to find n such that $nx/(n^2x^2+1) < \varepsilon$ for the same n.]

16. Prove that the series of term-sequence $\{f_n(x)\}$ is absolutely and uniformly convergent in a range (a, b) of x if, for each value of x in it, $|f_n(x)| \leqslant M_n$, a constant, and the series of term-sequence $\{M_n\}$ is convergent.

[The result follows by comparing $\left|\sum_{r=n}^{n+p} f_r(x)\right|$ with $\sum_{r=n}^{n+p} M_r$. The criterion is known as Weierstrass's test.]

17. Prove that the series of term-sequence $\left\{\frac{\sin nx}{n^p}\right\}$, $(p>1)$, is uniformly convergent in any range of x.

EXAMPLES VII

APPROXIMATIONS. ORDERS OF MAGNITUDE

1. Prove that the fractional error made in writing $(a+x)^2=a^2+2ax$ is of the second order in x. If $|x/a|<1/10^m$, (m a positive integer), shew that the fractional error is less than $1/(8\times 10^{2m-1})$ whether x is positive or negative. Hence shew that $(3\cdot 004)^2=9\cdot 024$ with a fractional error less than $1\cdot 8\times 10^{-6}$.

2. If x, y are of the same order of magnitude shew that the fractional error made in writing $(a+x)(b+y)=ab(1+x/a+y/b)$ is of the second order of smallness. Shew that if $|x/a|<|1/10^m$, $|y/b|<1/10^n$, (m, n positive integers), the fractional error is less than $1/(8\times 10^{m+n-1})$ whether x, y are positive or negative. Verify that the fractional error made in writing $3\cdot 004\times 4\cdot 005=12\cdot 031$ is less than $1\cdot 7\times 10^{-6}$.

3. Establish the identity $\dfrac{1}{a+x}=\dfrac{1}{a}\left(1-\dfrac{x}{a}\right)+\dfrac{1}{a}\dfrac{1}{1+x/a}\dfrac{x^2}{a^2}$, and hence prove that the fractional error made in writing $\dfrac{1}{a+x}=\dfrac{1}{a}\left(1-\dfrac{x}{a}\right)$ is of the second order in x.

If $|x/a|<1/10^m$ shew that the absolute error is less than $1/(a\times 9\times 10^{2m-1})$ whether x/a is positive or negative.

Prove that, in the same case, the fractional error made in writing $x/(a+x)=x/a$ is less than $1/10^m$. Hence shew that the fractional error made in writing $0\cdot 02/5\cdot 02=0\cdot 004$ is less than 4×10^{-3}.

4. Shew that $\dfrac{a+x}{b+y}=\dfrac{a}{b}\left\{1+\dfrac{x}{a}-\dfrac{y}{b}-\dfrac{xy}{ab}+\dfrac{1+x/a}{1+y/b}\left(\dfrac{y}{b}\right)^2\right\}$.

Investigate barriers to the absolute and fractional errors made in writing $\dfrac{a+x}{b+y}=\dfrac{a}{b}\left(1+\dfrac{x}{a}-\dfrac{y}{b}\right)$ when $|x/a|<1/10^m$, $|y/b|<1/10^n$. Employ the approximation to calculate $7\cdot 03/5\cdot 02$ and estimate barriers to the errors.

5. Employ the identities in Ex. 2, Set III, to determine barriers to the absolute and fractional errors made in writing $\sqrt{(a+x)}=\sqrt{a}(1+\tfrac{1}{2}x/a)$ when $|x/a|<1/10^m$. Evaluate $\sqrt{8\cdot 987}$ by this means and estimate barriers to the errors.

6. Shew that the fractional error made in writing $\dfrac{1}{\sqrt{(a+x)}}=\dfrac{1}{\sqrt{a}}\left(1-\tfrac{1}{2}\dfrac{x}{a}\right)$ is of the second order in x.

[In the expression for the error rationalize the numerator.]

7. Shew that if p, q are positive integers, $(a+x)^{\frac{p}{q}}$ is equal to

$$a^{\frac{p}{q}}+\frac{p}{q}xa^{\frac{p}{q}-1}-x^2a^{\frac{p}{q}-2}\frac{p(y^{q-2}+2y^{q-3}+\ldots+q-1)-q(y^{p-2}+2y^{p-3}+\ldots+p-1)}{q(y^{q-1}+y^{q-2}+\ldots+1)^2},$$

where $y=(1+x/a)^{1/q}$.

Hence find upper barriers to the absolute and fractional errors made in writing $(a+x)^{\frac{p}{q}}=a^{\frac{p}{q}}+\dfrac{p}{q}xa^{\frac{p}{q}-1}$ when $|x/a|<1/10^m$, (m a positive integer).

EXAMPLES VIII

LIMITING VALUES OF FUNCTIONS

1. Shew that the following functions are potentially continuous at the values indicated, and find the completing value in each case :

(i) $\dfrac{1-\sqrt{(1+x)}}{x}$, $(x=0)$; (ii) $\dfrac{x^4-1}{x^2-1}$, $(x=1)$; (iii) $\dfrac{x^2-a^2}{x-a}$, $(x=a)$;

(iv) $\dfrac{x^6-2x^4+2x-1}{x^3-3x+2}$, $(x=1)$; (v) $\dfrac{(x^3-1)(x^4-1)}{(x-1)(x^2-1)}$, $(x=\pm 1)$;

(vi) $\dfrac{\sqrt{(1+x)}-\sqrt{(1-x)}}{x}$, $(x=0)$; (vii) $\dfrac{\sqrt{(1+x+x^2)}-1}{x}$, $(x=0)$;

(viii) $\dfrac{\sqrt{(1+x)}-\sqrt{(1+x^2)}}{\sqrt{(1-x)}-\sqrt{(1-x^2)}}$, $(x=0)$; (ix) $\dfrac{1+\cos \pi x}{\tan^2 \pi x}$, $(x=1)$;

(x) $\dfrac{2}{\sin^2 x}-\dfrac{1}{1-\cos x}$, $(x=0)$; (xi) $\sec x+\tan x$, $(x=\frac{3}{2}\pi)$.

2. Shew that the function $\dfrac{\sqrt{(x+x^2)}-\sqrt{x}}{x\sqrt{x}}$ is upper potentially continuous at $x=0$, the completing value being $\frac{1}{2}$.

3. Shew that if $n>m>0$, the function $\dfrac{1-\sqrt{(1-x^n)}}{1-\sqrt{(1-x^m)}}$ is upper potentially continuous at $x=0$. In what case is the potential continuity there ordinary ?

4. Investigate the behaviour of the following functions as $x \to 0$:

(i) x^{-m}, $(m>0)$; (ii) $1/x^2$; (iii) $1/\sqrt{x}$; (iv) $\operatorname{cosec} x$; (v) $\dfrac{1}{x}\sin\dfrac{1}{x}$.

5. Investigate the behaviour of the functions $\dfrac{1}{x+a}+\dfrac{1}{x-a}$, $\dfrac{1}{x+a}-\dfrac{1}{x-a}$ as x converges to the values $-a$, a.

6. Investigate the behaviour of the following functions as $x \to \infty$:

(i) $\sqrt{x}\{\sqrt{(x+a)}-\sqrt{x}\}$; (ii) $a\cos^2 x+b\sin^2 x$; (iii) $x-I(x)$;

(iv) $x\cos^2 x+x^2\sin^2 x$; (v) $x^2\sin\dfrac{1}{x}$; (vi) $\dfrac{x-\sin x}{x+\sin x}$.

EXAMPLES IX

DIFFERENCES

1. Demonstrate the following difference formulae, where on the left-hand sides the increment is due to an increment Δx, or h, of x :

(i) $\Delta \dfrac{1}{cx+d}=-\dfrac{c\Delta x}{(cx+d)(cx+d+c\Delta x)}$;

(ii) $\Delta \dfrac{ax+b}{cx+d}=\dfrac{(ad-bc)\Delta x}{(cx+d)(cx+d+c\Delta x)}$;

(iii) $\Delta x^m=h\left\{\binom{m}{1}x^{m-1}+\binom{m}{2}x^{m-2}h+\ldots+h^{m-1}\right\}$, ($m$ a positive integer) ;

(iv) $\Delta \sin x=2\cos(x+\frac{1}{2}\Delta x)\sin(\frac{1}{2}\Delta x)$;

(v) $\Delta \tan x = \dfrac{\sin \Delta x}{\cos x \cos (x+\Delta x)}$;

(vi) $\Delta \{x(x-h)(x-2h)\ldots(x-mh+h)\} = mhx(x-h)(x-2h)\ldots(x-mh+2h)$;

(vii) $\Delta \dfrac{1}{x(x+h)(x+2h)\ldots(x+mh-h)} = -\dfrac{mh}{x(x+h)(x+2h)\ldots(x+mh)}$.

2. Prove that if $y=\frac{1}{2}(a+x)-\surd(ax)$, where $0<a<x$, $\Delta y/\Delta x - \frac{1}{2}\{1-\surd(a/x)\}$ is positive when $\Delta x>0$, and hence that y is strictly up-way.

3. If $P_n(x)=a_0+a_1x+\ldots+a_nx^n$, prove that $\Delta^n P_n(x)=a_nh^n n!$.

4. Establish the results:

$$\Delta^n\{x(x-h)(x-2h)\ldots(x-mh+h)\}=\frac{m!}{(m-n)!}h^n x(x-h)\ldots(x-mh+nh+h),$$

$$\Delta^n \frac{1}{x(x+h)(x+2h)\ldots(x+mh-h)}$$
$$=(-1)^n\frac{(m+n-1)!}{(m-1)!}h^n\frac{1}{x(x+h)\ldots(x+mh+nh-h)}.$$

5. Shew that $\Delta^n \sin x = 2^n \sin(x+\frac{1}{2}nh+\frac{1}{2}n\pi)\sin^n \frac{1}{2}h$, and deduce the value of $\Delta^n \cos x$.

6. If $f(r)=\dfrac{\alpha(\alpha+1)\ldots(\alpha+r)}{\beta(\beta+1)\ldots(\beta+r)}$ and $\Delta r=1$, shew that

$$\Delta^n f(r)=f(r)\frac{(\alpha-\beta)(\alpha-\beta-1)\ldots(\alpha-\beta-n+1)}{(\beta+r+1)(\beta+r+2)\ldots(\beta+r+n)}.$$

7. Prove that if $\{u_n\}$ is a sequence such that $u_{n+1}-au_n=0$, (a constant), then $u_n=a^nu_0$, and that if $u_{n+1}-au_n=f(n)$, then $u_n=a^nu_0+\sum_{r=0}^{n-1}a^{n-r-1}f(r)$.

If $f(n)=k^n$, shew that u_n is equal to $a^nu_0+(a^n-k^n)/(a-k)$ if $k\neq a$, and to $a^nu_0+na^{n-1}$ if $k=a$.

8. If $\{u_n\}$ is a sequence such that $u_{n+1}u_n+au_n+b=0$, (a, b constants), verify that

(i) $u_n=\dfrac{u_0(\lambda^{n+1}-\mu^{n+1})-\lambda\mu(\lambda^n-\mu^n)}{u_0(\lambda^n-\mu^n)-\lambda\mu(\lambda^{n-1}-\mu^{n-1})}$ if $\frac{1}{4}a^2-b>0$,

where λ, μ are the roots of the equation $x^2+ax+b=0$;

(ii) $u_n=R\dfrac{u_0\sin(n+1)\varphi-R\sin n\varphi}{u_0\sin n\varphi-R\sin(n-1)\varphi}$ if $\frac{1}{4}a^2-b<0$,

where $R=\surd b$, $\surd b\cos\varphi=-\frac{1}{2}a$, $\surd b\sin\varphi=\surd(b-\frac{1}{4}a^2)$;

(iii) $u_n=-\frac{1}{2}a\dfrac{(n+1)u_0+\frac{1}{2}na}{nu_0+\frac{1}{2}(n-1)a}$ if $\frac{1}{4}a^2-b=0$.

9. Shew that the equation $u_{n+1}u_n+au_{n+1}+bu_n+c=0$ is reduced to the form $v_{n+1}v_n+\beta v_n+\gamma=0$ by putting $u_n+a=v_n$, where $\beta=b-a$, $\gamma=c-ab$. Hence solve the equation in one of the forms given in the preceding Example.

10. If $u_{n+2}+au_{n+1}+bu_n=0$, verify that

(i) $(\lambda-\mu)u_n=u_1(\lambda^n-\mu^n)-u_0\lambda\mu(\lambda^{n-1}-\mu^{n-1})$ if $\frac{1}{4}a^2-b>0$;

(ii) $u_n\sin\varphi=R^{n-1}\{u_1\sin n\varphi-u_0R\sin(n-1)\varphi\}$ if $\frac{1}{4}a^2-b<0$;

(iii) $u_n=\{nu_1+\frac{1}{2}(n-1)au_0\}(-\frac{1}{2}a)^{n-1}$ if $\frac{1}{4}a^2-b=0$;

where λ, μ, R, φ are as given in Ex. 8.

11. If $6u_{n+2} - 5u_{n+1} + u_n = 0$ and $u_0 = 1$, $u_1 = \frac{1}{6}$, prove that $u_n = -\frac{1}{2^n} + \frac{2}{3^n}$. Hence shew that the sum-sequence $\{s_n\}$ has the focus 1.

[Employing the notation of Art. 61, the equation can be written $(6E^2 - 5E + 1)u_n = 0$, or $(3E-1)(2E-1)u_n = 0$. On putting $(2E-1)u_n = v_n$, so that $(3E-1)v_n = 0$, we get $v_n = v_0(\frac{1}{3})^n$, and hence $2u_{n+1} - u_n = v_0(\frac{1}{3})^n$, Similarly, putting $(3E-1)u_n = w_n$, we get $3u_{n+1} - u_n = w_0(\frac{1}{2})^n$. Hence $u_{n+1} = w_0(\frac{1}{2})^n - v_0(\frac{1}{3})^n$, and v_0, w_0 are determined from the fact that $u_0 = 1$. $u_1 = \frac{1}{6}$.]

12. If $u_{n+2} + au_{n+1} + bu_n = f(n)$, verify that

(i) $(\lambda - \mu)u_n = A\lambda^n + B\mu^n + \sum_{r=0}^{n-2} (\lambda^{n-r-1} - \mu^{n-r-1})f(r)$ if $\frac{1}{4}a^2 - b > 0$;

(ii) $u_n \sin\varphi = R^n(A\cos n\varphi + B\sin n\varphi) + \sum_{r=0}^{n-2} R^{n-r-2}\sin(n-r-1)\varphi . f(r)$ if $\frac{1}{4}a^2 - b < 0$;

(iii) $u_n = (A + nB)b^{n/2} + \sum_{r=0}^{n-2} (n-r-1)(-\frac{1}{2}a)^{n-r-2} f(r)$ if $\frac{1}{4}a^2 - b = 0$;

and determine, in each case, the constants A, B in terms of u_0, u_1.

13. If $u_n - 4u_{n-1} + 5u_{n-2} - 2u_{n-3} = 0$ and $u_0 = 1$, $u_1 = 0$, $u_2 = -5$, prove that $u_n = 3(n+1) + 2 - 2^{n+2}$.

14. If $u_{n+2} + 2ku_{n+1} + u_n = 0$, $(k^2 < 1)$, and $u_0 = 0$, $u_m = 0$, prove that $u_n = A\sin\frac{nr\pi}{m}$, where r is a positive integer, n lies between 0 and m, and A is a constant.

[We find that $k = -\cos r\pi/m$).]

15. At a regularly recurring market two firms purchase each other's goods. The value of the goods purchased by each is proportional to the value of the goods sold to the other at the preceding market. Investigate formulae for the values of the goods bought by each firm at any market.

Consider also the case where the value of the goods purchased by one firm is proportional to the average value of the goods sold by that firm at the two preceding markets.

16. The position of each of n blocks which are at equal distances a apart along a straight line is elastically controlled so that a force kx is required to displace a block any distance x along the line. Each adjacent pair of blocks is now connected by a link of length a, the tension required to extend a link by x being Kx. If a force F is applied to one end block in the line of the blocks, shew that the consequent displacement of the other end block is $\frac{F}{k}\frac{\lambda - 1/\lambda}{\lambda^n - 1/\lambda^n}$, where $\lambda = 1 + \frac{k}{2K} + \sqrt{\left(\frac{k}{K} + \frac{k^2}{4K^2}\right)}$.

Obtain an expression for the displacement of the first block.

CHAPTER II

DIFFERENTIATION OF FUNCTIONS OF A SINGLE VARIABLE. APPLICATION TO ALGEBRAIC FUNCTIONS

65. Introductory Remarks.

The primary concern of the Differential Calculus is, as we have already remarked, the investigation of the instantaneous rate of variation of one quantity with respect to another. Throughout the earlier part of this Book we shall be concerned chiefly with functions of a single variable (monometric functions), and in the present Chapter we investigate conditions for the existence, and rules for the formation, of the instantaneous rate of variation of such a function with respect to its argument. Much of the matter here given is of a general analytical nature and the results established apply to a large class of functions. The theorems demonstrated are illustrated by means of ordinary algebraic functions, with the general character of which the student is supposed acquainted. The fundamental transcendental (non-algebraic) functions receive separate treatment in Chaps. VI, VII, where they are defined and have their principal properties investigated. Simple geometrical applications of the Calculus are introduced in the present Chapter and are illustrated by means of graphs of algebraic functions.

66. Average and Instantaneous Rates.

As explained in the introduction to Chap. I, the instantaneous rate of increase of one quantity with respect to a second quantity at a particular value of the latter is defined by means of a consideration of the average rates of increase of the first quantity for increments of the second quantity which start at the particular value of that quantity. In order that the instantaneous rate may exist, it is necessary that the values of the average rates should close up to equality when the increments of the second quantity which are considered are continually and indefinitely diminished. The value to which the average rates close up is then defined to be the instantaneous rate in question. We proceed to introduce the recognized

symbolism for the process described and thereby to remove any doubt as to the meaning of the language employed.

The two quantities considered are represented by two mathematical variables y, x, the second, x, being the *independent variable* or *argument* with respect to which the instantaneous rate of increase of the first, the *dependent variable* or *function*, y, is to be defined. Let x_1 be the particular value of x for which the rate is considered, and let y_1 be the corresponding value of y. The increment $x - x_1$ of the independent variable from x_1 to a typical value x (greater or less than x_1) is denoted by Δx, and the corresponding increment, $y - y_1$, of y by Δy, as in Art. 60. 1. If we introduce the functional symbol $f(x)$ for y, we have

$$\Delta y = y - y_1 = f(x) - f(x_1) = f(x_1 + \Delta x) - f(x_1), \quad\text{.............(1)}$$

and the corresponding average rate is then represented by any one of the forms

$$\frac{\Delta y}{\Delta x},\ \frac{y - y_1}{x - x_1},\ \frac{f(x) - f(x_1)}{x - x_1},\ \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}. \quad\text{.........(2)}$$

To express the condition of the closing up to equality of the average rates, we introduce another pair of increments $x' - x_1$, or $\Delta' x$, and $y' - y_1$, or $\Delta' y$, of the two variables, the corresponding average rate being $\Delta' y/\Delta' x$. We consider the difference

$$\frac{\Delta y}{\Delta x} - \frac{\Delta' y}{\Delta' x} \quad\text{..............................(3)}$$

of any (every) such pair of average rates, with the proviso that the magnitudes of Δx, $\Delta' x$ are less than a certain positive number h. If, now, we choose a positive number ε, as small as we please, the condition in question requires that for some positive value of h (depending on ε) the magnitude

$$\left|\frac{\Delta y}{\Delta x} - \frac{\Delta' y}{\Delta' x}\right| \quad\text{..............................(4)}$$

of the difference of the two rates should be less than ε for any (every) such pair. Symbolically, the condition is expressed by

$$\left|\frac{\Delta y}{\Delta x} - \frac{\Delta' y}{\Delta' x}\right| < \varepsilon \text{ (arbitrary) if } |\Delta x| < h,\ |\Delta' x| < h \text{ (appropriate).} \quad (5)$$

Now (i), $\dfrac{\Delta y}{\Delta x}$ or $\dfrac{f(x) - f(x_1)}{x - x_1}$ is a function of x which is defined in the range of definition of $f(x)$ except for the value x_1 which makes the denominator vanish; (ii), the condition of closing up just given is

as in Art. 54, the condition that the function $\Delta y/\Delta x$ should be potentially continuous at the value x_1 of x; (iii), the value defined as the instantaneous rate is the completing value at x_1 of this potentially continuous function. For, if we denote this value by N_1, we have, as in the Article referred to,

$$\left|\frac{\Delta y}{\Delta x} - N_1\right| < \varepsilon' \text{ (arbitrary) if } |\Delta x| < h' \text{ (appropriate)}, \quad \ldots\ldots(6)$$

so that N_1 is the value to which the average rates close up.

Thus we can state that *the existence of an instantaneous rate at x_1 requires that the average rates with starting value x_1 should be a function which is potentially continuous at x_1, and the value of the instantaneous rate is then the completing value at x_1 of that function.*

With the symbolism of Art. 54, the completed function is denoted by $\left(\frac{\Delta y}{\Delta x}\right)^*$ and the completing value at x_1 by $\left(\frac{\Delta y}{\Delta x}\right)^*_{x_1}$, but the symbol for the instantaneous rate which is actually adopted is much neater. It is given a little later.

If, instead of the language of potential continuity, the language of limits is employed, the symbol for the instantaneous rate as a completing value is replaced by the limit-symbol

$$\lim_{x \to x_1} \frac{\Delta y}{\Delta x}, \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

It should be carefully observed that these definitions require the consideration of both positive and negative values of Δx, the typical value x lying in a two-sided neighbourhood of x_1. Instantaneous rates defined by the consideration of a one-sided neighbourhood are dealt with in Art. 76.

In the case of ordinary algebraic functions, with which we are at present concerned, algebraic transformations, valid provided $\Delta x \neq 0$, of the quotient $\Delta y/\Delta x$ often lead very simply to a function of x which is actually continuous at x_1, and the value there of the continuous function formed is the completing value of this quotient, that is, is the instantaneous rate sought. In the case of the transcendental functions, which we consider in Chaps. VI, VII, special investigations are required to establish the potential continuity of $\Delta y/\Delta x$ and to find its completing value.

We may observe here that the condition of potential continuity of $\Delta y/\Delta x$ requires that y should be a continuous function of x at x_1, for otherwise there would be some number, ε' say, for which

$|\Delta y| \geqslant \varepsilon'$ for a value of $|\Delta x|$ however small, and hence $|\Delta y/\Delta x|$ would take indefinitely large values.

It must also be pointed out that the condition of continuity of the function y or $f(x)$ at x_1 is not of itself sufficient to ensure the potential continuity of the function $\Delta y/\Delta x$ there. It is, in fact, possible to define functions which, while being continuous in a range, do not possess the above property at any point of the range.[1]

We now proceed to consider some elementary examples of the determination of instantaneous rates. The first two of these are, in substance, trivial but are inserted to illustrate the mode of argument in the simplest symbolic cases.

Ex. 1. *The function* $y = c$ (*constant*). Here

$$y - y_1 = c - c = 0, \qquad (8)$$

and therefore
$$\frac{\Delta y}{\Delta x} = 0. \qquad (9)$$

The right-hand side of the last equation is a continuous function of x and has the value 0 at x_1, which is therefore the instantaneous rate there.

Ex. 2. *The function* $y = x$. Here

$$y - y_1 = x - x_1, \qquad (10)$$

and therefore
$$\frac{\Delta y}{\Delta x} = 1. \qquad (11)$$

The right-hand side of the last equation is a continuous function of x and has the value 1 at x_1, which is therefore the instantaneous rate there.

Ex. 3. *The function* $y = x^2$. Here

$$y - y_1 = x^2 - x_1^2 = (x - x_1)(x + x_1), \qquad (12)$$

and therefore
$$\frac{\Delta y}{\Delta x} = x + x_1. \qquad (13)$$

The right-hand side of the last equation is a continuous function of x and has the value $2x_1$ at x_1, which is therefore the instantaneous rate there.

The student should shew directly that the ε-condition for the existence of the completing value is satisfied in this and the next Example.

Ex. 4. *The function* $y = x^3$. Here

$$y - y_1 = x^3 - x_1^3 = (x - x_1)(x^2 + xx_1 + x_1^2), \qquad (14)$$

and therefore
$$\frac{\Delta y}{\Delta x} = x^2 + xx_1 + x_1^2. \qquad (15)$$

The right-hand side of the last equation is a continuous function of x and has the value $3x_1^2$ at x_1, which is therefore the instantaneous rate at x_1.

Ex. 5. Shew that for the functions x^4, x^5 the instantaneous rates at x_1 are $4x_1^3$, $5x_1^4$, respectively.

[1] See, for example, Pierpont, *Functions of a Real Variable*, Vol. II, p. 498.

67. Introduction of Standard Notation for Instantaneous Rates.

In the preceding discussion the starting point x_1 was any point within the range of continuity of the function y or $f(x)$. We now discontinue the practice of using a suffixed letter for such a point and employ instead the letter x. The increment of the argument will be from x to x', say, the corresponding increment of the function being from y to y'. These are denoted, as before, by Δx, Δy. The instantaneous rate of change of y with respect to x at the typical point x will now be symbolized by

$$\left(\frac{\Delta y}{\Delta x}\right)_x^*, \quad \text{or} \quad \lim_{x' \to x} \frac{\Delta y}{\Delta x}, \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \quad \ldots\ldots\ldots\ldots\ldots(1)$$

In the Differential Calculus, the process of finding the instantaneous rate for a given function y is described as *finding the derivative of the function* y or *differentiating the function* y for the chosen value of x. The instantaneous rate is called the *derivative* or the *differential coefficient* of the function y with respect to x at the chosen value. The regular symbol employed for the derivative is the modification of $\frac{\Delta y}{\Delta x}$ arrived at by changing Δ (the Greek D) into d, that is, it is $\frac{dy}{dx}$. This change is to be regarded as indicating that a limit-process has been applied to $\frac{\Delta y}{\Delta x}$ in which the increments are indefinitely diminished. In symbols,

$$\frac{dy}{dx} = \left(\frac{\Delta y}{\Delta x}\right)_x^* = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

It must be emphasized that, on account of the way in which $\frac{dy}{dx}$ is introduced, no meaning is here attached to the separate symbols dy, dx. The symbol for the derivative[1] is also written in the 'solidus' form dy/dx.

The symbolism here described is called the *Leibniz notation* for the derivative. If y is replaced by its functional symbol $f(x)$, we have the form $\frac{df(x)}{dx}$, which is often changed to $\frac{d}{dx}f(x)$, the corresponding 'solidus' form being $df(x)/dx$. Other notations for the derivative are introduced in Chap. III, Art. 86. The most important of these is the modification $f'(x)$ of the symbol $f(x)$. This is freely used at a later stage.

[1] The symbol is read 'dy by dx,' not 'dy divided by dx.'

The value of the derivative at any special value of x, say x_1, may be indicated by the addition of a suffix to the general derivative symbol. Thus we write for it $\left(\frac{dy}{dx}\right)_{x=x_1}$, or simply $\left(\frac{dy}{dx}\right)_{x_1}$. The notation $f'(x_1)$ is also employed for the same purpose.

Ex. 1. Prove that if y is a differentiable function of x, then

$$\frac{d(-y)}{dx} = -\frac{dy}{dx}. \quad \text{.................................(3)}$$

If a function has a definite derivative at each point of a range, it is said to be *differentiable in the range.* The functions considered above, namely c, x, x^2, x^3, x^4, x^5, are differentiable functions in any range of x, and we have the following summary :

Function	c	x	x^2	x^3	x^4	x^5
Derivative	0	1	$2x$	$3x^2$	$4x^3$	$5x^4$

The notion of the dimensions of formulae, explained in Art. 64. 2, is applicable to the derivative dy/dx. We have seen in that Sub-article that if y is of n dimensions relative to x, $\Delta y/\Delta x$ is of $n-1$ dimensions, that is, $\Delta y/\Delta x$ is multiplied by k^{n-1} when x is replaced by kx for all relevant values of x. Therefore the limit of $\Delta y/\Delta x$ when $\Delta x \to 0$ is multiplied by k^{n-1} when x is replaced by kx. Thus the derivative dy/dx is of dimensions $n-1$ relative to x. In particular, if x, y are of the same dimensions, dy/dx is of zero dimensions.

68. Illustrations from Mechanics and Physics.

We now introduce the Leibniz notation for certain of the instantaneous rates employed as illustrations in Art. 1, Chap. I. Thus the *velocity* of a particle moving along a straight line is defined as the instantaneous time-rate of increase of displacement of the particle. Let x denote the algebraic distance of the particle from an origin O on the line at the (instant of) time t. Then dx/dt is the symbol for the rate of increase of x with respect to t at any instant, and this therefore symbolizes the velocity, v say, of the particle, so that

$$v = \frac{dx}{dt}. \quad \text{...............................(1)}$$

In the same way, dv/dt is the symbol for the time-rate of increase of the velocity, that is, it is the symbol for the acceleration, α say, so that

$$\alpha = \frac{dv}{dt}. \quad \text{...............................(2)}$$

Again, if w is the work done by the propelling force on a particle

from some fixed instant up to the instant t, dw/dt is the time-rate of work or power, p say, at that instant, so that

$$p=\frac{dw}{dt}. \qquad \text{.............................(3)}$$

In the same way, the force exerted, F say, is given by

$$F=\frac{dw}{dx}, \qquad \text{.............................(4)}$$

when the work is specified in terms of the position x reached by the particle.

69. Geometrical Interpretation of the Derivative.

Let AB (Fig. 26) be the graph of the function y in a certain range of x relative to given rectangular axes Ox, Oy, the scales along these axes being the same. Let P_1, (x_1, y_1), be a chosen point and P, (x, y), any neighbouring point on the curve (on either side of P_1). Also let the secant through P_1, P cut Ox in R, and denote the angle xRP, that is, the *slope* of the chord P_1P to Ox, by φ, as shewn. Drawing P_1M_1, PM perpendicular to Ox and P_1L parallel to Ox to meet MP in L, clearly $\angle LP_1P=\varphi$. Also $P_1L=OM-OM_1$, $LP=MP-M_1P_1$, so that, in terms of the unit of scale,

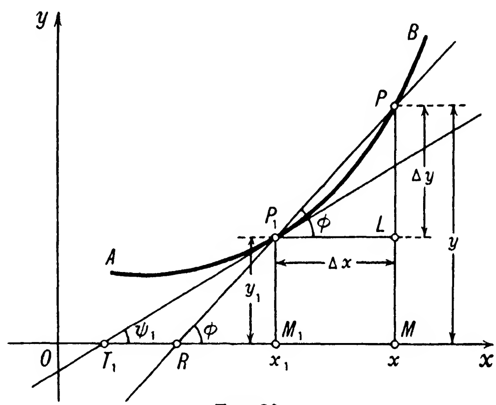


FIG. 26.

$$P_1L=x-x_1=\Delta x, \quad LP=y-y_1=\Delta y, \qquad \text{................(1)}$$

and hence

$$\frac{\Delta y}{\Delta x}=\frac{LP}{P_1L}=\tan\angle LP_1P=\tan\varphi. \qquad \text{...................(2)}$$

The closing up to equality of the values of $\Delta y/\Delta x$ as $x \to x_1$ is thus geometrically equivalent to the closing up to equality of the values of the trigonometrical tangents of the slopes of the chords such as P_1P as P moves up to P_1 from either side. The *slopes* of the chords will then also close up to equality, and this is equivalent to the statement that there is a definite tangent line P_1T_1 at P_1 and that its

slope is the common value to which the others converge. If this slope is denoted by ψ_1, we thus have the relation

$$\left(\frac{dy}{dx}\right)_{x_1}=\tan \psi_1. \quad \text{.............................(3)}$$

The quantity $\tan \psi_1$ is called the *gradient* of the tangent line at P_1, or simply the *gradient of the curve at* P_1, relative to Ox.[1] Dropping the suffix, we have, for any typical point on the curve AB,

$$\frac{dy}{dx}=\tan \psi, \quad \text{.................................(4)}$$

where the axes of reference are rectangular and the scales along them the same.

70. Convention for the Measurement of the Slope ψ.

In arriving at the above result (4), the slope ψ of the tangent to the curve at the point considered was taken to be an acute angle. We now introduce a convention for the measurement of ψ under which the result is always true (for rectangular axes). *The slope ψ of the tangent line at any point, relative to Ox, is defined as the smallest angle of counter-clockwise rotation required to bring this axis into parallelism with the tangent line.*[2] The angle ψ therefore lies in the range $(0, \pi)$.

For the case illustrated in Fig. 26, y increases with x in the neighbourhood of P_1, $(dy/dx)_{x_1}$ is positive and so also is $\tan \psi_1$. A second case is illustrated in Fig. 27. Here y decreases in the neighbourhood of P_1 as x increases, $(dy/dx)_{x_1}$ is negative and so also is $\tan \psi_1$.

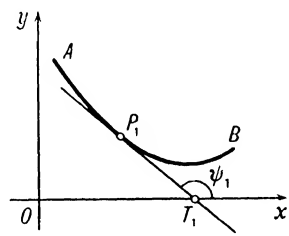


FIG. 27.

The letter g is frequently employed as the symbol for the gradient at any point, and the functional form $g(x)$ is used when it is desired to indicate its dependence on the value of the abscissa x. Thus we write[3]

$$\frac{dy}{dx}=\tan \psi = g = g(x). \quad \text{.........................(1)}$$

[1] The slope and gradient of a curve relative to Oy are defined in Art. 78. Without specification, the slope and gradient are always to be taken as referring to Ox.

[2] This convention introduces only the *orientation* of the tangent relative to Ox. Subsequently we shall introduce a more general convention for the measurement of ψ in which a *direction* (or *sense*) is assigned to the tangent; see Chap. XII, Art. 280. The above result (4) is, however, unaffected.

[3] The symbol g, or $g(x)$, is often employed below simply as a substitute for dy/dx, even when there is no reference to a graph.

The value of the gradient at a special value x_1, say, may now be denoted either by g_1 or by $g(x_1)$.

The sine and cosine of the slope ψ follow at once from (1). Thus, employing the symbol (sig g) to denote the algebraic sign of g, as in Art. 52. 1,

$$\sin\psi = \frac{|\tan\psi|}{\sqrt{(1+\tan^2\psi)}} = \frac{|g|}{\sqrt{(1+g^2)}} = (\operatorname{sig} g)\frac{g}{\sqrt{(1+g^2)}}, \quad \text{........(2)}$$

this sine being always positive, and

$$\cos\psi = \frac{\sin\psi}{\tan\psi} = (\operatorname{sig} g)\frac{1}{\sqrt{(1+g^2)}}. \quad \text{..................(3)}$$

If the gradient vanishes at a point on a curve, this point is referred to as a *point of zero gradient* on the curve, and the curve is said to have zero gradient there. Since $\tan\psi$ now vanishes, the slope ψ is equal either to 0 or to π and the tangent is parallel to Ox. It is then often described as 'horizontal'. At a point of zero gradient the relation $dy/dx=0$ holds, according to (1).

A second special case is that in which the tangent at P_1 is parallel to Oy, or is 'vertical,' as represented in Figs. 28, 29. In Fig. 28, the slope ψ at any neighbouring point P is less than $\frac{1}{2}\pi$, and there is a definite positive gradient (or derivative) at each such point. As P moves up to P_1, the corresponding values of the slope converge to $\frac{1}{2}\pi$ and those of the gradient (or derivative), while remaining positive, increase indefinitely. We thus adopt the convention that, at P_1, $\tan\psi_1 = +\infty$, and write $(dy/dx)_{x_1} = +\infty$. If we consider the ratios of increments with P_1 as starting point, these results correspond to the fact that *all* the values of $\Delta y/\Delta x$ can be made to exceed any number, however large, by taking the range of values of $|\Delta x|$ sufficiently small.

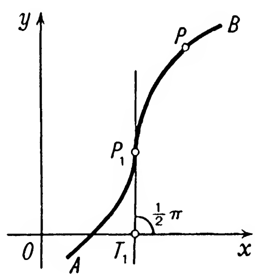


FIG. 28.

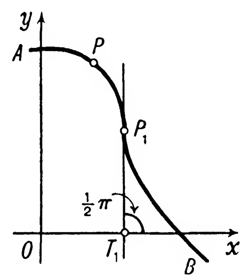


FIG. 29.

For the case illustrated in Fig. 29 there is a definite negative gradient (or derivative) at any point P in the neighbourhood of P_1, and as P moves up to P_1 the corresponding values of the gradient (or derivative) diverge to $-\infty$. By convention we say that in this case $\tan\psi_1 = -\infty$, and we write $(dy/dx)_{x_1} = -\infty$.

In the above cases the increase (decrease) of $\Delta y/\Delta x$ is *systematic* as $|\Delta x| \to 0$, and we express the results $dy/dx = +\infty$, $dy/dx = -\infty$ in words by saying that at P_1 the function has a positive infinite derivative or a negative infinite derivative, as the case may be. The general term *definite-infinite* is employed to describe such a derivative.

If the curve is as shewn in Fig. 30, the ordinate y is not a single-valued function of x in the neighbourhood of P_1. Such cases do not fall within the range of the present discussion, and are postponed until Art. 76.

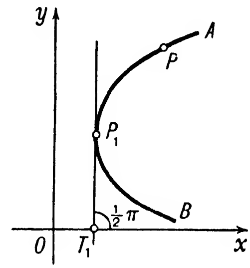


FIG. 30.

Ex. 1. *The functions* x^2, x^3, x^4, x^5. When $x=0$ the gradients of the graphs of these functions are all zero and the tangents therefore horizontal. This follows from the fact that in each case $dy/dx=0$ when $x=0$.

Ex. 2. Find the gradient and slope of the curve $y=x^2$ at each of the points whose abscissae are $-3, -1, 1, 3$.

Since $dy/dx=2x$, we have $\tan\psi=2x$, and hence the respective gradients sought are $-6, -2, 2, 6$. The corresponding slopes are, in radians, 1·73595, 2·03444, 1·10715, 1·40565, or, in degree measure, 99° 27′ 44″, 116° 33′ 54″, 63° 26′ 6″, 80° 32′ 16″.

Ex. 3. Find the angles at which the curves $y=x^2$, $y=x^3$ intersect.

The abscissae of the points of intersection are 0, 1. For the quadratic parabola $y=x^2$, we have $dy/dx=2x$, and for the cubic parabola $y=x^3$, $dy/dx=3x^2$. When $x=0$ the derivatives vanish, and when $x=1$ they have the respective values 2, 3.

Let θ_1, θ_2 be the corresponding acute angles of intersection of the (tangents to the) curves. Then evidently $\theta_1=0$. Also, if ψ_1, ψ_2 are the slopes of the respective curves when $x=1$, we have $\tan\psi_1=2$, $\tan\psi_2=3$. The tables now give $\psi_1=1{\cdot}10715$, $\psi_2=1{\cdot}24905$, and hence

$$\theta_2=\psi_2-\psi_1=0{\cdot}14190\approx 8^\circ\, 8'. \quad \text{.........................(4)}$$

This procedure for the evaluation of θ_2 requires two references to the tables. One reference only is required if we employ the formula

$$\tan\theta_2=\frac{\tan\psi_2-\tan\psi_1}{1+\tan\psi_1\tan\psi_2}, \quad \text{.........................(5)}$$

which in this case gives

$$\tan\theta_2=\frac{3-2}{1+6}=\frac{1}{7}. \quad \text{.........................(6)}$$

71. Fundamental Theorems for the Differentiation of Functions built up of Elementary Functions.

The definition of the derivative of a function y for the value x_1 of the argument as the completing value there of the potentially continuous function $\Delta y/\Delta x$, enables us to establish in a simple manner a

system of rules for the differentiation of sums, products and quotients of differentiable functions by applying the general theorems of Art. 24, dealing with the continuity of combinations of continuous functions.

71. 1. THEOREM I. *Derivative of a sum.* Let u, v be two functions[1] of x each possessing a definite derivative at the point x_1. Denote their sum by y, so that $y=u+v$. If Δu, Δv, Δy are the respective increments in u, v, y corresponding to an increment Δx from x_1 as starting point, we have

$$\Delta y=\Delta u+\Delta v, \qquad (1)$$

and hence, provided $\Delta x \neq 0$,

$$\frac{\Delta y}{\Delta x}=\frac{\Delta u}{\Delta x}+\frac{\Delta v}{\Delta x}. \qquad (2)$$

Now the function on the right-hand side of (2) is the sum of two functions each of which is potentially continuous at x_1, since u, v each have a definite derivative there. It is therefore potentially continuous there and its completing value is the sum of the completing values of the two terms, namely, $\left(\frac{du}{dx}\right)_{x_1}+\left(\frac{dv}{dx}\right)_{x_1}$. Hence the function $\Delta y/\Delta x$ is potentially continuous at x_1 and its completing value there is this sum. It follows that there is a definite derivative of the function y at x_1 and that its value is given by

$$\left(\frac{dy}{dx}\right)_{x_1}=\left(\frac{du}{dx}\right)_{x_1}+\left(\frac{dv}{dx}\right)_{x_1}. \qquad (3)$$

Dropping the suffix, we have, for any typical point x,

$$\frac{d(u+v)}{dx}=\frac{du}{dx}+\frac{dv}{dx}, \qquad (4)$$

which is the symbolic statement of the theorem to be demonstrated. Thus the sum of two differentiable functions is itself a differentiable function and its derivative is the sum of the separate derivatives.

In the same way, if we start with $y=u-v$, we find

$$\frac{d(u-v)}{dx}=\frac{du}{dx}-\frac{dv}{dx}, \qquad (5)$$

a result which also follows by writing $y=u+(-v)$ and using the result of Ex. 1, Art. 67, which makes $d(-v)/dx=-dv/dx$.

[1] The argument x can be indicated in the symbols for the functions, if desired, by writing them as $u(x)$, $v(x)$.

71. 2. THEOREM II. *Derivative of a product.* Consider next the product uv, which we denote by y. With the above meanings for Δu, Δv, Δy, we have

$$\begin{aligned}\Delta y &= (u_1 + \Delta u)(v_1 + \Delta v) - u_1 v_1 \\ &= \Delta u \,.\, v + u_1 \,.\, \Delta v. \qquad \text{.............................(6)}\end{aligned}$$

Therefore, provided $\Delta x \neq 0$,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} v + u_1 \frac{\Delta v}{\Delta x}. \qquad \text{............................(7)}$$

Now the function on the right-hand side of (7) is potentially continuous at x_1, being the sum of two functions which are potentially continuous there. This follows from the fact that $\Delta u/\Delta x$, $\Delta v/\Delta x$ are potentially continuous at x_1, while v is continuous and u_1 constant (and therefore continuous). The completing value of the right-hand side is therefore $\left(\frac{du}{dx}\right)_{x_1} v_1 + u_1 \left(\frac{dv}{dx}\right)_{x_1}$, which is also the completing value of the potentially continuous function $\Delta y/\Delta x$ at x_1. There is thus a definite derivative at x_1 given by

$$\left(\frac{dy}{dx}\right)_{x_1} = \left(\frac{du}{dx}\right)_{x_1} v_1 + u_1 \left(\frac{dv}{dx}\right)_{x_1}. \qquad \text{..................(8)}$$

Dropping the suffix, we have, for any typical point x,

$$\frac{d(uv)}{dx} = \frac{du}{dx} v + u \frac{dv}{dx}, \qquad \text{.............................(9)}$$

which is the symbolic statement of the theorem to be demonstrated. Thus the product of two differentiable functions is itself a differentiable function and its derivative is formed by differentiating each factor in turn, keeping the other factor unchanged, and then adding the two results so obtained.

71. 3. THEOREM III. *Derivative of a reciprocal.* Consider next the function $1/v$ and suppose that at the point x_1 considered, v does not vanish. We then have

$$y_1 = \frac{1}{v_1}, \quad \Delta y = \frac{1}{v} - \frac{1}{v_1} = -\frac{\Delta v}{v v_1}, \qquad \text{..................(10)}$$

and hence, provided $\Delta x \neq 0$,

$$\frac{\Delta y}{\Delta x} = -\frac{1}{v v_1} \frac{\Delta v}{\Delta x}. \qquad \text{..........................(11)}$$

The function on the right-hand side of (11) is potentially con-

tinuous at x_1, its completing value being $-\dfrac{1}{v_1^2}\left(\dfrac{dv}{dx}\right)_{x_1}$, and, as before, there is a definite derivative of y at x_1 given by

$$\left(\frac{dy}{dx}\right)_{x_1} = -\frac{1}{v_1^2}\left(\frac{dv}{dx}\right)_{x_1}. \qquad \text{.........................(12)}$$

Dropping the suffix, we have, for any typical value x,

$$\frac{d(1/v)}{dx} = -\frac{1}{v^2}\frac{dv}{dx}, \qquad \text{...........................(13)}$$

provided that v does not vanish. This is the symbolic statement of the theorem in question. Thus the reciprocal of a differentiable function is itself a differentiable function, provided that the original function does not vanish at the point concerned.

71. 4. THEOREM IV. *Derivative of a quotient.* We now consider the quotient $y=u/v$, $(v\neq 0)$. This is brought back to the product of two differentiable functions by writing it in the form

$$y = u\frac{1}{v}. \qquad \text{............................(14)}$$

We now have

$$\begin{aligned}\frac{dy}{dx} &= \frac{du}{dx}\frac{1}{v} + u\frac{d(1/v)}{dx}, \quad \text{(by Theorem II)},\\ &= \frac{du}{dx}\frac{1}{v} + u\left(-\frac{1}{v^2}\frac{dv}{dx}\right), \quad \text{(by Theorem III)},\\ &= \frac{du}{dx}\frac{1}{v} - \frac{u}{v^2}\frac{dv}{dx}, \qquad \text{............................(15)}\end{aligned}$$

which is a symbolic statement of the theorem in question.

The result is often written in the symmetrical form

$$\frac{d\left(\dfrac{u}{v}\right)}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}, \qquad \text{.........................(16)}$$

which is, perhaps, rather easier to memorize. It is, however, not always so convenient as (15), and the student should practise differentiating quotients as often by the direct application of the rule for a product as by employing the above rule (16).

71. 5. *Generalizations of the above theorems.* (i) Consider the sum $y=u+v+w$, where u, v, w are differentiable functions of x. Writing

$$y = z + w, \quad \text{where} \quad z = u + v, \qquad \text{....................(17)}$$

we have, by Theorem I,

$$\frac{dy}{dx}=\frac{dz}{dx}+\frac{dw}{dx}=\frac{d(u+v)}{dx}+\frac{dw}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}. \quad \text{..........(18)}$$

The generalization of this result to the case of any finite number of differentiable functions is immediate, and we conclude that the derivative of the sum of any finite number of such functions is the sum of the separate derivatives.

(ii) Consider next the product $y=uvw$. Writing

$$y=zw, \quad \text{where} \quad z=uv, \quad \text{........................(19)}$$

we have, by Theorem II,

$$\frac{dy}{dx}=\frac{dz}{dx}w+z\frac{dw}{dx}=\frac{d(uv)}{dx}w+uv\frac{dw}{dx}=\left(\frac{du}{dx}v+u\frac{dv}{dx}\right)w+uv\frac{dw}{dx}$$

$$=\frac{du}{dx}vw+u\frac{dv}{dx}w+uv\frac{dw}{dx}. \quad \text{..(20)}$$

The generalization of this result to any finite number of differentiable functions is also immediate. We conclude that the derivative of the product of any finite number of such functions is found by differentiating each factor in turn, leaving the other factors unchanged, and forming the sum of all the results so obtained. Thus we have

$$\frac{d(uvw\ldots)}{dx}=\frac{du}{dx}vw\ldots+u\frac{dv}{dx}w\ldots+uv\frac{dw}{dx}\ldots+\ldots \quad \text{......(21)}$$

This result may also be written in the more compact form

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}+\frac{1}{w}\frac{dw}{dx}+\ldots, \quad \text{......................(22)}$$

obtained by dividing throughout by the product $uvw\ldots$, assumed not zero, and writing y in place of it on the left-hand side.

71. 6. *Special cases of the general theorems.* (i) *The function* $u+c$, (*c constant*). We have, by Theorem I,

$$\frac{d(u+c)}{dx}=\frac{du}{dx}+\frac{dc}{dx}, \quad \text{..........................(23)}$$

and since $dc/dx=0$, as in Ex. 1, Art. 66, this becomes

$$\frac{d(u+c)}{dx}=\frac{du}{dx}. \quad \text{................................(24)}$$

The geometrical interpretation of this result is that if the corre-

sponding ordinates of two curves differ throughout by a constant, the corresponding gradients are equal.

(ii) *The function cu, (c constant).* Theorem II now gives

$$\frac{d(cu)}{dx}=\frac{dc}{dx}u+c\frac{du}{dx}=c\frac{du}{dx}, \qquad \text{.....................(25)}$$

since dc/dx vanishes. Hence the derivative of a constant multiple of a differentiable function is the product of the constant multiplier and the derivative of the function.

Interpreted geometrically, this means that if all the ordinates of a curve are changed in a given ratio, the gradient is changed in the same ratio.

Ex. 1. *The function $au+b$, (a, b constants).* Here we have

$$\frac{d(au+b)}{dx}=\frac{d(au)}{dx}+\frac{db}{dx}=a\frac{du}{dx}. \qquad \text{.........................(26)}$$

In particular, if $u=x$, so that $du/dx=dx/dx=1$, we have

$$\frac{d(ax+b)}{dx}=a. \qquad \text{.................................(27)}$$

Ex. 2. *The function $a_0+a_1x+a_2x^2+a_3x^3$, (a_0, a_1, a_2, a_3 constants).* Here

$$\left.\begin{aligned} &\frac{da_0}{dx}=0, \quad \frac{d(a_1x)}{dx}=a_1\frac{dx}{dx}=a_1, \\ &\frac{d(a_2x^2)}{dx}=a_2\frac{dx^2}{dx}=2a_2x, \quad \frac{d(a_3x^3)}{dx}=a_3\frac{dx^3}{dx}=3a_3x^2. \end{aligned}\right\} \qquad \text{.........(28)}$$

Hence

$$\begin{aligned}\frac{d(a_0+a_1x+a_2x^2+a_3x^3)}{dx}&=\frac{da_0}{dx}+\frac{d(a_1x)}{dx}+\frac{d(a_2x^2)}{dx}+\frac{d(a_3x^3)}{dx} \\ &=a_1+2a_2x+3a_3x^2. \qquad \text{.........................(29)}\end{aligned}$$

Ex. 3. *The function $(1+x+x^2)(1-x)$.* By the product rule, Theorem II, we have

$$\begin{aligned}\frac{d(1+x+x^2)(1-x)}{dx}&=\frac{d(1+x+x^2)}{dx}(1-x)+(1+x+x^2)\frac{d(1-x)}{dx} \\ &=(1+2x)(1-x)+(1+x+x^2)(-1)=-3x^2. \qquad \text{......(30)}\end{aligned}$$

The function is actually equal to $1-x^3$, and the result obtained follows at once by differentiating this expression.

Ex. 4. *The function $1/(1+x)$, ($x\neq -1$).* Write

$$y=\frac{1}{1+x}=\frac{1}{u}, \qquad \text{...................................(31)}$$

where $u=1+x$, so that $du/dx=1$. Then, by Theorem III, we have

$$\frac{dy}{dx}=-\frac{1}{u^2}\frac{du}{dx}=-\frac{1}{u^2}, \qquad \text{..........................(32)}$$

or
$$\frac{d}{dx}\left(\frac{1}{1+x}\right)=-\frac{1}{(1+x)^2}. \qquad (33)$$

In a similar way we find
$$\frac{d}{dx}\left(\frac{1}{1-x}\right)=\frac{1}{(1-x)^2},\ (x \neq 1). \qquad (34)$$

Ex. 5. *The function* $(1-x)/(1+x)$, $(x \neq -1)$. Write this in the form $y=u/v$, where $u=1-x$, $v=1+x$, so that $du/dx=-1$, $dv/dx=1$. Then, by the rule (16) for the derivative of a quotient, we have

$$\frac{dy}{dx}=\frac{(1+x)\dfrac{d(1-x)}{dx}-(1-x)\dfrac{d(1+x)}{dx}}{(1+x)^2}=\frac{(1+x).(-1)-(1-x).1}{(1+x)^2}=-\frac{2}{(1+x)^2}. \qquad (35)$$

Again, treating y as the product of $(1-x)$ by $1/(1+x)$, we have, by Theorem II,
$$\frac{dy}{dx}=\frac{d(1-x)}{dx}\frac{1}{1+x}+(1-x)\frac{d}{dx}\left(\frac{1}{1+x}\right)=(-1)\frac{1}{1+x}+(1-x)\frac{-1}{(1+x)^2}, \qquad (36)$$
which leads to the same result.

We may also write
$$y=-\frac{1+x-2}{1+x}=-1+\frac{2}{1+x}, \qquad (37)$$

and the derivative follows at once from Ex. 4 above.

Ex. 6. Find the angles of intersection of the graph of the function
$$y=(x-1)(x-2)(x-3) \qquad (38)$$
with Ox, and the abscissae of the points of zero gradient.

By the generalization of the rule for the derivative of a product, as expressed by (20) above, we have
$$\frac{dy}{dx}=\frac{d(x-1)}{dx}(x-2)(x-3)+(x-1)\frac{d(x-2)}{dx}(x-3)+(x-1)(x-2)\frac{d(x-3)}{dx}$$
$$=(x-2)(x-3)+(x-1)(x-3)+(x-1).(x-2)=3x^2-12x+11. \qquad (39)$$

The curve cuts Ox when $x=1, 2, 3$, and for these values we have, in turn, $dy/dx=2, -1, 2$. The corresponding slopes are therefore $1{\cdot}10715$, $\frac{3}{4}\pi$, $1{\cdot}10715$ radians, or $63^\circ 26'$, 135°, $63^\circ 26'$.

The abscissae of the points of zero gradient are given by
$$3x^2-12x+11=0, \qquad (40)$$
and are therefore $x=2-\frac{1}{3}\sqrt{3}$, $x=2+\frac{1}{3}\sqrt{3}$, that is, $x=1{\cdot}423$ and $x=2{\cdot}577$, approximately.

Ex. 7. Find the values of the constants a, b, c of the function $y=ax^2+bx+c$ in order that its graph may pass through the points $(1, 2)$, $(3, 1)$, and, in addition, have zero gradient where $x=\frac{3}{2}$.

Since the points $(1, 2)$, $(3, 1)$ are on the curve, we have
$$2=a+b+c, \qquad (41)$$
$$1=9a+3b+c. \qquad (42)$$

Also $dy/dx=2ax+b$, and this vanishes when $x=\frac{3}{2}$, so that
$$0=3a+b. \qquad (43)$$

From (41), (42), (43) we find $a = -\frac{1}{2}$, $b = \frac{3}{2}$, $c = 1$, and the equation of the curve is thus $y = -\frac{1}{2}x^2 + \frac{3}{2}x + 1$.

Ex. 8. From a point P, (x, y), on a curve $y = f(x)$ the perpendicular PN is drawn to the tangent to the curve through a fixed point P_1, (x_1, y_1), on it. It is required to find the rate of increase of the distance P_1N with respect to x as P moves along the curve.[1]

Let AB (Fig. 31) be the graph of the function $f(x)$ and suppose that at P_1 the gradient $\tan \psi_1$ is positive, so that ψ_1 lies in the range $(0, \frac{1}{2}\pi)$. Let the positive sense for the measurement of P_1N be taken towards the right, as indicated, and denote the variable distance P_1N by x'. Drawing P_1L parallel to Ox, we observe that the sum of the projections of P_1L and LP on P_1N reproduces P_1N, and therefore

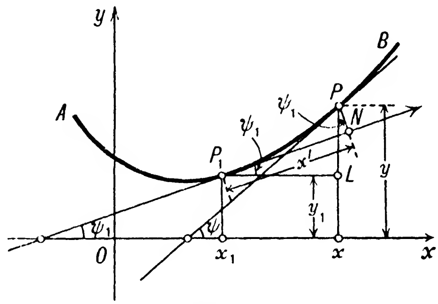


FIG. 31.

$$x' = (x - x_1) \cos \psi_1 + (y - y_1) \sin \psi_1. \quad \text{.......................(44)}$$

This expresses x' as the sum of two differentiable functions of x, and since x_1, y_1, ψ_1 are fixed, we have

$$\frac{dx'}{dx} = \cos \psi_1 + \frac{dy}{dx} \sin \psi_1 = \cos \psi_1 + \tan \psi \sin \psi_1 = \cos(\psi - \psi_1) \sec \psi, \quad \text{.....(45)}$$

where $\tan \psi$ is the gradient at P.

We shall require the value of this derivative at P_1, which is obtained by putting $\psi = \psi_1$, and is therefore given by

$$\left(\frac{dx'}{dx}\right)_{x_1} = \sec \psi_1. \quad \text{....................................(46)}$$

Employing the result (3) of Art. 70, noticing that here g_1 is positive, we can write this result in the form

$$\left(\frac{dx'}{dx}\right)_{x_1} = \sqrt{(1 + g_1^2)}. \quad \text{..............................(47)}$$

The student should verify that if g_1 is negative, the positive sense for the measurement of x' being again towards the right, the result obtained is

$$\left(\frac{dx'}{dx}\right)_{x_1} = -\sec \psi_1 = \sqrt{(1 + g_1^2)}. \quad \text{...................(48)}$$

72. Differentiation of Powers of a Differentiable Function; Integral Indices. Derivative of x^m, (m an integer).

With the aid of the preceding theorems we are able to write down the derivative of a power of a differentiable function, u say, of x,

[1] The results of this Example are employed in Chap. V, Art. 133, in the investigation of a formula for the *curvature* of a curve.

when the index is an integer, positive or negative, the derivative du/dx being supposed known.

72. 1. *Positive integral index.* We consider first the function u^m, where m is a positive integer. Denoting this by y, we may write

$$y = u \,.\, u \,.\, u \ldots, \quad (m \text{ factors}). \qquad \ldots\ldots(1)$$

Hence, by the generalized rule for the derivative of a product, we have

$$\frac{dy}{dx} = \frac{du}{dx} \,.\, u \,.\, u \,.\, u \ldots + u \,.\, \frac{du}{dx} \,.\, u \,.\, u \ldots + u \,.\, u \,.\, \frac{du}{dx} \,.\, u \ldots + \ldots \quad \ldots(2)$$

In the expression on the right-hand side of (2) there are m terms each equal to the product of du/dx by u^{m-1}. Hence

$$\frac{du^m}{dx} = mu^{m-1}\frac{du}{dx}. \qquad \ldots\ldots(3)$$

As a particular case, take $u = x$ so that $du/dx = dx/dx = 1$. Then we have

$$\frac{dx^m}{dx} = mx^{m-1}. \qquad \ldots\ldots(4)$$

We may express this result in words by saying that the derivative of a power of x in which the index is a positive integer is obtained by reducing the index by 1 and multiplying the power so obtained by the original index. This result is known as the *index rule* for the derivative of a power of x. It is here proved only when the index is a positive integer, but we shall shew subsequently that it applies generally for *any* power of x.

72. 2. *Negative integral index.* Consider next the function u^{-m}, ($u \neq 0$, m a positive integer), which we write in the form

$$u^{-m} = \frac{1}{u^m} = \left(\frac{1}{u}\right)^m. \qquad \ldots\ldots(5)$$

If in (3) we replace u by $1/u$ and accordingly $\dfrac{du}{dx}$ by $\dfrac{d(1/u)}{dx}$, that is, by $-\dfrac{1}{u^2}\dfrac{du}{dx}$, we get the result

$$\begin{aligned}\frac{du^{-m}}{dx} = \frac{d\left(\dfrac{1}{u}\right)^m}{dx} &= m\left(\frac{1}{u}\right)^{m-1}\left(-\frac{1}{u^2}\frac{du}{dx}\right)\\ &= -m\frac{1}{u^{m-1}} \cdot \frac{1}{u^2}\frac{du}{dx}\\ &= -mu^{-m-1}\frac{du}{dx}, \qquad \ldots\ldots(6)\end{aligned}$$

which follows the same rule as (3) with $-m$ written in place of m.

In particular, we have

$$\frac{dx^{-m}}{dx} = -mx^{-m-1}, \quad (x \neq 0). \qquad \text{.......................(7)}$$

This proves the 'index rule' for the derivative of a power of x with a negative integral index.

72. 3. *Alternative method for obtaining the derivative of x^m when m is a positive integer.* The derivative of x^m in this case may be found in a variety of ways. One of these is now investigated.

For a starting point x_1 we have

$$\Delta y = x^m - x_1{}^m = (x - x_1)(x^{m-1} + x^{m-2}x_1 + x^{m-3}x_1{}^2 + \dots + x_1{}^{m-1}), \qquad \text{........(8)}$$

and hence

$$\frac{\Delta y}{\Delta x} = \frac{\Delta x (x^{m-1} + x^{m-2}x_1 + \dots + x_1{}^{m-1})}{\Delta x} = x^{m-1} + x^{m-2}x_1 + \dots + x_1{}^{m-1}, \qquad \text{......(9)}$$

provided $\Delta x \neq 0$. Now the polynomial $x^{m-1} + x^{m-2}x_1 + \dots + x_1{}^{m-1}$ is continuous at x_1, and, since it contains m terms, its value there is $mx_1{}^{m-1}$. This is therefore the completing value of the potentially continuous function $\Delta y/\Delta x$ at x_1. We conclude that

$$\left(\frac{dy}{dx}\right)_{x_1} = mx_1{}^{m-1}. \qquad \text{..............................(10)}$$

73. Derivative of a Polynomial.

Let

$$y = a_0x^m + a_1x^{m-1} + \dots + a_{m-1}x + a_m, \qquad \text{.................(1)}$$

where the a's are constants and m is a positive integer. Then

$$\begin{aligned}
\frac{dy}{dx} &= \frac{da_0x^m}{dx} + \frac{da_1x^{m-1}}{dx} + \dots + \frac{da_{m-1}x}{dx} + \frac{da_m}{dx} \\
&= a_0\frac{dx^m}{dx} + a_1\frac{dx^{m-1}}{dx} + \dots + a_{m-1}\frac{dx}{dx} + 0 \\
&= a_0mx^{m-1} + a_1(m-1)x^{m-1-1} + \dots + a_{m-1} \\
&= ma_0x^{m-1} + (m-1)a_1x^{m-2} + \dots + a_{m-1}. \qquad \text{.............(2)}
\end{aligned}$$

74. Derivative of a Rational Algebraic Function.

Expressing the function as the quotient of two polynomials, we have

$$y = \frac{a_0x^m + a_1x^{m-1} + \dots + a_m}{b_0x^n + b_1x^{n-1} + \dots + b_n}, \qquad \text{....................(1)}$$

where the a's and b's are constants and m, n are positive integers. The derivatives of the numerator and denominator are obtained as in the preceding Article, and the required derivative follows at once

by employing the rule for the derivative of a quotient. Thus, excluding the values of x, if any, for which the denominator vanishes, we have

$$\frac{dy}{dx}=\frac{f_1-f_2}{(b_0x^n+\ldots+b_n)^2}, \quad \ldots\ldots\ldots\ldots(2)$$

where

$$f_1\equiv(b_0x^n+\ldots+b_n)\{ma_0x^{m-1}+(m-1)a_1x^{m-2}+\ldots+a_{m-1}\}, \quad \ldots\ldots(3)$$

$$f_2\equiv(a_0x^m+\ldots+a_m)\{nb_0x^{n-1}+(n-1)b_1x^{n-2}+\ldots+b_{n-1}\}. \quad \ldots\ldots(4)$$

The theory of elementary (partial) fractions may be employed to simplify the form of the function y and hence of the derivative dy/dx, but as this theory is not required for the purpose of finding the derivative, it is unnecessary to give here the details of the process. This theory is referred to again when dealing with the subject of *Successive Derivatives* in Chap. III, and is fully expounded in Chap. X, Art. 241, when dealing with the problem of *Systematic Integration*, for which purpose it is essential. The relevant portions of that Article can be read in the present connection.

Ex. 1. *The functions* x^4, x^5, x^9, $\frac{1}{10}x^{10}$, $1/x$, $1/x^2$, $-1/(2x^2)$, $1/x^3$, $1/x^8$, $1/(9x^9)$.

The derivatives, obtained by applying the 'index rule,' are, in succession, $4x^3$, $5x^4$, $9x^8$, x^9, $-1/x^2$, $-2/x^3$, $1/x^3$, $-3/x^4$, $-8/x^9$, $-1/x^{10}$, the value $x=0$ being excluded from the ranges of the functions after the fourth one written.

Ex. 2. *The function* $(a-x)^2$. Writing $y=u^2$, where $u=a-x$, so that $du/dx=-1$, we have

$$\frac{dy}{dx}=2u\frac{du}{dx}=2(a-x)(-1)=-2(a-x). \quad \ldots\ldots\ldots\ldots(5)$$

Ex. 3. *The function* $(ax^2+bx+c)^m$, (*m an integer*). Writing $y=u^m$, where $u=ax^2+bx+c$, so that $du/dx=2ax+b$, we have

$$\frac{dy}{dx}=mu^{m-1}\frac{du}{dx}=m(ax^2+bx+c)^{m-1}(2ax+b). \quad \ldots\ldots\ldots\ldots(6)$$

Ex. 4. *The function* u^mv^n, (*u, v differentiable functions of x, and m, n integers*). If m (or n) is negative, we must exclude those values of x, if any, for which u (or v) vanishes. The product rule here gives

$$\frac{dy}{dx}=\frac{du^m}{dx}v^n+u^m\frac{dv^n}{dx}=mu^{m-1}\frac{du}{dx}v^n+nu^mv^{n-1}\frac{dv}{dx}=u^{m-1}v^{n-1}\left(m\frac{du}{dx}v+nu\frac{dv}{dx}\right). \quad (7)$$

Ex. 5. *The function* $(1-x)^m(1+x)^n$, (*m, n integers*). We must exclude the value $x=1$ if m is negative and $x=-1$ if n is negative. We find

$$\frac{dy}{dx}=m(1-x)^{m-1}.(-1).(1+x)^n+(1-x)^m.n(1+x)^{n-1}.1$$

$$=\{(n-m)-(n+m)x\}(1-x)^{m-1}(1+x)^{n-1}. \quad \ldots\ldots\ldots\ldots(8)$$

Ex. 6. *The function* $1\Big/\left(x+\dfrac{1}{x}\right)^2$, $(x\neq 0)$. Writing $y=1/u^2=u^{-2}$, where $u=x+1/x$, so that $du/dx=1-1/x^2$, we have

$$\frac{dy}{dx}=-2u^{-3}\frac{du}{dx}=-\frac{2}{\left(x+\dfrac{1}{x}\right)^3}\left(1-\frac{1}{x^2}\right)=-\frac{2x(x^2-1)}{(x^2+1)^3}. \quad \text{.............(9)}$$

Ex. 7. Find the coordinates of the points on the curve $y=x/(1-x^2)$ at which the gradients are equal to 1.

We here illustrate the intuitive method of obtaining the elementary (partial) fractions for y by starting with the obvious identity

$$(1+x)-(1-x)=2x. \quad \text{..................................(10)}$$

Division by $2(1-x^2)$ now gives

$$\frac{1}{2(1-x)}-\frac{1}{2(1+x)}=\frac{x}{1-x^2}, \quad \text{............................(11)}$$

which is the required expression. We thus have

$$\frac{dy}{dx}=\frac{1}{2(1-x)^2}+\frac{1}{2(1+x)^2}=\frac{1+x^2}{(1-x^2)^2}. \quad \text{........................(12)}$$

Equating this to 1, we obtain the equation $x^4-3x^2=0$, so that the abscissae of the required points are $-\sqrt{3}$, 0, $\sqrt{3}$. The corresponding values of y are $\frac{1}{2}\sqrt{3}$, 0, $-\frac{1}{2}\sqrt{3}$.

We observe that as $|x|\to 1$, $|y|\to\infty$ and $dy/dx\to\infty$. The graph of the function is most easily constructed by adding the corresponding ordinates of the graphs of the component functions $\dfrac{1}{2(1-x)}$, $-\dfrac{1}{2(1+x)}$, each of which is a rectangular hyperbola. Portions of these graphs and of the compounded graph are shown in Fig. 32.

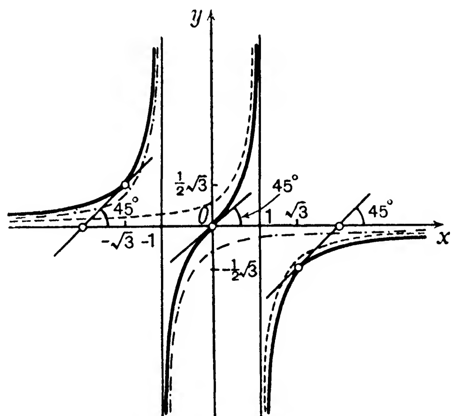


FIG. 32.
$y=x/(1-x^2)$.

75. Differentiation of Powers of x with Fractional Indices.

We have up to this stage developed a system of rules for the differentiation of a power of a differentiable function in which the index is an integer, and for the combination, by addition, subtraction, multiplication and division, of any finite number of such powers. We now proceed to obtain the generalization of these rules when the index is not restricted to have integral values but may be any rational number. In this investigation

the starting point is the differentiation of a power of the variable x in which the index is the reciprocal of a positive integer.

75. 1. *Differentiation of $x^{1/q}$, (q a positive integer).*[1] Suppose, at first, that x is positive. As explained in Art. 27. 1, the function $x^{1/q}$ is positive, that is, we are dealing with the positive qth root. For a starting point x_1, we have

$$\Delta y = x^{1/q} - x_1{}^{1/q} = \frac{x^{1/q} - x_1{}^{1/q}}{x^{q/q} - x_1{}^{q/q}}(x - x_1)$$

$$= \frac{x - x_1}{x^{(q-1)/q} + x^{(q-2)/q}x_1{}^{1/q} + \ldots + x^{1/q}x_1{}^{(q-2)/q} + x_1{}^{(q-1)/q}}$$

$$= \frac{\Delta x}{x^{(q-1)/q} + \ldots + x_1{}^{(q-1)/q}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

so that, provided $\Delta x \neq 0$,

$$\frac{\Delta y}{\Delta x} = \frac{1}{x^{(q-1)/q} + \ldots + x_1{}^{(q-1)/q}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now the function on the right-hand side of (2) is continuous at x_1, being the reciprocal of the sum of q non-vanishing functions, each of which is continuous at x_1, by Art. 27. 2. The function $\Delta y/\Delta x$ is therefore potentially continuous at x_1, and its completing value is the value of the above continuous function there. There is therefore a definite derivative of y at x_1, given by

$$\left(\frac{dy}{dx}\right)_{x_1} = \frac{1}{x_1{}^{(q-1)/q} + x_1{}^{(q-2)/q}x_1{}^{1/q} + \ldots + x_1{}^{(q-1)/q}} = \frac{1}{qx_1{}^{(q-1)/q}}. \quad \ldots(3)$$

Dropping the suffix, this gives, for any positive value of x,

$$\frac{dx^{\frac{1}{q}}}{dx} = \frac{1}{qx^{(q-1)/q}} = \frac{1}{q}x^{\frac{1}{q}-1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

This result again follows the 'index rule' (Art. 72) for the differentiation of x^m, where now $1/q$ is written for m, q being a positive integer.

The restriction of x to non-negative values is necessary if q is *even*. If q is *odd*, $x^{1/q}$ is continuous for *all* values of x, and it then possesses a finite derivative, given by (4), for all values of x except 0. With $x=0$ as starting point we have $\Delta y/\Delta x = x^{\frac{1}{q}}/x = x^{\frac{1}{q}-1}$, and since q is odd, it follows that, if $q > 1$, all the values of $\Delta y/\Delta x$ can be made to exceed any number, however large, by taking the range of values of $|\Delta x|$ sufficiently small. With the convention of Art.

[1] A simpler but less direct method is given in Art. 81, Ex. 2.

70, we write $(dy/dx)_0 = +\infty$ and say that the function has a positive infinite derivative for the value 0 of x.

The geometrical interpretation of this result is that the gradient is infinite at O and the tangent is perpendicular to Ox, that is, it coincides with Oy. These remarks are illustrated in Fig. 33, where portions of the graphs of the functions $y = x^{1/3}$, $y = x^{1/5}$ are drawn.

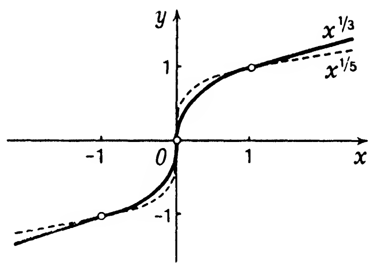


FIG. 33.

The question of a derivative at $x = 0$ of $x^{1/q}$ when q is *even* is discussed in the next Article.

75. 2. *Differentiation of $x^{p/q}$, (p, q positive integers)*. We next consider the power $x^{p/q}$, in which the index p/q is positive, rational and expressed in its lowest terms. When q is even, the function is continuous for non-negative values of x, and when q is odd, for all values of x.

We shall suppose, at first, that x is positive. Writing

$$y = x^{p/q} = (x^{1/q})^p = u^p, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

where $u = x^{1/q}$, we bring the discussion back to the case of a power of a differentiable function in which the index is a positive integer. Employing the result (3) of Art. 72. 1 and the above result (4), we get

$$\frac{dy}{dx} = pu^{p-1}\frac{du}{dx} = p\left(x^{\frac{1}{q}}\right)^{p-1}\frac{dx^{\frac{1}{q}}}{dx}$$

$$= px^{\frac{p-1}{q}} \cdot \frac{1}{q}x^{\frac{1}{q}-1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

or

$$\frac{dx^{\frac{p}{q}}}{dx} = \frac{p}{q}x^{\frac{p}{q}-1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

The derivative is again formed according to the 'index rule' for powers of x, with p/q now written in place of m.

75. 3. *Differentiation of $x^{-p/q}$, (p, q positive integers)*. Considering next the case of a negative rational index, we have the function

$$y = x^{-p/q}, \quad (x \text{ supposed positive}), \quad \ldots\ldots\ldots\ldots\ldots(8)$$

which we write in the form

$$y = \frac{1}{x^{p/q}} = \frac{1}{u}, \quad \text{where} \quad u = x^{p/q}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Employing the rule for the derivative of a reciprocal and the above result (7), we find

$$\frac{dy}{dx}=-\frac{1}{u^2}\frac{du}{dx}=-\frac{1}{(x^{p/q})^2}\frac{dx^{p/q}}{dx}$$

$$=-\frac{1}{x^{\frac{2p}{q}}}\cdot\frac{p}{q}x^{\frac{p}{q}-1}$$

$$=-\frac{p}{q}x^{-\frac{p}{q}-1}, \qquad \text{.....................(10)}$$

which follows the same rule as (7) with $-p/q$ written in place of p/q.

75. 4. We have thus proved that for a power of the variable x, (x positive), in which the index is *any rational number*, m say, the derivative is formed according to the 'index rule'

$$\frac{dx^m}{dx}=mx^{m-1}. \qquad \text{.............................(11)}$$

The same rule can be shewn to hold for the derivative of a power of x in which the index is *irrational* by the use of the method of sequences, on which the rigorous treatment of irrationals is based. A proof of this is given in Chap. XXI, Art. 398. We here assume this extension of the rule, and write, for example,

$$\frac{dx^{\sqrt{2}}}{dx}=\sqrt{2}x^{\sqrt{2}-1}, \qquad \text{..........................(12)}$$

where now x must be positive, (Art. 27. 6).

75. 5. *Derivative of x^m; extension of range of x.* Write $m=p/q$, where p, q are integers and q is positive.

(i) *Derivative for $x<0$.* (*a*) *q even.* In this case the function $x^{p/q}$ is not defined and there is no question of a derivative.

(*b*) *q odd.* Here $x^{p/q}$ is defined and the derivative is given by (11) for negative as well as positive values of x.

(ii) *Derivative at $x=0$.* (*a*) *q even.* In this case the function is defined on one side only of the starting point and therefore an ordinary derivative does not exist. The discussion is resumed in the next Article.

(*b*) *q odd, p positive.* Here $x^{p/q}$ is continuous at $x=0$ and $\Delta y/\Delta x=x^{\frac{p}{q}-1}$. If $p/q>1$, the last function is continuous at $x=0$ and has the value zero there. Thus $(dy/dx)_0=0$.

If $p/q<1$, the function $x^{\frac{p}{q}-1}$ is not continuous at $x=0$ and there is no finite derivative there. If, however, p is odd, the function is definite-infinite and we can write $(dy/dx)_0 = +\infty$. If p is even, the same function is not definite-infinite and there is not a definite-infinite derivative; the discussion of this case is resumed in the next Article.

(*c*) *q odd, p negative.* Here $x^{p/q}$ is undefined at $x=0$ and therefore an ordinary derivative does not exist for this value. The student should observe that this statement is not inconsistent with the existence of infinite limits of dy/dx when x approaches the value 0 from above or below.

Some of the results obtained above are illustrated in Fig. 34,

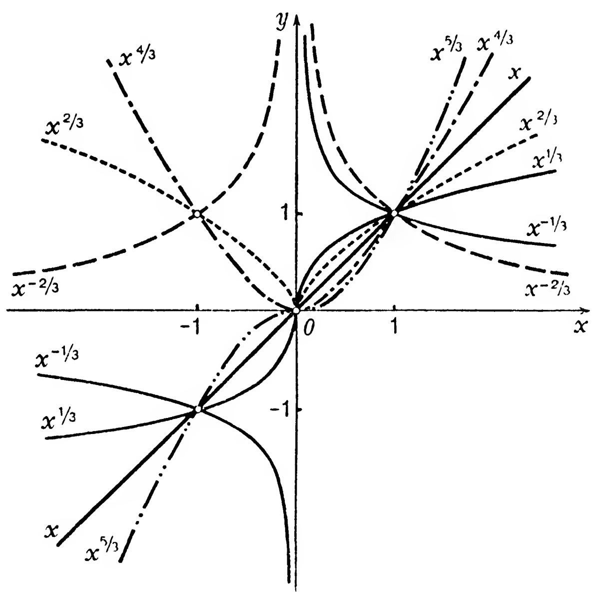


FIG. 34.

which shews portions of the graphs of the function $x^{p/q}$ for the values $\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, -\frac{1}{3}, -\frac{2}{3}$ of the index. We have the following results:

y	$x^{1/3}$,	$x^{2/3}$,	x,	$x^{4/3}$,	$x^{5/3}$,	$x^{-1/3}$,	$x^{-2/3}$.
dy/dx	$\frac{1}{3}x^{-2/3}$,	$\frac{2}{3}x^{-1/3}$,	1,	$\frac{4}{3}x^{1/3}$,	$\frac{5}{3}x^{2/3}$,	$-\frac{1}{3}x^{-4/3}$,	$-\frac{2}{3}x^{-5/3}$.

Ex. 1. *The function* $\sqrt{x}$, $(x>0)$. The index rule here gives

$$\frac{d\sqrt{x}}{dx} = \frac{dx^{1/2}}{dx} = \tfrac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}. \qquad \text{.........................(13)}$$

It is instructive to establish this result directly. For a starting point x_1, we find, without any trouble,

$$\Delta y = \frac{\Delta x}{\sqrt{x} + \sqrt{x_1}}, \quad \text{......(14)}$$

so that, if $\Delta x \neq 0$,

$$\frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x} + \sqrt{x_1}}. \quad \text{......(15)}$$

The function on the right-hand side of (15) is continuous at x_1, its value there being $1/(2\sqrt{x_1})$. This, therefore, is the completing value of the potentially continuous function $\Delta y/\Delta x$ and is the required value of the derivative.

Ex. 2. *The function* $1/\sqrt{x}$, $(x > 0)$. Here we have

$$\frac{d}{dx}\frac{1}{\sqrt{x}} = \frac{dx^{-1/2}}{dx} = -\tfrac{1}{2}x^{-3/2} = -\frac{1}{2x\sqrt{x}}. \quad \text{......(16)}$$

Ex. 3. *The function* $\dfrac{x^4}{\sqrt[5]{x}}$. We write $y = x^4 x^{-1/5} = x^{19/5}$, and the index rule gives

$$\frac{dy}{dx} = \frac{19}{5}x^{14/5}. \quad \text{......(17)}$$

Ex. 4. *The function* $\dfrac{1-\sqrt{x}}{1+\sqrt{x}}$, $(x > 0)$. We write $y = u/v$, where $u = 1 - \sqrt{x}$, $v = 1 + \sqrt{x}$, so that $du/dx = -1/(2\sqrt{x})$, $dv/dx = 1/(2\sqrt{x})$. Then

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} = \frac{(1+\sqrt{x})\left(-\dfrac{1}{2\sqrt{x}}\right) - (1-\sqrt{x})\dfrac{1}{2\sqrt{x}}}{(1+\sqrt{x})^2} = -\frac{1}{\sqrt{x}\,.\,(1+\sqrt{x})^2}. \quad (18)$$

Ex. 5. *The function* $(a\sqrt{x}+b)^2$, $(x > 0)$. To find the derivative we may either square out and differentiate the expression obtained, or we may proceed by writing $y = u^2$, where $u = a\sqrt{x} + b$, and employing the rule $dy/dx = 2u\,du/dx$. Since $du/dx = d(a\sqrt{x})/dx = a/(2\sqrt{x})$, this gives

$$\frac{dy}{dx} = 2(a\sqrt{x}+b)\frac{a}{2\sqrt{x}} = a^2 + \frac{ab}{\sqrt{x}}. \quad \text{......(19)}$$

76. Upper and Lower Derivatives.

The definition of a derivative given above involves a consideration of the values of the ratio $\Delta y/\Delta x$ for both positive and negative values of Δx. In geometrical language, the behaviour of $\Delta y/\Delta x$ is to be discussed in a neighbourhood extending on both sides of the starting point x_1. In some cases it is, however, necessary or desirable to restrict the neighbourhood to one side of the point x_1. An obvious case is that in which the range of the function terminates at the point. For example, the function $x^{3/2}$ is not defined for negative values of x, so that a neighbourhood of $x = 0$ must be confined to positive values.

When a neighbourhood is restricted in this way, we call the derivative obtained from a consideration of the values of $\Delta y/\Delta x$ an

upper derivative or a *lower derivative*[1] according as Δx is confined to positive or negative values. Except for the restriction of one-sidedness of the neighbourhood, the criteria for the existence of an upper or a lower derivative are exactly the same as those already discussed, and the language of potential continuity is equally applicable. Thus, if $\Delta y/\Delta x$ has upper (lower) potential continuity at x_1, its completing value there is the upper (lower) derivative of y at x_1. When necessary, an upper, or lower, derivative will be indicated by affixing (u), or (l), to the symbol dy/dx or $f'(x)$.

If the function is defined both above and below x_1, there may, of course, be both an upper and a lower derivative there. If these are unequal, a geometrical representation would shew a sudden change of direction (gradient) in the graph.[2] If both derivatives exist and are equal, an ordinary derivative exists at the point, for the conditions

$$\left|\frac{\Delta y}{\Delta x} - N\right| < \varepsilon \text{ if } 0 < x - x_1 < h, \text{ and } \left|\frac{\Delta y}{\Delta x} - N\right| < \varepsilon \text{ if } 0 < x_1 - x < h', \quad (1)$$

where ε is arbitrary and h, h' are appropriately chosen, involve the condition

$$\left|\frac{\Delta y}{\Delta x} - N\right| < \varepsilon \text{ if } |x - x_1| < h'', \quad \text{.....................(2)}$$

where h'' is less than or equal to the smaller of the numbers h, h'.

The term *differentiable in a range* is extended to apply to functions which possess one-sided derivatives at the ends and ordinary derivatives at all other points.

As an example, let us consider the function $x^{3/2}$, mentioned above. It is not defined for negative values of x, so that only an upper derivative can be considered at $x=0$. Now, with $x=0$ as starting point, we have $\Delta y/\Delta x = x^{3/2}/x = x^{1/2}$. The last function has upper continuity at $x=0$, its value there being 0, and therefore $\Delta y/\Delta x$ has upper potential continuity at $x=0$ and the upper derivative exists there and is equal to 0. We therefore write

$$\left(\frac{dx^{3/2}}{dx}\right)_0^{(u)} = 0. \quad \text{..............................(3)}$$

[1] It is sometimes convenient to use the terms *right-hand* and *left-hand* instead of *upper* and *lower*. The terms *upper right-hand derivative*, etc., used in more advanced theory, are concerned with focal bounds of *average* rates.

[2] Thus one-sided derivatives correspond to one-sided tangents at a point P of a curve $y=f(x)$, that is, to tangents which are defined by means of chords drawn from P to points of the curve on one side only of that point. We have often to consider a curve or a branch of a curve which terminates at a point P. The one-sided tangent in such a case is called an *end-tangent* at P to the curve or branch. The gradient of the one-sided or end-tangent is always equal to the corresponding upper or lower derivative dy/dx.

More generally, if $y = x^{p/q}$, the cases requiring consideration, when $x = 0$ is the starting point, are :

(i) *q even, p positive.* (a) If $p > q$, $x^{\frac{p}{q}-1}$ has upper continuity at $x = 0$, its value there being 0. There is therefore an upper derivative, namely 0, at this value, and we write

$$\left(\frac{dx^{p/q}}{dx}\right)_0^{(u)} = 0, \quad \text{.......................(4)}$$

(b) If $p < q$, $x^{\frac{p}{q}-1}$ is definite-infinite for $x = +0$, and we write

$$\left(\frac{dx^{p/q}}{dx}\right)_0^{(u)} = +\infty. \quad \text{.......................(5)}$$

(ii) *q odd, p even and $0 < p < q$.* Here $x^{\frac{p}{q}-1}$ changes sign with x, so that we have a positive infinite upper derivative and a negative infinite lower derivative at $x = 0$. Thus

$$\left(\frac{dx^{p/q}}{dx}\right)_0^{(u)} = +\infty, \quad \left(\frac{dx^{p/q}}{dx}\right)_0^{(l)} = -\infty. \quad \text{..............(6)}$$

As a further example of one-sided derivatives, we take the case of a curve with a vertical tangent at P_1 (Fig. 35), postponed from Art. 70. We shall have to consider two single-valued functions, one corresponding to the upper and one to the lower part of the curve. At P_1 we may consider two upper derivatives, one for each of the two functions. For the upper part of the curve, $\Delta y/\Delta x$, or the tangent of the slope of the chord P_1P, is positive and is greater than any assigned number if the magnitude of Δx is less than a certain value. The corresponding upper derivative is therefore said to be equal to $+\infty$. In the same way, for the lower part of the curve, $\Delta y/\Delta x$ is negative and the upper derivative is $-\infty$.

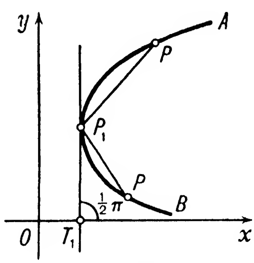


FIG. 35.

Ex. 1. *The function* $\sqrt{x}$. With $x = 0$ as starting point we have $\Delta y/\Delta x = x^{-1/2}$, and hence $\Delta y/\Delta x$ is positive and greater that any assigned number M, say, if $x < 1/M^2$. We conclude that

$$\left(\frac{d\sqrt{x}}{dx}\right)_0^{(u)} = +\infty. \quad \text{.......................(7)}$$

This is in agreement with the general result (5) above.

Ex. 2. *The function* $|x|$. With $x = 0$ as starting point we here have

$$\Delta y/\Delta x = \pm x/x = \pm 1, \quad \text{.......................(8)}$$

the upper (lower) sign being taken if x is positive (negative). Therefore

$$\left(\frac{d\,|x|}{dx}\right)_0^{(u)}=1,\quad \left(\frac{d\,|x|}{dx}\right)_0^{(l)}=-1. \quad\text{.......................(9)}$$

The ordinary derivative is not defined at $x=0$, for we cannot make the difference $\left|\frac{\Delta y}{\Delta x}-\frac{\Delta' y}{\Delta' x}\right|$ less than any assigned positive number for *all* values of Δx, $\Delta' x$ which lie in any neighbourhood including $x=0$. Portion of the graph of the function is shown in Fig. 36.

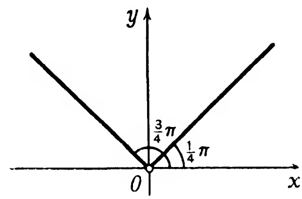


FIG. 36.
$y=|x|$.

77. Change of Independent Variable in a Derivative.

The process of *changing the independent variable of a function* has already been explained and illustrated in Art. 29. Thus, if $f(u)$ is a given function of a variable u, and if u is expressed as a function $\varphi(x)$, say, of a variable x, the independent variable u of the given function is changed to x when u is replaced by $\varphi(x)$. If we denote the given function by y, we have the equations $y=f(u)$, $u=\varphi(x)$, and the symbolic expression for y as a function of the final variable x is

$$y=f\{\varphi(x)\}. \quad\text{..............................(1)}$$

The variable u then takes the rôle of intermediary variable, and we say that y is given *as a function of a function of* x.

We now investigate a theorem for the derivative of such a function y, when the independent variable is changed from u to x, in terms of the derivative of y with respect to u and that of u with respect to x. The result established leads to a rule for the derivative of a function of a function which is constantly applied throughout the Calculus. As a particular case, we shew below how the rule supplies the final step in the generalization of the rule for the derivative of a power of a differentiable function when the index is any real number.

Since y, u are supposed differentiable functions of their respective arguments u, x, they are *continuous* functions of these arguments, and hence, by Art. 30, y is a continuous function of x when expressed in the form (1). We now shew that y is a differentiable function of x and that its derivative with respect to x is given by

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}. \quad\text{..............................(2)}$$

Let y_1, u_1 be the values of y, u corresponding to the value x_1 of x, and let Δy, Δu, Δx be corresponding increments from these values.

Then the functions $\Delta y/\Delta u$, $\Delta u/\Delta x$, $\Delta y/\Delta x$ are continuous functions of u, x, x, respectively, in neighbourhoods of the respective values u_1, x_1, x_1, but at these points they are undefined.

Now, provided $x \neq x_1$, $u \neq u_1$, that is, $\Delta x \neq 0$, $\Delta u \neq 0$, we have, by Algebra,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}. \quad \text{.............................(3)}$$

Since y has a definite derivative with respect to u at u_1, $\Delta y/\Delta u$ is a potentially continuous function of u at u_1, and hence of x at x_1. For a similar reason, the function $\Delta u/\Delta x$ is a potentially continuous function of x at x_1, and if we denote the completed functions by $(\Delta y/\Delta u)^*$, $(\Delta u/\Delta x)^*$, which are such that $(\Delta y/\Delta u)^*_{x_1} = (dy/du)_{x_1}$, and $(\Delta u/\Delta x)^*_{x_1} = (du/dx)_{x_1}$, we may write, in place of (3),

$$\frac{\Delta y}{\Delta x} = \left(\frac{\Delta y}{\Delta u}\right)^* \left(\frac{\Delta u}{\Delta x}\right)^*, \quad \text{...........................(4)}$$

where, as before, $\Delta u \neq 0$, $\Delta x \neq 0$.

This result is also true if Δu (and therefore also Δy) is zero, provided that Δx is not zero, for we then have $(\Delta u/\Delta x)^* = 0$, and both sides of the equation (4) vanish.[1]

Now the function on the right-hand side of (4) is continuous at x_1 and hence the function $\Delta y/\Delta x$ is potentially continuous there, its completing value being that of the right-hand side at x_1. There is thus a derivative of y with respect to x at x_1 given by

$$\left(\frac{dy}{dx}\right)_{x_1} = \left(\frac{dy}{du}\right)_{x_1} \left(\frac{du}{dx}\right)_{x_1}. \quad \text{..........................(5)}$$

Dropping the suffix, we can thus assert that if the relation between y and x is expressed in the form $y = f\{\varphi(x)\}$, or in the equivalent form $y = f(u)$ where $u = \varphi(x)$, the derivative of y with respect to x at any point of the range is given by the equation

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}. \quad \text{..............................(6)}$$

If we denote $f\{\varphi(x)\}$ by $\chi(x)$, (6) can be written in the form

$$\chi'(x) = f'(u)\varphi'(x), \quad \text{............................(7)}$$

where the accents denote differentiations with respect to the arguments indicated.

[1] The possible occurrence of $\Delta u = 0$ when $\Delta x \neq 0$ is the critical point in a rigorous proof. Although u is a continuous function of x, the algebraic definition of continuity allows u to be equal to u_1, not only at x_1, but at $x_1 + \Delta x$, where Δx is as small as we please; see Ex. 3 below.

The theorem may be conveniently stated in the form of a rule.

Rule for the derivative of a function of a function. If a function y is expressed as a function $f(u)$ of a function u of x, and if the derivative of each of these functions with respect to its argument is known, then the derivative of y with respect to x is known, being the product of the derivatives of the two functions.

The result (6) can evidently be extended to cases where a further change of independent variable is made, that is, where y is expressed as a function of a function of a function of x. The symbolic expression of y is then of the form

$$y=f[\varphi\{\psi(x)\}], \quad\text{.................................(8)}$$

or its equivalent

$$y=f(u) \text{ where } u=\varphi(v) \text{ and } v=\psi(x), \quad\text{................(9)}$$

and we have

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}. \quad\text{..............................(10)}$$

There are corresponding results when more than three functions are involved.

77. 1. If we *start* with a given function, $F(x)$ say, of x, which is denoted by y, the restrictions under which we can introduce

(i) a single intermediary variable u, given by $u=\varphi(x)$, and write $y=f(u)$, or

(ii) two intermediary variables u, v, given by $u=\varphi(v)$, $v=\psi(x)$, and write $y=f\{\varphi(v)\}$, or $y=f(u)$, and so on,

are of a very general kind. They are discussed in the theory of *inversion* of functions. Such considerations, however, do not arise at this stage, the practical application of the rule being here to cases where, as in the Examples below, a suitable form of expression for y is found at once *by inspection*. The student should follow out and compare the different suitable forms which suggest themselves.

77. 2. *Application to the differentiation of u^m.* We suppose that m is any real number and that u is a positive differentiable function of x. According to the 'index rule' of Art. 75. 4, we have

$$\frac{du^m}{du}=mu^{m-1}. \quad\text{..............................(11)}$$

The rule for the derivative of a function of a function now gives

$$\frac{du^m}{dx}=\frac{du^m}{du}\frac{du}{dx}=mu^{m-1}\frac{du}{dx}. \quad\text{.....................(12)}$$

This is the generalization of the results (3), (6) of Art. 72 to the case in which m is any real number.

In the case of rational indices of odd denominator, u may take negative values and its derivative is then given by (12). There may also be either ordinary or one-sided derivatives when $u=0$; compare Arts. 75. 5, 76.

Ex. 1. Let $y=\sqrt{u}$ where $u=ax+b$. The expression for y as a function of x is

$$y=\sqrt{(ax+b)}, \quad\text{......(13)}$$

provided $ax+b\geqslant 0$. If $u>0$, we have

$$dy/du=1/(2\sqrt{u}), \quad du/dx=a, \quad\text{......(14)}$$

and the rule (6) gives

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\frac{1}{2\sqrt{u}}a=\frac{a}{2\sqrt{(ax+b)}}. \quad\text{......(15)}$$

When $x=-b/a$ the function (13) has a positive infinite upper derivative if a is positive and a negative infinite lower derivative if a is negative.

The following particular cases of the above should be noted:

$$\frac{d\sqrt{(1+x)}}{dx}=\frac{1}{2\sqrt{(1+x)}}, \quad \frac{d\sqrt{(1-x)}}{dx}=-\frac{1}{2\sqrt{(1-x)}}. \quad\text{......(16)}$$

Ex. 2. Let $y=\sqrt{u}$ where $u=ax^2+bx+c$. Here we have

$$y=\sqrt{(ax^2+bx+c)}, \quad\text{......(17)}$$

provided $ax^2+bx+c\geqslant 0$. Since $du/dx=2ax+b$, the rule (6) now gives

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\frac{1}{2\sqrt{u}}(2ax+b)=\frac{2ax+b}{2\sqrt{(ax^2+bx+c)}}. \quad\text{......(18)}$$

An important particular case is that in which $u=a^2-x^2$, so that

$$y=\sqrt{(a^2-x^2)}, \quad\text{......(19)}$$

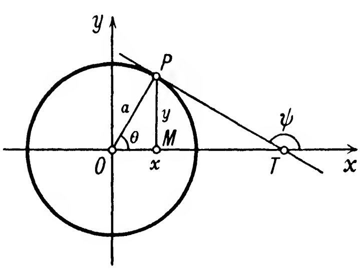


FIG. 37.

where $|x|\leqslant|a|$. We now have

$$\frac{dy}{dx}=-\frac{x}{\sqrt{(a^2-x^2)}}=-\frac{x}{y}. \quad\text{......(20)}$$

Interpreted geometrically, this gives the gradient at any point on the part of the circle $x^2+y^2=a^2$ for which y is positive. If P (Fig. 37) is the point (x, y), ψ the gradient at P and θ the angle MOP, we have

$$\tan\psi=-x/y=-\cot\theta=-\tan(\tfrac{1}{2}\pi-\theta), \quad\text{......(21)}$$

so that $\psi-\theta=\frac{1}{2}\pi$, as is evident from the Figure.

The student should verify that the formula $dy/dx=-x/y$ is also applicable to any point on the lower half of the circle, for which $y=-\sqrt{(a^2-x^2)}$.

For the ellipse $x^2/a^2+y^2/b^2=1$, the ordinate y is given by

$$y=\pm\frac{b}{a}\sqrt{(a^2-x^2)}, \quad\text{......(22)}$$

and with the aid of the above results we obtain the formula

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}. \qquad\qquad (23)$$

Ex. 3. Let $y=f(u)$ and suppose that u is a function of x which has a constant value, c say, for some range of x. Then if $f(u)$ is defined for the value c of u, y becomes a (constant) function of x in the range indicated and is expressed by

$$y=f(c)=C, \text{ say}. \qquad\qquad (24)$$

Since $du/dx=0$ in this range and dy/du is finite, by assumption, the rule (6) gives

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\frac{dy}{du}\cdot 0=0. \qquad\qquad (25)$$

This result, of course, follows from (24) by differentiation. The Example is introduced merely to illustrate some of the remarks on p. 146.

Ex. 4. Let $y=\sqrt{u}$ where $u=\dfrac{1+v}{1-v}$ and $v=\sqrt{x}$. As in Ex. 2, p. 44, the expression for y as a function of x is

$$y=\sqrt{\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)}, \qquad\qquad (26)$$

provided x lies in the range (0, 1]. We have

$$\frac{dy}{du}=\frac{1}{2\sqrt{u}},\quad \frac{du}{dv}=\frac{2}{(1-v)^2},\quad \frac{dv}{dx}=\frac{1}{2\sqrt{x}}, \qquad\qquad (27)$$

and (10) here gives

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}=\frac{1}{2\sqrt{u}}\frac{2}{(1-v)^2}\frac{1}{2\sqrt{x}}=\frac{1}{2\sqrt{x}\,.\,(1+\sqrt{x})^{1/2}(1-\sqrt{x})^{3/2}}. \qquad (28)$$

Ex. 5. Let $y=\sqrt{|x|}$, so that $y=\sqrt{x}$ when $x>0$ and $y=\sqrt{(-x)}$ when $x<0$. We have

$$\frac{d\sqrt{x}}{dx}=\frac{1}{2\sqrt{x}},\quad \frac{d\sqrt{(-x)}}{dx}=\frac{d\sqrt{(-x)}}{d(-x)}\frac{d(-x)}{dx}=\frac{1}{2\sqrt{(-x)}}(-1), \qquad (29)$$

so that

$$\frac{d\sqrt{|x|}}{dx}=\pm\frac{1}{2\sqrt{|x|}}, \qquad (30)$$

the positive (negative) sign being taken if x is positive (negative). The derivative is discontinuous when $x=0$. There is a positive infinite upper derivative and a negative infinite lower derivative there. This is illustrated in Fig. 38, which shews portion of the graph of the function.

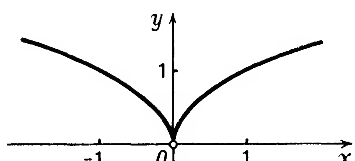


FIG. 38.
$y=\sqrt{|x|}$.

Ex. 6. Let $y=\{1+\sqrt{(1+x^2)}\}^{1/2}$. Introducing two intermediary variables u, v, let $y=\sqrt{u}$, $u=1+\sqrt{v}$, $v=1+x^2$. The rule (10) now leads to the derivative

$$\frac{dy}{dx}=\frac{x}{2\{1+\sqrt{(1+x^2)}\}^{1/2}\sqrt{(1+x^2)}}. \qquad\qquad (31)$$

Ex. 7. Let $y=\dfrac{1}{\sqrt{x}+\sqrt{(x+1)}}$. Rationalizing the denominator by multiplying numerator and denominator by the non-zero term $\sqrt{x}-\sqrt{(x+1)}$, we get the transform

$$y=\sqrt{(x+1)}-\sqrt{x}, \quad\text{.............................(32)}$$

the derivative of which can be written down, namely

$$\frac{dy}{dx}=\frac{1}{2\sqrt{(x+1)}}-\frac{1}{2\sqrt{x}}. \quad\text{.............................(33)}$$

Ex. 8. One side of a triangle is increasing at a uniform rate v units of length per unit time. Find the rate of increase of the area of the triangle at any instant.

Denote the lengths of the sides at any instant by a, b, c, and take a as the variable. The formula

$$S=\sqrt{\{s(s-a)(s-b)(s-c)\}}=\tfrac{1}{4}\sqrt{(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)}, \quad\text{...(34)}$$

where $s=\frac{1}{2}(a+b+c)$, gives the area in terms of the sides. The rate of variation of S with respect to the time t is given by the rule (6), which here becomes $\dfrac{dS}{dt}=\dfrac{dS}{da}\dfrac{da}{dt}$.

Writing (34) in the form $S=\frac{1}{4}\sqrt{\alpha}$, we have

$$\frac{dS}{da}=\frac{dS}{d\alpha}\frac{d\alpha}{da}=\frac{1}{8\sqrt{\alpha}}(4c^2a+4ab^2-4a^3)=\frac{a}{8S}(b^2+c^2-a^2). \quad\text{...........(35)}$$

Now $\qquad S=\frac{1}{2}bc\sin A \quad$ and $\quad b^2+c^2-a^2=2bc\cos A, \quad$ ....(36)

where A is the angle opposite the side a. Therefore $dS/da=\frac{1}{2}a\cot A$, and, writing v for da/dt, we have

$$\frac{dS}{dt}=\tfrac{1}{2}va\cot A. \quad\text{................................(37)}$$

78. Differentiation of Inverse Functions. Rule for the Interchange of Dependent and Independent Variables in a Derivative.

The question of the inversion of a functional relation of the form

$$y=f(x), \quad\text{..................................(1)}$$

expressing y as a continuous function of x, to define x as a function of y, has already been discussed in Arts. 36, 37. It is there shewn that by suitable restrictions as to range, so that y is a strictly one-way function of x, the inverse function x is a continuous, single-valued, strictly one-way function of y. We now shew that if y is a differentiable function of x, the inverse function x is a differentiable function of y, the derivatives being connected by the relation

$$\frac{dx}{dy}=\frac{1}{dy/dx}. \quad\text{..............................(2)}$$

Suppose that x_1 is a point in a range within which the inversion has been made, so that, by assumption, there is a definite neighbourhood of x_1 within which y is strictly one-way. Then if $x_1+\Delta x$ is in

this neighbourhood, the corresponding increment Δy of y is different from zero, and we have, by Algebra,

$$\frac{\Delta x}{\Delta y}=\frac{1}{\Delta y/\Delta x}. \qquad\qquad (3)$$

Now $\Delta x/\Delta y$ is a continuous function of y in the neighbourhood referred to, but at the point y_1 it takes on the indeterminate form $0/0$. Equation (3) shews, however, that it is potentially continuous there, for $\Delta y/\Delta x$ is a non-vanishing, potentially continuous function there, and hence also its reciprocal.[1] The completing value of $\dfrac{1}{\Delta y/\Delta x}$ is simply $\dfrac{1}{(dy/dx)_{x_1}}$, and this is therefore the value of the derivative sought. Thus we have

$$\left(\frac{dx}{dy}\right)_{y_1}=\frac{1}{(dy/dx)_{x_1}}, \qquad\qquad (4)$$

or, dropping the suffixes,

$$\frac{dx}{dy}=\frac{1}{dy/dx}. \qquad\qquad (5)$$

Thus the derivative of x with respect to y is the reciprocal of the derivative of y with respect to x. This result is quoted as *the rule for the interchange of the dependent and independent variables in a derivative.*

The proof does not apply if dy/dx vanishes at the point concerned, for then $\dfrac{1}{dy/dx}$ is infinite and such a value cannot be taken as the completing value of a function. However, it is convenient to regard the relation (5) as still holding in such a case, and to say that if dy/dx converges to zero as we approach the point in question, $|\,dx/dy\,|$ diverges to infinity there, and vice versa.

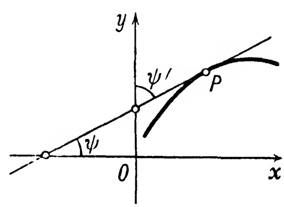


FIG. 39.

The result (5) has a simple geometrical interpretation. Referring to Fig. 39, the tangent at P has a slope ψ relative to Ox, this slope being measured according to the convention of Art. 70. We now define the slope and gradient of the curve relative to the axis Oy. This slope, which we denote by ψ', is the smallest angle of *clockwise* rotation required to bring the y-axis into parallelism with the tangent line, and therefore lies in the range $(0, \pi)$. The value of its tangent

[1] This follows since y is a differentiable function of x at x_1, provided $(dy/dx)_{x_1}\neq 0$.

is the gradient of the curve at the point considered relative to Oy, so that we now have

$$\tan\psi' = \frac{dx}{dy}. \qquad \text{.......(6)}$$

The relation (5) is therefore expressed by the equation

$$\tan\psi' = 1/\tan\psi. \qquad \text{.......(7)}$$

Now this is evident geometrically ; for the case illustrated in the Figure we have $\psi' = \frac{1}{2}\pi - \psi$, while if ψ lies in the range $(\frac{1}{2}\pi, \pi)$, so that ψ' also lies in this range, we have $\psi' = \frac{3}{2}\pi - \psi$. In either case,

$$\tan\psi' = \cot\psi = 1/\tan\psi. \qquad \text{.......(8)}$$

If the tangent line is horizontal, $\psi = 0$ and $\psi' = \frac{1}{2}\pi$. We then write $\tan\psi = 0$, $|\tan\psi'| = \infty$, consistently with the rule.

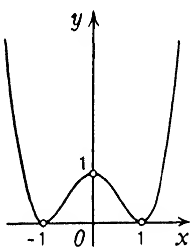


FIG. 40.
$y = (1 - x^2)^2$.

Ex. 1. *The function* $y = (1 - x^2)^2$. It is easy to shew that y is strictly down-way in the ranges $[-\infty, -1)$, $(0, 1)$, and strictly up-way in the ranges $(-1, 0)$, $(1, \infty]$ of x. The corresponding inverse functions are expressed by

$$\left.\begin{array}{ll} x = -\sqrt{(1+\sqrt{y})}, & x = \sqrt{(1-\sqrt{y})}, \\ x = -\sqrt{(1-\sqrt{y})}, & x = \sqrt{(1+\sqrt{y})}. \end{array}\right\} \qquad \text{.......(9)}$$

Since $dy/dx = -4x(1-x^2)$, we have, for any pair of values (x, y) which satisfy the given equation,

$$\frac{dx}{dy} = -\frac{1}{4x(1-x^2)}. \qquad \text{.......(10)}$$

The nature of the graph of the function y is indicated in Fig. 40.

79. Derivative when the Functional Relation is given Parametrically.

The formula for the derivative of a function of a function, Art. 77, can be transformed to a formula for the derivative of y with respect to x when x and y are defined as functions of an intermediary variable or parameter, t say, by equations of the forms

$$x = f(t), \quad y = \varphi(t). \qquad \text{.......(1)}$$

Here $f(t)$, $\varphi(t)$ are single-valued differentiable functions in the relevant range of t, and we suppose further that $f(t)$ is strictly one-way in the range, so that the relation $x = f(t)$ can be inverted to give $t = \arg f(x)$. The expression of y in terms of x then becomes

$$y = \varphi\{\arg f(x)\}. \qquad \text{.......(2)}$$

According to the rule of the preceding Article, if dx/dt does not vanish for the value of t considered, t has a derivative with respect to x given by

$$\frac{dt}{dx} = \frac{1}{dx/dt}. \qquad \text{.......(3)}$$

Also, the rule for the derivative of a function of a function gives

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}. \qquad (4)$$

It follows that for the case considered we have

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}, \qquad (5)$$

which is the result to be obtained. Stated as a rule, it asserts that *the derivative of y with respect to x is the ratio of the derivatives of y and x with respect to t.*

Ex. 1. *Motion of a particle in a plane.* If a particle is moving along a plane curve whose equation is $y=f(x)$, its coordinates (x, y) are each functions of the time t, and we have

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}, \qquad (6)$$

which expresses a well-known result of Kinematics.

In particular, if the curve is a straight line AB, of inclination φ to Ox, dy/dx has the constant value $\tan\varphi$, and therefore

$$\frac{dy/dt}{dx/dt}=\tan\varphi \qquad (7)$$

throughout the motion. If A is the point (x_1, y_1) and P, (x, y), is any point on the line, we have, on writing $AP=r$,

$$x-x_1=r\cos\varphi, \quad y-y_1=r\sin\varphi, \qquad (8)$$

and therefore

$$\frac{dx}{dt}=\frac{dr}{dt}\cos\varphi=v\cos\varphi, \quad \frac{dy}{dt}=\frac{dr}{dt}\sin\varphi=v\sin\varphi, \qquad (9)$$

where v is the velocity of the particle at P. These equations give, on squaring and adding,

$$v^2=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2. \qquad (10)$$

80. Connection between the Derivative and the Gradient in Oblique Coordinates.

Let the equation of the curve AB (Fig. 41) referred to the oblique axes Ox, Oy, of obliquity ω, be $y=f(x)$, and let ψ be the slope at P, (x, y), relative to Ox. The quantity $\tan\psi$ is still the gradient of the curve, but it differs from dy/dx. A connection between these quantities can be readily established. If we introduce

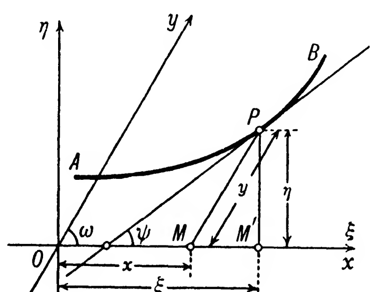


FIG. 41.

the rectangular axes $O\xi$, $O\eta$, as shewn, we have, for the coordinates ξ, η,

$$\xi = x + y\cos\omega, \quad \eta = y\sin\omega, \quad \text{.....................(1)}$$

which express ξ, η in terms of x, y, and hence, on account of the relation $y = f(x)$, in terms of x. Therefore

$$\tan\psi = \frac{d\eta}{d\xi} = \frac{d\eta/dx}{d\xi/dx} = \frac{\sin\omega\,\dfrac{dy}{dx}}{1+\cos\omega\,\dfrac{dy}{dx}}, \quad \text{...................(2)}$$

so that

$$\frac{dy}{dx} = \frac{\tan\psi}{\sin\omega - \cos\omega\tan\psi} \quad \text{...........................(3)}$$

$$= \frac{\sin\psi}{\sin(\omega-\psi)}. \quad \text{....................................(4)}$$

The relation (3) shews that the gradient is zero when $dy/dx = 0$, as in the case of rectangular coordinates.

The sine and cosine of the slope ψ follow from (2). We have

$$\sin\psi = \frac{|\tan\psi|}{\sqrt{(1+\tan^2\psi)}} = \left(\operatorname{sig}\frac{dy}{dx}\right)\frac{\sin\omega\,\dfrac{dy}{dx}}{\sqrt{\left\{1+2\cos\omega\,\dfrac{dy}{dx}+\left(\dfrac{dy}{dx}\right)^2\right\}}}, \quad \text{....(5)}$$

$$\cos\psi = \frac{\sin\psi}{\tan\psi} = \left(\operatorname{sig}\frac{dy}{dx}\right)\frac{1+\cos\omega\,\dfrac{dy}{dx}}{\sqrt{\left\{1+2\cos\omega\,\dfrac{dy}{dx}+\left(\dfrac{dy}{dx}\right)^2\right\}}}, \quad \text{..............(6)}$$

where $\left(\operatorname{sig}\dfrac{dy}{dx}\right)$ means the algebraic sign of dy/dx.

80. 1. *Working notions.* The formulae for the gradients in rectangular and oblique coordinates are easily recovered in the following intuitive manner.

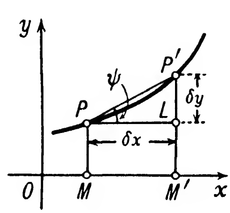


FIG. 42.

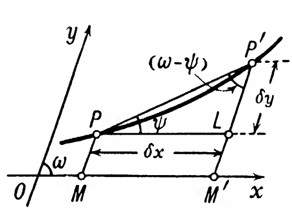


FIG. 43.

Referring to Figs. 42, 43, we take two neighbouring points P, P' on either curve, whose coordinates differ by (small) amounts[1] δx, δy. The inclination

[1] The use of δ when the increment is *small* has been referred to on p. 96.

of the chord PP' to Ox is denoted by ψ, and we have, from the triangle PLP' in Fig. 42, the relation $\delta y/\delta x = \tan\psi$, and from the corresponding triangle in Fig. 43, the relation $\dfrac{\delta y}{\sin\psi} = \dfrac{\delta x}{\sin(\omega-\psi)}$. From the latter we get

$$\frac{\delta y}{\delta x} = \frac{\sin\psi}{\sin(\omega-\psi)} = \frac{1}{\sin\omega\cot\psi - \cos\omega}, \quad \text{.....................(7)}$$

and hence

$$\tan\psi = \frac{\sin\omega\,\dfrac{\delta y}{\delta x}}{1+\cos\omega\,\dfrac{\delta y}{\delta x}}. \quad \text{...(8)}$$

If we now think of ψ as the slope *of the tangent at* P, the above relations are approximately true, and lead to the results already established when we allow P' to converge to P. While thinking of the approximate relations, we write down the accurate results

$$\frac{dy}{dx} = \tan\psi, \qquad \frac{\sin\omega\,\dfrac{dy}{dx}}{1+\cos\omega\,\dfrac{dy}{dx}} = \tan\psi, \quad \text{.....................(9)}$$

corresponding to the two types of coordinates.

The above process of recovering the formulae is described as the *use of infinitesimals*, the term indicating that all the differences employed are to be regarded as indefinitely small. We shall make frequent use of this method in the sequel.

Ex. 1. From a point P, (x, y), on a curve $y=f(x)$ referred to oblique axes Ox, Oy, of obliquity ω, the perpendicular PN is drawn to the tangent to the curve through a fixed point P_1, (x_1, y_1), on it. It is required to find the rate of increase of P_1N with respect to x as P moves along the curve.

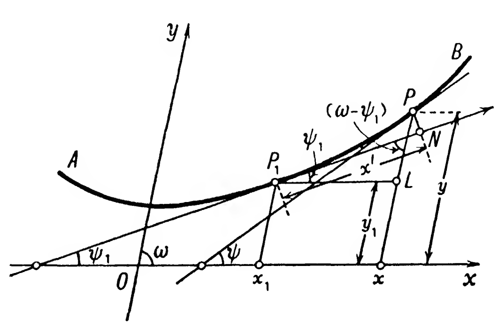


FIG. 44.

Let AB (Fig. 44) be the graph of the function $f(x)$ and suppose that at P_1 the gradient $\tan\psi_1$ is positive. Let the positive sense for the measurement of P_1N, (which we denote by x'), be taken towards the right, as indicated, and draw P_1L parallel to Ox. Since the sum of the projections of P_1L, LP on P_1N reproduces P_1N, we have, on observing that the (acute) angle between LP and P_1N is $\omega-\psi_1$,

$$x' = (x-x_1)\cos\psi_1 + (y-y_1)\cos(\omega-\psi_1). \quad \text{..................(10)}$$

Hence, differentiating and using (4), we get

$$\frac{dx'}{dx}=\cos\psi_1+\frac{dy}{dx}\cos(\omega-\psi_1)$$

$$=\frac{dy}{dx}\left\{\cos\psi_1\frac{\sin(\omega-\psi)}{\sin\psi}+\cos(\omega-\psi_1)\right\}$$

$$=\frac{dy}{dx}\frac{\sin\omega\cos(\psi-\psi_1)}{\sin\psi}$$

$$=\sqrt{\left\{1+2\cos\omega\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^2\right\}}\cos(\psi-\psi_1), \quad \text{.............(11)}$$

where the transformation of the last line follows from (5), observing that for the case illustrated dy/dx is positive.

When P coincides with P_1, the above result takes the form

$$\left(\frac{dx'}{dx}\right)_{x_1}=\sqrt{\left\{1+2\cos\omega\left(\frac{dy}{dx}\right)_{x_1}+\left(\frac{dy}{dx}\right)_{x_1}^{2}\right\}}, \quad \text{.................(12)}$$

which is employed subsequently in Chap. V, Art. 137.

The student should verify that the results (11), (12) are also valid when $\tan\psi_1<0$, that is, when ψ_1 lies in the range $(\frac{1}{2}\pi, \pi)$, the positive sense for the measurement of x' being again towards the right.

81. Implicit Functions. The Process of 'Differentiating an Equation'.

The question of the definition of a function y of x by means of a relation of the form

$$f(x, y)=0 \quad \text{.................................(1)}$$

between the two variables has already been introduced in Art. 44. The general question of the differentiability of such a function y must, however, be postponed, as it belongs to the theory of dimetric functions. We merely state here that when the existence of a differentiable function y is assured, the derivative dy/dx can be found without expressing y explicitly in terms of x, (a procedure which may indeed be quite impracticable), and a general formula for it can be stated in terms of *partial derivatives*; compare the following Article. The principle, however, by which the derivative dy/dx is obtained is sufficiently illustrated in the following examples in which the general theory is not required. The formulae obtained for the derivatives will, in general, involve both x and y, and it may be impracticable to express them in terms of x alone.

To consider a simple example, take

$$x^2+y^2-a^2=0 \quad \text{...........................(2)}$$

as the relation between x and y, (a constant). If x lies in the range $(-a, a)$, the existence of two differentiable functions, given by

$$y=\sqrt{(a^2-x^2)} \quad \text{and} \quad y=-\sqrt{(a^2-x^2)}, \quad \text{..............(3)}$$

is assured and, choosing either of them, the left-hand side of equation (2) becomes a differentiable function of x. But according to this relation, this function is actually zero throughout the range and therefore its derivative with respect to x is also zero in the range. The derivatives of the successive terms of the function are given by $dx^2/dx=2x$, $dy^2/dx=2y\,dy/dx$, $da^2/dx=0$, and we therefore have the result

$$2x+2y\frac{dy}{dx}=0, \quad\text{.......................(4)}$$

so that

$$\frac{dy}{dx}=-\frac{x}{y}. \quad\text{.......................(5)}$$

This is in agreement with the result (20) of Ex. 2, Art. 77.

If the equation connecting x and y is written in the equivalent form

$$x^2=a^2-y^2, \quad\text{.......................(6)}$$

both sides are differentiable functions of x, and, being always equal to each other in the range, their derivatives with respect to x are also equal in the range. The result obtained by equating these derivatives is plainly equivalent to that obtained by taking all the terms of the equation (6) to the left-hand side. The same remarks apply when (2) is written in any of its other equivalent forms.

The above process is clearly applicable to any equation of the form (1) as soon as we know how to express the derivative of $f(x, y)$ with respect to x in terms of x, y and dy/dx. It is described as the process of *differentiating an equation* and will be freely used in the sequel.

Ex. 1. *The equation* $y^2=ax^3$. When ax is positive, this equation defines two differentiable functions of x, viz. $y=\pm\sqrt{(ax^3)}$. Their derivatives can be written down with the aid of the 'index rule' of Art. 75. 4, and are given by $dy/dx=\pm\frac{3}{2}\sqrt{(ax)}$.

As an illustration of the process of the present Article, differentiating the given equation leads to the relation $dy^2/dx=d(ax^3)/dx$, that is,

$$2y\frac{dy}{dx}=3ax^2, \quad\text{..................(7)}$$

whence

$$\frac{dy}{dx}=\frac{3ax^2}{2y}, \quad\text{..................(8)}$$

which is easily shewn to include the above values for dy/dx.

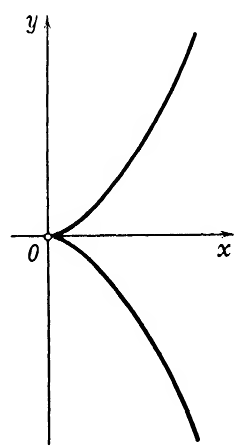


FIG. 45.
$y^2=ax^3$.

The graph of the given equation is called the *semi-cubical parabola*, and is shown in Fig. 45, corresponding to a positive value of a.

Ex. 2. *Derivative of* $x^{1/q}$. If $y=x^{1/q}$, $(x>0)$, we have $y^q-x=0$. Differentiating the last equation, we get

$$qy^{q-1}\frac{dy}{dx}-1=0, \quad\text{.......................................(9)}$$

or

$$\frac{dy}{dx}=\frac{1}{q}y^{-q+1}=\frac{1}{q}x^{\frac{1}{q}-1}. \quad\text{.................................(10)}$$

In a similar manner the derivative of $x^{p/q}$ is found by differentiating the equation $y^q-x^p=0$.

Ex. 3. *The equation* $\quad x^3+y^3=3axy, \quad (a>0).$ ....(11)

This defines a single differentiable function when $x<0$, three distinct differentiable functions when $0<x<4^{1/3}a$, and a single differentiable function when $x>4^{1/3}a$. The explicit forms for y are, however, unwieldy[1], and in order to find the derivative dy/dx at any point we apply the above process of differentiating the equation as it stands with respect to x. Whichever the single-valued function y under consideration, each side of (11) is a differentiable function of x in the corresponding range, and we thus have

$$\frac{dx^3}{dx}+\frac{dy^3}{dx}=3a\frac{d(xy)}{dx}, \quad\text{.................................(12)}$$

which gives

$$3x^2+3y^2\frac{dy}{dx}=3a\left(y+x\frac{dy}{dx}\right), \quad\text{..........................(13)}$$

whence

$$\frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}, \ (x,\ y\neq 0), \quad\text{....................(14)}$$

which expresses the derivative dy/dx in terms of a, x, y.

The constant a can, if desired, be eliminated between this result and (11), for we have $a=(x^3+y^3)/(3xy)$; the result obtained is

$$\frac{dy}{dx}=\frac{y(y^3-2x^3)}{x(2y^3-x^3)}. \quad\text{.........(15)}$$

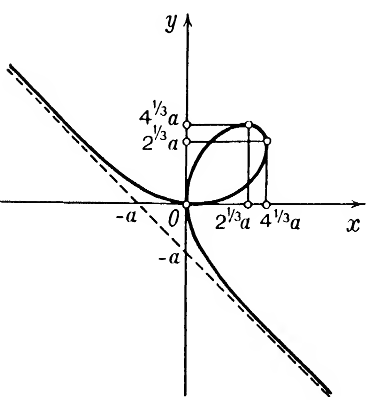


FIG. 46.
$x^3+y^3=3axy$.

The graph of the equation (11) is called the *Folium of Descartes*. Portion of it is shown[2] in Fig. 46, corresponding to a positive value of a. We observe from (14) that $dy/dx=0$ when $ay=x^2$, and that $|dy/dx|=\infty$ when $y^2=ax$. Thus the tangent is horizontal where $x^3+x^6/a^3=3x^3$, that is, where $x=2^{1/3}a$, $y=4^{1/3}a$, and is vertical where $y^6/a^3+y^3=3y^3$, that is, where $x=4^{1/3}a$, $y=2^{1/3}a$. Near the origin the curve approximates to the form of two parabolas $3ay=x^2$, $3ax=y^2$, touching the axes Ox, Oy, respectively.

[1] For example, the real explicit form for y corresponding to values of x which lie outside the range $(0,\ 4^{1/3}a)$ is

$$y=[-\tfrac{1}{2}x^3+\sqrt{\{\tfrac{1}{4}x^3(x^3-4a^3)\}}]^{1/3}+[-\tfrac{1}{2}x^3-\sqrt{\{\tfrac{1}{4}x^3(x^3-4a^3)\}}]^{1/3}.$$

[2] The line $x+y+a=0$, represented in the Fig. by the broken line, is an *asymptote* to the curve.

Ex. 4. *The equation* $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, $(a>0)$. If $|x|\leqslant a$, this equation defines two differentiable functions y of x, one positive and whose range of values is $(0, a)$, the other negative and whose range of values is $(-a, 0)$. For either function we have $dx^{\frac{2}{3}}/dx+dy^{\frac{2}{3}}/dx=0$, that is,

$$\tfrac{2}{3}x^{-\frac{1}{3}}+\tfrac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx}=0, \quad \ldots\ldots\ldots(16)$$

so that

$$\frac{dy}{dx}=-\left(\frac{y}{x}\right)^{\frac{1}{3}}. \quad \ldots\ldots\ldots\ldots(17)$$

The graph of the given equation is called the *astroid* and is shewn in Fig. 47.

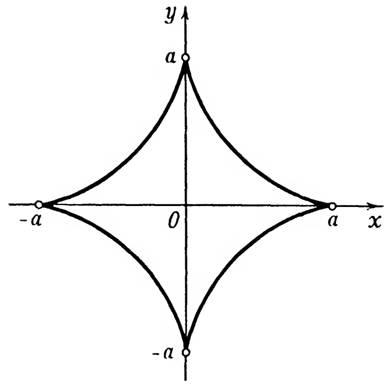


FIG. 47.
$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

Ex. 5. *The equation* $\sum\limits_{r,s=1}^{m,n} A_{rs}x^r y^s=0$. Here m, n are positive integers and the function on the left-hand side of the equation is thus a rational integral function of x and y. On equating to zero its derivative with respect to x, we get

$$\sum_{r,s=1}^{m,n} A_{rs}\left(rx^{r-1}y^s+sx^r y^{s-1}\frac{dy}{dx}\right)=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(18)$$

giving

$$\frac{dy}{dx}=-\frac{\sum\limits_{r,s=1}^{m,n} A_{rs}rx^{r-1}y^s}{\sum\limits_{r,s=1}^{m,n} A_{rs}sx^r y^{s-1}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19)$$

Ex. 6. *The equation* $\sum\limits_{r,s=1}^{m,n} A_{rs}f_r(x)\varphi_s(y)=0$. This is a generalization of the previous Example to the case where the variable factors in the terms forming the sum are any differentiable functions of their respective arguments. We now find

$$\sum_{r,s=1}^{m,n} A_{rs}\left\{\frac{df_r(x)}{dx}\varphi_s(y)+f_r(x)\frac{d\varphi_s(y)}{dy}\frac{dy}{dx}\right\}=0, \quad \ldots\ldots\ldots\ldots(20)$$

and hence

$$\frac{dy}{dx}=-\frac{\sum\limits_{r,s=1}^{m,n} A_{rs}\dfrac{df_r(x)}{dx}\varphi_s(y)}{\sum\limits_{r,s=1}^{m,n} A_{rs}f_r(x)\dfrac{d\varphi_s(y)}{dy}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(21)$$

82. Introduction of Partial Derivatives.

In the preceding Example we have seen that if y is a given function of x, the derivative of the function

$$\sum_{r,s=1}^{m,n} A_{rs}\,f_r(x)\varphi_s(y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

with respect to x is

$$\sum_{r,s=1}^{m,n} A_{rs} \frac{df_r(x)}{dx} \varphi_s(y) + \sum_{r,s=1}^{m,n} A_{rs} f_r(x) \frac{d\varphi_s(y)}{dy} \frac{dy}{dx}. \quad \text{..........(2)}$$

Now $\sum_{r,s=1}^{m,n} A_{rs} \frac{df_r(x)}{dx} \varphi_s(y)$ is the expression we should get for the derivative of (1) with respect to x if we ignored the fact that y is a function of x and treated it, and therefore the functions $\varphi_s(y)$, as constant. In the same way, $\sum_{r,s=1}^{m,n} A_{rs} f_r(x) \frac{d\varphi_s(y)}{dy}$ is the expression we should get for the derivative of (1) with respect to y if we ignored the fact that x and y are related and treated x, and therefore the functions $f_r(x)$, as constant.

A special nomenclature and symbolism are used for the two derivatives obtained according to the conventions mentioned. They are called *partial derivatives* with respect to x and y, respectively, and are indicated by replacing the usual form of the italic d by the 'round' form ∂ in the derivative symbol. Thus

$$\frac{\partial}{\partial x}\left\{\sum_{r,s=1}^{m,n} A_{rs} f_r(x)\varphi_s(y)\right\} = \sum_{r,s=1}^{m,n} A_{rs} \frac{df_r(x)}{dx} \varphi_s(y), \quad \text{..........(3)}$$

and

$$\frac{\partial}{\partial y}\left\{\sum_{r,s=1}^{m,n} A_{rs} f_r(x)\varphi_s(y)\right\} = \sum_{r,s=1}^{m,n} A_{rs} f_r(x) \frac{d\varphi_s(y)}{dy}. \quad \text{..........(4)}$$

The derivative of the function (1) with respect to x, where y is a function of x, can now be written in the form

$$\frac{\partial}{\partial x}\left\{\sum_{r,s=1}^{m,n} A_{rs} f_r(x)\varphi_s(y)\right\} + \frac{\partial}{\partial y}\left\{\sum_{r,s=1}^{m,n} A_{rs} f_r(x)\varphi_s(y)\right\} \frac{dy}{dx}. \quad \text{.....(5)}$$

The process of differentiating the equation

$$\sum_{r,s=1}^{m,n} A_{rs} f_r(x)\varphi_s(y) = 0 \quad \text{.........................(6)}$$

with respect to x thus leads to the equation

$$\frac{\partial}{\partial x}\left\{\sum_{r,s=1}^{m,n} A_{rs} f_r(x)\varphi_s(y)\right\} + \frac{\partial}{\partial y}\left\{\sum_{r,s=1}^{m,n} A_{rs} f_r(x)\varphi_s(y)\right\} \frac{dy}{dx} = 0, \quad \text{...(7)}$$

and the value of dy/dx is given by

$$\frac{dy}{dx} = -\frac{\dfrac{\partial}{\partial x}\left\{\sum_{r,s=1}^{m,n} A_{rs} f_r(x)\varphi_s(y)\right\}}{\dfrac{\partial}{\partial y}\left\{\sum_{r,s=1}^{m,n} A_{rs} f_r(x)\varphi_s(y)\right\}}. \quad \text{.................(8)}$$

With the aid of the general theory of dimetric functions, it can be shewn that when, in any manner, y is defined as a differentiable function of x, the derivative of the function $f(x, y)$ with respect to x is

$$\frac{\partial f(x, y)}{\partial x}+\frac{\partial f(x, y)}{\partial y}\frac{dy}{dx}; \qquad \text{.........................(9)}$$

in particular, the result of differentiating the equation $f(x, y)=0$ is

$$\frac{\partial f(x, y)}{\partial x}+\frac{\partial f(x, y)}{\partial y}\frac{dy}{dx}=0, \qquad \text{......................(10)}$$

which leads to the result

$$\frac{dy}{dx}=-\frac{\partial f(x, y)/\partial x}{\partial f(x, y)/\partial y}, \qquad \text{.........................(11)}$$

exactly as in the restricted case just considered.

The student should verify this rule for the other Examples above, first writing the given equations in the form $f(x, y)=0$.

83. Simple Geometrical Results associated with the Derivative.

We now investigate some simple properties of graphs of functions, the coordinate system being rectangular. The association of these properties with the derivative depends on the connection between the gradient of the curve at any point and the derivative, viz. $\tan\psi = dy/dx$.

83. 1. *The equation of the tangent.* Let P, (x, y), be any point on the curve $y=f(x)$. The tangent at P is the line through (x, y) whose gradient is equal to dy/dx. If Q, (ξ, η), is a point on this tangent line, the gradient is also equal to $(\eta-y)/(\xi-x)$, and on equating this to dy/dx and cross-multiplying, we obtain

$$\eta-y=(\xi-x)\frac{dy}{dx} \qquad \text{..............................(1)}$$

$$=(\xi-x)f'(x). \qquad \text{............................(2)}$$

The same relation is obtained whatever the position of Q on the tangent, and, treating (1) or (2) as an equation in the variables (ξ, η), it is the equation of the tangent at P. The coordinates (ξ, η) are usually described as *current coordinates along the line.* The letters (X, Y) are sometimes used for the same purpose.

If the curve is specified by equations of the form $x=f(t)$, $y=\varphi(t)$,

where t is an auxiliary independent variable (parameter), we have, as in Art. 79,

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\varphi'(t)}{f'(t)}, \quad (f'(t)\neq 0), \quad \dots\dots\dots\dots\dots\dots(3)$$

and the equation of the tangent then takes the more symmetrical form

$$\frac{\eta-\varphi(t)}{\varphi'(t)}=\frac{\xi-f(t)}{f'(t)}, \quad (\varphi'(t)\neq 0). \quad \dots\dots\dots\dots\dots\dots(4)$$

83. 2. *The equation of the normal.* The normal at P is the line through P perpendicular to the tangent there. The product of its gradient by that of the tangent line is therefore[1] -1, and, employing (ξ, η) as current coordinates on the normal, its equation becomes

$$\eta-y=-\frac{1}{dy/dx}(\xi-x), \quad \dots\dots\dots\dots\dots\dots(5)$$

that is,

$$\xi-x+(\eta-y)\frac{dy}{dx}=0. \quad \dots\dots\dots\dots\dots\dots(6)$$

Ex. 1. *The rectangular hyperbola $xy=c^2$, (c constant).* On differentiating the equation with respect to x, we have

$$x\frac{dy}{dx}+y=0, \quad \dots\dots\dots\dots\dots\dots(7)$$

so that $dy/dx=-y/x$. The equation of the tangent at (x, y) is thus

$$\eta-y=(\xi-x)(-y/x), \quad \dots\dots\dots\dots\dots\dots(8)$$

or

$$\xi y+\eta x=2c^2. \quad \dots\dots\dots\dots\dots\dots(9)$$

The equation of the normal is found to be

$$\xi x-\eta y=x^2-y^2. \quad \dots\dots\dots\dots\dots\dots(10)$$

Ex. 2. *The ellipse $x^2/a^2+y^2/b^2=1$.* On differentiating the equation, or proceeding as in Ex. 2, Art. 77, we find $\dfrac{dy}{dx}=-\dfrac{b^2x}{a^2y}$, and the equation of the tangent is therefore

$$\eta-y=(\xi-x)\left(-\frac{b^2x}{a^2y}\right), \quad \dots\dots\dots\dots\dots\dots(11)$$

that is,

$$\frac{\xi x}{a^2}+\frac{\eta y}{b^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \quad \dots\dots\dots\dots\dots\dots(12)$$

For the normal we find

$$\frac{\xi-x}{b^2x}=\frac{\eta-y}{a^2y}. \quad \dots\dots\dots\dots\dots\dots(13)$$

83. 3. *Intercepts on the axes made by the tangent.* Let the tangent

[1] If ψ, φ are the slopes of the tangent and normal to Ox, we have $\varphi=\psi\pm\frac{1}{2}\pi$ according as $\varphi\gtrless\psi$, and therefore $\tan\varphi=-\cot\psi=-1/\tan\psi$.

at P cut Ox in T and Oy in U (Fig. 48). The distances OT, OU are called the *intercepts* on the axes Ox, Oy, respectively, made by the tangent. If, in the equation (2) for the tangent, we put $\eta=0$, corresponding to the point T, we have

$$-y=(\xi-x)f'(x), \quad \ldots(14)$$

whence, solving for ξ, we get, as the expression for the x-intercept,[1]

$$OT=x-y/f'(x). \quad \ldots(15)$$

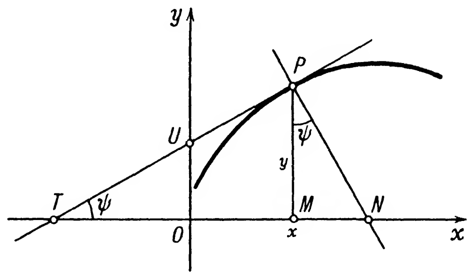


FIG. 48.

Similarly, if we put $\xi=0$, corresponding to the point U, we have

$$\eta-y=-xf'(x), \quad \ldots\ldots(16)$$

whence we get, as the expression for the y-intercept,

$$OU=y-xf'(x). \quad \ldots\ldots(17)$$

Ex. 3. *The astroid* $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. As in Ex. 4, Art. 81, we have

$$dy/dx=-(y/x)^{\frac{1}{3}},$$

and the intercepts OT, OU are therefore given by

$$OT=x-y(-x^{\frac{1}{3}}/y^{\frac{1}{3}})=x+x^{\frac{1}{3}}y^{\frac{2}{3}}=a^{\frac{2}{3}}x^{\frac{1}{3}}, \quad \ldots\ldots(18)$$

$$OU=y-x(-y^{\frac{1}{3}}/x^{\frac{1}{3}})=y+x^{\frac{2}{3}}y^{\frac{1}{3}}=a^{\frac{2}{3}}y^{\frac{1}{3}}. \quad \ldots\ldots(19)$$

We observe that

$$OT^2+OU^2=a^{\frac{4}{3}}(x^{\frac{2}{3}}+y^{\frac{2}{3}})=a^2, \quad \ldots\ldots(20)$$

which shews that the length of the portion of the tangent intercepted between the axes is constant. This is a characteristic feature of the curve.

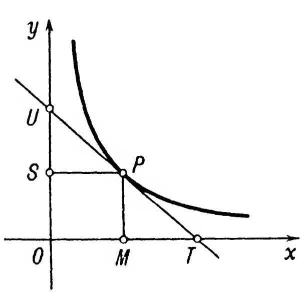


FIG. 49.

Ex. 4. *The rectangular hyperbola* $xy=c^2$. Here we find $OT=2x$, $OU=2y$. Referring to Fig. 49, where PM, PS are the perpendiculars from $P,(x,y)$, to the axes, we thus have $OT=2OM$, $OU=2OS$, so that P is the middle point of TU.

Further, $OT.OU=4xy=4c^2$, which shews that the area of the triangle OTU is constant for all positions of P on the curve.

83. 4. *The 'tangent,' 'normal,' 'subtangent' and 'subnormal.'* Let the normal at P (Fig. 48) meet Ox in N and let MP be the ordinate to P. Then $OM=x$, $MP=y$, $\angle MTP=\angle MPN=\psi$, where $\tan\psi=dy/dx$.

[1] OT is here the algebraic distance of T to the right of O and OU that of U above O.

In books on Geometry, the lengths PT, PN, TM, MN are called respectively the *tangent, normal, subtangent, subnormal* of the curve, corresponding to the point P. We have the following results:

$$\text{(i) } PT = |y \operatorname{cosec} \psi| = |y| \frac{\sqrt{\{1+(dy/dx)^2\}}}{|dy/dx|}, \quad \text{.........(21)}$$

which gives the (length of the) tangent.

$$\text{(ii) } PN = |y \sec \psi| = |y| \sqrt{\{1+(dy/dx)^2\}}, \quad \text{...........(22)}$$

which gives the (length of the) normal.

$$\text{(iii) } TM = |y \cot \psi| = \left|\frac{y}{dy/dx}\right|, \quad \text{.........................(23)}$$

which gives the (length of the) subtangent.

$$\text{(iv) } MN = |y \tan \psi| = \left|y \frac{dy}{dx}\right|, \quad \text{............................(24)}$$

which gives the (length of the) subnormal.

Ex. 5. *The parabola* $y^2 = 4ax$. Here we find $y\, dy/dx = 2a$, so that the subnormal is constant.

Ex. 6. *The ellipse* $x^2/a^2 + y^2/\lambda^2 = 1$, ($a$, λ *positive*). We have

$$dy/dx = -(\lambda^2 x)/(a^2 y), \quad \text{............................(25)}$$

and the subtangent is given by

$$TM = \left|\frac{y}{dy/dx}\right| = \left|\frac{a^2y^2}{\lambda^2 x}\right| = \left|\frac{a^2}{x}\left(1 - \frac{x^2}{a^2}\right)\right| = \frac{a^2 - x^2}{|x|}. \quad \text{..............(26)}$$

This result is independent of λ, and it follows that all the ellipses obtained from the given equation by keeping a constant and varying λ have the same subtangent, that is, the tangents to the several curves at points having the same abscissa all meet the x-axis in the same point.

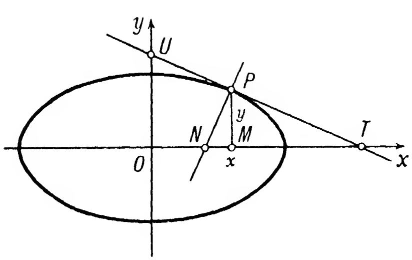


FIG. 50.

Each member of the family of ellipses obtained in this way has $2a$ as the length of its axis along Ox. They are all described as *orthogonal projections* of the circle $x^2 + y^2 = a^2$. If y, y^* denote the corresponding ordinates MP, MP^* of an ellipse and the circle, we have $y = \lambda y^*/a$, and if we imagine the plane containing the circle rotated about Ox through an angle $\arccos(\lambda/a)$, the projection of P^* on the original plane is P. This construction breaks down when $\lambda > a$, but the same descriptive term is applied as when $\lambda < a$.

Referring to Fig. 50 and employing the result (26), we have

$$OT = OM + MT = x + \frac{a^2 - x^2}{x} = \frac{a^2}{x}, \quad \text{.....................(27)}$$

so that $$OM\,.\,OT=a^2, \qquad (28)$$

which is a well-known property of the ellipse.

The subnormal corresponding to the point P is given by

$$NM=\left|y\frac{dy}{dx}\right|=\frac{\lambda^2 x}{a^2}, \qquad (29)$$

and hence $$ON=OM-MN=x-\lambda^2x/a^2=x(1-\lambda^2/a^2). \qquad (30)$$

If $\lambda<a$, we may write this

$$ON=e^2x, \qquad (31)$$

where $e=\sqrt{(1-\lambda^2/a^2)}$, the *eccentricity*.

84. Transformation of Coordinates. Change of Rectangular Axes.

When a curve is to be drawn relative to a specified pair of coordinate axes, or when some property of a curve is to be investigated, it is often convenient to transform the equation of the curve by choosing a new pair of rectangular axes to which it is to be referred. In the most general change of rectangular axes, the origin of coordinates is shifted and the orientations of the axes are altered. The transformation can plainly be regarded as effected in two successive stages, one in which the origin is shifted without change of orientation of the axes, and the other in which the origin is unchanged and the axes are rotated.

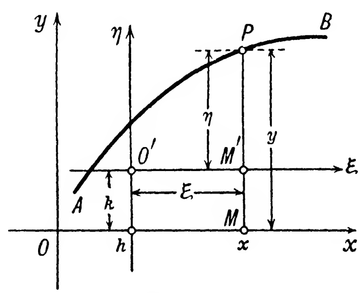


FIG. 51.

84. 1. *Change of origin.* Let P, (x, y), be a point on the curve AB (Fig. 51), whose equation we may suppose given in the form

$$f(x, y)=0, \qquad (1)$$

referred to rectangular axes Ox, Oy, and let O', (h, k), be the new origin. The axes $O'\xi$, $O'\eta$ are drawn parallel to Ox, Oy, and the coordinates of P in the new system are (ξ, η). Drawing PM perpendicular to Ox, we have the relations

$$x=h+\xi, \quad y=k+\eta. \qquad (2)$$

Replacing x, y in (1) by these values, we get $f(h+\xi, k+\eta)=0$, or

$$\varphi(\xi, \eta)=0, \text{ say}, \qquad (3)$$

which is therefore the equation of the curve referred to the new axes.

When there is no risk of confusion, it is unnecessary to maintain symbolically the distinction between the old and the new variables, and we then write

$$\varphi(x, y)=0 \quad \text{.................................(4)}$$

as the equation of the curve referred to the new axes $O'x$, $O'y$.

Ex. 1. *The parabola* $y=ax^2+bx+c$. Writing this in the form

$$y-\frac{4ac-b^2}{4a}=a\left(x+\frac{b}{2a}\right)^2, \quad \text{.................................(5)}$$

and transferring the origin to the point $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$, we obtain, as the modified equation,

$$\left(\frac{4ac-b^2}{4a}+\eta\right)-\frac{4ac-b^2}{4a}=a\left\{\left(-\frac{b}{2a}+\xi\right)+\frac{b}{2a}\right\}^2, \quad \text{.................(6)}$$

that is,

$$\eta=a\xi^2. \quad \text{.....................................(7)}$$

We may write this as $y=ax^2$ on changing the meanings of x and y, so that these letters now refer to the new axes.

84. 2. *Change of orientation of the axes.* Suppose next that the transformation of coordinates corresponds to referring the curve to axes $O\xi$, $O\eta$ (Fig. 52) having the same origin O as the initial set Ox, Oy but orientations different from them. Denote the inclination of $O\xi$ to Ox (or of $O\eta$ to Oy) by θ, and let P have coordinates (x, y), (ξ, η) relative to the two systems. Since OM is equal to the algebraic sum of the projections of OM', $M'P$ on Ox, we have

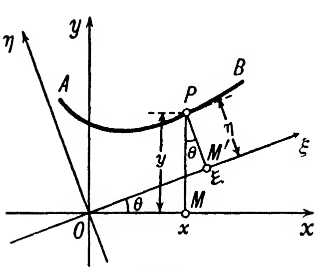


FIG. 52.

$$x=\xi\cos\theta-\eta\sin\theta, \quad \text{..........(8)}$$

and since MP is the sum of the projections of OM', $M'P$ on Oy,

$$y=\xi\sin\theta+\eta\cos\theta. \quad \text{.........................(9)}$$

If the equation of the curve relative to the original axes is given in the form (1), that relative to the new axes is

$$f(\xi\cos\theta-\eta\sin\theta, \xi\sin\theta+\eta\cos\theta)=0, \quad \text{...............(10)}$$

or

$$\varphi(\xi, \eta)=0, \text{ say.} \quad \text{............................(11)}$$

As before, we can revert to x, y for the 'new' coordinates, and write (11) in the form $\varphi(x, y)=0$ when there is no risk of confusion.

The transformation here described must evidently leave unchanged the forms of the expressions for the square of distance from O to any point P, and for the angle between any two lines OP, OP'. We have

$$\begin{aligned} OP^2 = x^2 + y^2 &= (\xi \cos\theta - \eta \sin\theta)^2 + (\xi \sin\theta + \eta\cos\theta)^2 \\ &= \xi^2\cos^2\theta - 2\xi\eta\cos\theta\sin\theta + \eta^2\sin^2\theta \\ &\qquad + \xi^2\sin^2\theta + 2\xi\eta\sin\theta\cos\theta + \eta^2\cos^2\theta \\ &= \xi^2 + \eta^2, \quad \ldots\ldots(12) \end{aligned}$$

which verifies the first statement. To verify the second, we investigate the invariance of the expression for the cosine of the angle POP'. Let P' have coordinates (x', y'), (ξ', η') in the two systems and let α be the angle POP'. Then, as in Ex. 1, Art. 16, we have

$$\cos\alpha = \frac{xx' + yy'}{OP \,.\, OP'}. \quad \ldots\ldots(13)$$

But
$$\begin{aligned} xx' + yy' &= (\xi\cos\theta - \eta\sin\theta)(\xi'\cos\theta - \eta'\sin\theta) \\ &\qquad + (\xi\sin\theta + \eta\cos\theta)(\xi'\sin\theta + \eta'\cos\theta) \\ &= \xi\xi'\cos^2\theta - (\xi\eta' + \xi'\eta)\cos\theta\sin\theta + \eta\eta'\sin^2\theta \\ &\qquad + \xi\xi'\sin^2\theta + (\xi\eta' + \xi'\eta)\sin\theta\cos\theta + \eta\eta'\cos^2\theta \\ &= \xi\xi' + \eta\eta', \quad \ldots\ldots(14) \end{aligned}$$

and the verification is thus complete.

The expressions $x^2 + y^2$, $xx' + yy'$ are hence said to be *invariant for rectangular transformation without change of origin.*

Ex. 2. *The rectangular hyperbola* $xy = c^2$. Let the new axes bisect the angles between the original set, so that θ may be taken to be $\frac{1}{4}\pi$. The formulae (8), (9) of transformation are now

$$x = (\xi - \eta)/\sqrt{2}, \quad y = (\xi + \eta)/\sqrt{2}, \quad \ldots\ldots(15)$$

and the given equation transforms to

$$\xi^2 - \eta^2 = 2c^2, \quad \ldots\ldots(16)$$

which we can write in the form $x^2 - y^2 = c^2$, on changing the meanings of x, y, c.

85. Connection between the Derivatives in the General Rectangular Transformation. Invariance Properties.

The general rectangular transformation is expressed by

$$x = h + \xi\cos\theta - \eta\sin\theta, \quad y = k + \xi\sin\theta + \eta\cos\theta, \quad \ldots\ldots(1)$$

which give x, y as functions of the variable ξ, say, when η is a function of ξ. We thus have

$$\frac{dy}{dx} = \frac{dy/d\xi}{dx/d\xi} = \frac{\sin\theta + \dfrac{d\eta}{d\xi}\cos\theta}{\cos\theta - \dfrac{d\eta}{d\xi}\sin\theta} = \frac{\tan\theta + \dfrac{d\eta}{d\xi}}{1 - \dfrac{d\eta}{d\xi}\tan\theta}. \quad \ldots\ldots(2)$$

This result follows at once from the relation between the slopes of the tangent to the axes Ox, $O\xi$.

As in the case of the transformation by rotation of the coordinate axes, the general transformation (1) leaves unchanged the expressions for the square of the distance between any two points and for the angle between any two lines in the plane. Thus, if the coordinates of P_1, P_2 are (x_1, y_1), (x_2, y_2) and (ξ_1, η_1), (ξ_2, η_2) in the respective systems,

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = (\xi_2 - \xi_1)^2 + (\eta_2 - \eta_1)^2. \quad \text{............(3)}$$

Again, if the coordinates of two other points P_1', P_2' are (x_1', y_1'), (x_2', y_2') and (ξ_1', η_1'), (ξ_2', η_2') in the two systems, and if α is the angle between P_1P_2 and $P_1'P_2'$, we have

$$\begin{aligned} P_1P_2 \,.\, P_1'P_2' \,.\cos\alpha &= (x_2 - x_1)(x_2' - x_1') + (y_2 - y_1)(y_2' - y_1') \\ &= (\xi_2 - \xi_1)(\xi_2' - \xi_1') + (\eta_2 - \eta_1)(\eta_2' - \eta_1'). \quad \text{....(4)} \end{aligned}$$

The verification of the two invariants is left to the student.

EXAMPLES X

DIRECT DETERMINATION OF DERIVATIVES. SIMPLE GRADIENTS

1. If $y = x^3$ and Δx is measured from $x = 2$, shew that $\Delta y/\Delta x$ differs from 12 by less than $1/10^m$, (m a positive integer), if $|\Delta x| < 1/(7 \times 10^m)$.

2. If $y = 1/x$, shew that $\Delta y/\Delta x - (-1/x^2) = \Delta x/\{x^2(x+\Delta x)\}$, and hence that if Δx is measured from $x = 2$, $\Delta y/\Delta x$ differs from $-\frac{1}{4}$ by less than $1/10^m$ if $|\Delta x| < 4/10^m$.

3. Shew that if m is a positive integer greater than 2 and $x_1 + \Delta x < 2x_1$, $(x_1 > 0)$,

$$\left|\frac{(x_1+\Delta x)^m - x_1{}^m}{\Delta x} - m x_1{}^{m-1}\right| < \varepsilon \text{ (arbitrary) if } |\Delta x| < \frac{2\varepsilon}{m(m-1)(2x_1)^{m-2}}.$$

[Employ the result of Ex. 1, Set III, p. 106.]

4. If $y = (x-a)/(x+a)$, find the difference between $\Delta y/\Delta x$ and $2a/(x+a)^2$, and hence shew that the latter expression is the derivative of y with respect to x.

5. For the parabola $y = x^2$ find

(i) the abscissae of the points at which the slopes are $\frac{1}{6}\pi$, $\frac{1}{4}\pi$, $\frac{1}{3}\pi$;

(ii) where the tangents from the point $(1, 0)$ touch the curve and the gradients of these tangents;

(iii) the point of the curve nearest to the line through $(0, 1)$ and making an angle of $45°$ with Ox. [$\frac{1}{6}\sqrt{3}$, $\frac{1}{2}$, $\frac{1}{2}\sqrt{3}$; $x = 0$, $x = 2$, 0, 4; $x = \frac{1}{2}$.]

6. Find where the straight line $y=x+1$ cuts the parabola $y=x^2$ and the acute angles at which they intersect. [$x=\frac{1}{2}(1\pm\sqrt{5})$; $27°\ 50'$, $83°\ 58'$.]

7. Find where the circle $x^2+y^2=4$ intersects the parabola $y=x^2$ and the acute angles at which they intersect. [$x=\pm 1{\cdot}2496$; $73°\ 8'$.]

EXAMPLES XI

DERIVATIVES OF RATIONAL FUNCTIONS

1. Find the derivative of each of the following functions:

$$2x-3,\quad 1+x+x^2+x^3,\quad (2x-3)(5x+1),\quad x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4,\quad x(1-x^2),$$

$$(x^3+1)(x^4+2),\quad (x+a)(x^2+a^2)(x^3+a^3),\quad x^2+1+1/x^2,\quad 1/(2x-3),$$

$$1/(1+x^2),\quad 1/(a+bx+cx^2),\quad 1\Big/\left(x+\frac{1}{x}\right),\quad \frac{1}{x(x-1)},\quad \frac{3x-4}{2x+5},\quad \frac{x}{1+x+x^2},$$

$$\frac{1}{(ax+b)(cx+d)},\quad \frac{2-5x^2}{3+2x},\quad \frac{ax^2+b}{cx^2+d},\quad \frac{1+x+x^2}{1-x+x^2},\quad \frac{ax^2+bx+c}{Ax^2+Bx+C},\quad \frac{x^4-1}{x^4+1},\quad \frac{x+1}{x^2+2x},$$

$$\frac{x^2+1}{x^3-1},\quad \frac{x^m}{a^n+x^n},\quad (m,\ n \text{ integers}).$$

2. Find the derivative of each of the following functions, m, n being integers:

$$(ax+b)^2,\quad (ax^2+bx+c)^3,\quad (a_0+a_1x+a_2x^2+\ldots+a_nx^n)^m,\quad \sum_{r=1}^{n}(x_r-x)^2,$$

$$(6x^2+2x+5)^4,\quad (x+1)^5,\quad x^2(3x+5)^4,\quad (x+1/x)^2,\quad (1+1/x+1/x^2)^3,$$

$$\frac{1}{(ax^2+bx+c)^2},\quad \frac{1}{(1-x)^6},\quad \frac{2}{(x+1)^3},\quad \frac{x^3}{(2x+9)^2},\quad \frac{1}{(ax^2+bx)^m},\quad \left(\frac{ax+b}{cx+d}\right)^2,\quad \frac{(ax^2+b)^m}{(cx^2+d)^n}.$$

3. Find the gradient at any point of the parabola $y=x^2-2x+3$. Find where the tangent is parallel to Ox and where it makes angles of $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$ with Ox. Sketch the curve. [$x=1$, $x=\frac{3}{2}$; $x=\frac{1}{2}$.]

4. Find the constants a, b, c in the equation $y=ax^2+bx+c$ so that the parabola may pass through the point (3, 3) and have zero gradient at the point (1, 2). [$\frac{1}{4}$, $-\frac{1}{2}$, $\frac{9}{4}$.]

5. The constants a, b, c, d in the equation $y=(ax^2+bx+c)/(x+d)$ are to be adjusted so that the following conditions are satisfied: (i) $y=2$ for $x=0$, (ii) $y\to\infty$ as $x\to 1$, (iii) $dy/dx=4$ when $x=0$, and (iv) $dy/dx\to 1$ when $x\to\infty$. Find the function and sketch its graph. Find also the gradients at the points where it crosses Ox. [The constants are 1, -2, -2, -1.]

6. Find the perpendicular distance p of the point P, (h, k), from the line $ax+by+c=0$.

[Employ the relation $p=|PN\cos\psi|$, where N is the point where the ordinate through P cuts the line.]

7. A point P, (x, y), is taken on the curve $y=f(x)$ and the perpendicular PN is drawn from it to a line ANB passing through A, (h, k), and of inclination φ to Ox. Find the rate of increase of AN with respect to x as P moves along the curve. [$\cos(\psi-\varphi)\sec\psi$.]

8. Shew that if y is the reciprocal of z, the ratio of the rates of increase of y and z with respect to x is equal and opposite to the ratio of y to z.

9. A particle is projected into a resisting medium, the distance s penetrated after time t being given by $s = 20t - \frac{1}{10}t^3$. What is the total penetration? [108·87.]

10. If m is a positive integer, shew (i) that x^{-m} has not a derivative at $x = 0$; (ii) that if m is odd, the derivative of x^{-m} diverges to $-\infty$ as $x \to 0$, while if m is even, the derivative diverges to $-\infty$ or $+\infty$ according as $x \to +0$ or $x \to -0$; (iii) that the derivative of x^{-m} converges to 0 as $x \to \pm\infty$.

11. Shew that if the relation between x and y is expressed by an equation linear in each variable, then y is equal to a fraction whose numerator and denominator are linear in x, and vice versa. Find the derivative of y with respect to x and that of x with respect to y independently, and shew by algebraic transformation that one derivative is the reciprocal of the other.

12. Shew that two functions of x which differ by a constant have equal derivatives, and that two functions which differ by an expression linear in x have derivatives which differ by a constant.

13. If the relation between y and x is expressed by the equality of two quotients, one of which has numerator and denominator linear in y alone and the other numerator and denominator linear in x alone, find the direct expression of y in terms of x. Shew that the relation between x and y can be expressed by equating the difference of the reciprocals of two expressions, linear in x and y, respectively, to a constant. Find the derivative of y with respect to x.

EXAMPLES XII

DERIVATIVES OF FRACTIONAL POWERS. FUNCTION OF A FUNCTION

1. If $y = x^{1/3}$, shew that $\dfrac{\Delta y}{\Delta x} = \dfrac{1}{x^{2/3} + x^{1/3}x_1^{1/3} + x_1^{2/3}}$, and hence that $(dy/dx)_1 = 1/(3x_1^{2/3})$.

2. Find the derivative of each of the following functions:

$$x^{3/2},\quad x^{n/m},\quad x^{-1/3},\quad x\sqrt{(x\sqrt{x})},\quad x^{-3/4},\quad (1+x)\sqrt{x},\quad \sqrt{x}+ax^{5/2},\quad \sqrt{x}+1/\sqrt{x},$$

$$\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2,\quad \frac{x^2+5x+3}{\sqrt{x}},\quad \sqrt{\frac{x}{a}}+\sqrt{\frac{a}{x}},\quad \sqrt{(x^3)}+\frac{2}{x}-\frac{3}{x^2}+\frac{4}{\sqrt{(x^5)}},\quad \frac{\sqrt{x}}{1+x},$$

$$\frac{a\sqrt{x}+b}{c\sqrt{x}+d},\quad \frac{x^{3/2}}{1+x^{1/2}},\quad \frac{1-x^{-1/4}}{x^{1/5}},\quad \left(1+\frac{1}{\sqrt{x}}\right)^2-2\sqrt{x},\quad (a\sqrt{x}+b)^{-2},\quad (ax^{3/2}+bx^{1/2})^3,$$

$$\left(\frac{a\sqrt{x}+b}{c\sqrt{x}+d}\right)^3,\quad \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)\left(\frac{2}{x}-\frac{7}{x^3}\right).$$

3. Find the derivative of each of the following functions, where $m, n, \ldots$ are any real numbers:

$$\sqrt{(3-2x)},\ (2-3x)^{2/3},\ (ax+b)^{1/n},\ (ax^2+bx+c)^{m/n},\ (a\sqrt{x}+b)^{-1/3},\ x\sqrt{(1+x)},$$

$$(ax^{m/n}+bx^{p/q})^{r/s},\ \{x(a+x)\}^{1/4},\ \sqrt{x}+\sqrt{(x+\sqrt{x})},\ \sqrt{(x+1)}\sqrt{(x+2)},\ (a^2+x^2)^{3/4},$$

$x^2\sqrt{(a^2-x^2)}$, $(4x^2-3x+2)^{-2/5}$, $(a^3-x^3)^{1/3}$, $x+\sqrt{(x^2+1)}$, $x+\sqrt{(x^2-1)}$, $(\frac{1}{3}x^{3/2}+\frac{1}{5}x^{5/2})^{1/2}$, $(1-x)\sqrt{(1+x^2)}$, $\{x+\sqrt{(1-x^2)}\}^n$, $x^m(ax^n+b)^p$, $(u^m+v^n)^{-p}$, $(x^2+1)\sqrt{(x^3-x)}$, $x\sqrt{(x^2-3x+2)}$, $\sqrt{\left(\dfrac{x+2}{x+1}\right)}$, $\dfrac{u^m+v^n}{u^mv^n}$, $\left(\dfrac{a-2x}{a+3x}\right)^{5/2}$, $\left(\dfrac{ax^2+b}{cx^2+d}\right)^{1/m}$, $\dfrac{1}{(4x^2+3x-2)^{1/3}}$, $\left(\dfrac{ax^2+bx+c}{Ax^2+Bx+C}\right)^{m/n}$, $\left(\dfrac{a\sqrt{x}+b}{c\sqrt{x}+d}\right)^{2/3}$, $\dfrac{1}{\sqrt{(1-x^2)}}$, $\dfrac{x}{\sqrt{(1+x^2)}}$, $\dfrac{\sqrt{(a+x)}}{\sqrt{a}+\sqrt{x}}$, $\left\{\dfrac{x}{x+\sqrt{(1-x^2)}}\right\}^{2/3}$, $\dfrac{x^3}{3(1+x^2)^{3/2}}$, $\dfrac{x+1}{(x-1)^{2/3}}$, $\dfrac{1+\sqrt{(1-x)}}{x}$, $\dfrac{x}{(2x^4+5x+1)^{1/4}}$, $\left\{\dfrac{x}{\sqrt{(1-x^2)}}\right\}^3$, $\dfrac{1}{(1+x)^{1/2}(1-x)^{3/2}}$, $\left\{x\left(\dfrac{x+b}{x+c}\right)^{2/3}+d\right\}^{3/2}$, $\dfrac{x+\sqrt{(x^2-a^2)}}{x-\sqrt{(x^2-a^2)}}$, $\dfrac{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}$, $\dfrac{\sqrt{(a-x)}+\sqrt{(a+x)}}{\sqrt{(a-x)}-\sqrt{(a+x)}}$, $\left(\dfrac{a^n+x^n}{b^n+x^n}\right)^m$, $\sqrt{\left(\dfrac{1+x^{1/2}}{1-x^{3/2}}\right)}$.

4. If $P=(u_1^{p_1}u_2^{p_2}\dots u_n^{p_n})/(v_1^{q_1}v_2^{q_2}\dots v_n^{q_n})$, where u_r, v_r are differentiable functions of x and p_r, q_r are any real numbers, prove that

$$\frac{dP}{dx}=P\sum_{r=1}^{n}\left\{\frac{p_r}{u_r}\frac{du_r}{dx}-\frac{q_r}{v_r}\frac{dv_r}{dx}\right\}.$$

5. Give a geometrical interpretation of the result $\dfrac{d}{dx}f(mx)=mf'(mx)$.

6. Shew that the function defined by the formula $\dfrac{\sqrt{(1+x)}-1}{x}$ when $x\neq 0$ and whose value is $\frac{1}{2}$ when $x=0$, has the derivative $-\frac{1}{8}$ when $x=0$.

7. If u, v are functions of x and $du/dv=r$, shew that the derivative of u/v with respect to uv is $\dfrac{rv-u}{v^2(rv+u)}$.

8. Find the derivatives of $f(-x)$, $f(x^2)$ with respect to x, the derivative $f'(x)$ being supposed known.

Find the derivative of $\sqrt{\left(\dfrac{1+x}{1-x}\right)}$ with respect to $\sqrt{x}$. $\left[\dfrac{2\sqrt{x}}{(1+x)^{1/2}(1-x)^{3/2}}.\right]$

9. Find the rate of increase of the volume of a cuboid whose edges are at any instant increasing at the rates u, v, w units of length per unit time.

[$uyz+vzx+wxy$, where x, y, z are the lengths of the edges.]

10. What is the rate of increase of the ordinate of the circle $x^2+y^2=4$ if that of the abscissa is V? [$\pm Vx/\sqrt{(4-x^2)}$.]

11. Sand drops from a chute at the rate of 3 cu. ft. per min. and forms a heap on the ground in the shape of a circular cone of vertical angle 90°. At what rate is the height of the cone increasing after 4 mins.?

[0·1879 ft. per min.]

12. Water is poured at a constant rate into a can of conical form, the height being 6 in. and the radii of the ends 2 in., 4 in. The can is filled in 5 secs. Find the rates of increase of the depth of the water when the can is filled to half the height and when half full. [1·244, 1·028 in. per sec.]

13. A straight line PQ has its ends on two straight lines APB, CQD, of inclinations θ, φ to Ox, and its ends move with velocities u, v along the lines.

Find the rate of change of the gradient of the line PQ and the rate of increase of the length l of PQ.

$\left[\frac{1}{l\cos^2\psi}\{v\sin(\varphi-\psi)-u\sin(\theta-\psi)\},\ v\cos(\varphi-\psi)-u\cos(\theta-\psi)\right.$, where ψ is the slope of PQ.$\left.\right]$

14. Two points P_1, P_2 move in a known manner in a plane and at time t they have coordinates (x_1, y_1), (x_2, y_2), respectively. Find the rate of increase of the length P_1P_2.

If Q is on the line joining P_1, P_2 and divides it in the variable ratio $\lambda : (1-\lambda)$, find the velocity of Q.

15. A line P_1P_2 of constant length moves with its ends one on each of two given curves. Shew that if v_1, v_2 are the velocities of its ends and φ_1, φ_2 the inclinations of the line to the respective tangents, $|v_1/v_2| = |\cos\varphi_2/\cos\varphi_1|$.

16. Two bars AB, DC turn about fixed centres A, D and are connected by a link BC. If the lines of the bars meet, when produced, at I, shew that the magnitudes of the velocities of B, C are in the ratio $IB : IC$.

Deduce that if P is the point of BC which is at the foot of the perpendicular from I on it at the instant considered, the velocity of P is along BC.

17. Two particles A, B move with constant velocities u, v, one along each of two non-intersecting straight lines at right-angles to each other and whose shortest distance apart is h. At a certain instant the particles are at the ends A_0, B_0 of the shortest line joining the two lines. Find their rate of separation at any subsequent instant. $[(u^2+v^2)t/\{(u^2+v^2)t^2+h^2\}^{1/2}.]$

18. A rod AB of length l moves with its ends one on each of the axes Ox, Oy. If A moves with constant velocity V along Ox, what is the velocity of B along Oy when the rod makes an angle θ with Ox ?

If the axes are inclined at angle ω, find the corresponding velocity at any instant. $[-V\cot\theta,\ V\cos\theta\sec(\theta+\omega).]$

19. The sides of a triangle are increasing at the uniform rate v units of length per unit time. Find the rate of increase of the area A of the triangle at any instant. $\left[\tfrac{1}{4}Av\left(\frac{3}{s}+\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}\right)\right.$, where $s=\tfrac{1}{2}(a+b+c)$.$\left.\right]$

20. In a lazy-tongs with n cells find the ratio between the rate of decrease of the breadth of the cells and the rate of increase of their total length.

$\left[\frac{\sqrt{(a^2-\frac{1}{4}x^2)}}{\frac{1}{2}nx}\right.$, where $2a$ is the length of each link and x is the breadth at any instant.$\left.\right]$

21. A point P moves along a straight line MN with velocity V. Find the rate at which the sum of its distances from two points A, B, on the same side of the line, is increasing. Shew that this rate vanishes when

$$A'P = A'B' \,.\, A'A/(A'A+B'B),$$

where A', B' are the projections of A, B on MN.

[The inclinations of PA, PB to the line MN are then equal.]

22. The elasticity of a gas of volume v in which the pressure is p is measured by the increase of pressure per unit fractional decrease of volume. Find the elasticity when (i) $pv=C$, (ii) $p(v-a)=C$, (iii) $pv^\gamma=C$, (iv) $\left(p+\dfrac{a}{v^2}\right)(v-b)=C$, where C, a, b, γ are constants.

[p, $pv/(v-a)$, γp, $(pv-a/v+2ab/v^2)/(v-b)$.]

23. The tangent is drawn at any point P of a curve in space. Shew that the cosine of its inclination to Ox is $1\Big/\sqrt{\left\{1+\left(\dfrac{dy}{dx}\right)^2+\left(\dfrac{dz}{dx}\right)^2\right\}}$.

If the coordinates of P are expressed in terms of a parameter t, shew that this becomes $\dfrac{|\,dx/dt\,|}{\sqrt{\{(dx/dt)^2+(dy/dt)^2+(dz/dt)^2\}}}$.

A circle of radius a with centre at the origin lies in a plane through Oy making an angle α with the plane yOz. Shew that the cosine of the inclination of the tangent at any point (x, y, z) of the circle to Ox is $(y/a)\sin\alpha$.

EXAMPLES XIII

DERIVATIVES OF IMPLICIT FUNCTIONS

1. Employ the rule for the interchange of the dependent and independent variables in a derivative to deduce the result $\dfrac{d\sqrt{x}}{dx}=\dfrac{1}{2\sqrt{x}}$, $(x>0)$, from the result $\dfrac{dx^2}{dx}=2x$.

Deduce, similarly, the result $\dfrac{dx^{\frac{1}{m}}}{dx}=\dfrac{1}{m}x^{\frac{1}{m}-1}$, $(x>0$, m a positive integer), from the result $\dfrac{dx^m}{dx}=mx^{m-1}$.

2. Find the value of dy/dx when x, y are specified parametrically by the equations: (i) $x=\dfrac{6t}{1+t^2}$, $y=\dfrac{1-t^2}{1+t^2}$; (ii) $x=\dfrac{1-t^2}{1+t^2}$, $y=\dfrac{t(1-t^2)}{1+t^2}$; (iii) $x=t-t^3$, $y=1-t^4$; (iv) $x=2t^2-4t+3$, $y=3t^3-4t^2+5$.

3. Find the perpendicular distance p of the point P, (h, k), from the line $ax+by+c=0$, the axes being of obliquity ω.

[Use the result $p=|\,PN\sin(\omega-\psi)\,|$, where N is the point where the ordinate through P cuts the line.]

4. A point P, (x, y), is taken on the curve $y=f(x)$, referred to axes of obliquity ω, and the perpendicular PN is drawn from it to the line ANB, passing through A, (h, k), and of inclination φ to Ox. Find the rate of increase of AN with respect to x as P moves along the curve.

[$\sin\omega\cos(\psi-\varphi)\operatorname{cosec}(\omega-\psi)$.]

5. Employ the process of differentiating an equation to find dy/dx in the following cases:

(i) $\sqrt{(ax^2+bx+c)}-\sqrt{(ay^2+by+c)}=\alpha(x-y)$; (ii) $\dfrac{x}{x-y}=\dfrac{1}{\sqrt{(x+y)}}$;

(iii) $x^3+ax^2y+bxy^2+y^3=c$; (iv) $x^m+y^m=a^m$; (v) $\sqrt{(x^2+y^2)}+\sqrt{(x^2-y^2)}=a$;

(vi) $y\sqrt{x}=1+x$; (vii) $\sqrt{\left(\dfrac{x+y}{x-y}\right)}+\sqrt{\left(\dfrac{x^2+y^2}{x^2-y^2}\right)}=x+y$; (viii) $x+y=\sqrt{\left(\dfrac{x^2-ay}{y^2-ax}\right)}$;

(ix) $x^m y^n=(x+y)^{m+n}$; (xi) $x^2+y^2=xy(x+y)$.

6. If p, v are functions of θ such that pv/θ and pv^γ are both constant, (γ constant), express the derivatives of p and v with respect to θ in terms of p, v, θ, γ. $\left[-\dfrac{p\gamma}{\theta(1-\gamma)}, \dfrac{v}{\theta(1-\gamma)}.\right]$

7. Find the value of dy/dx when $x^2+xy+y^2=1$ and the points where this derivative is zero or infinite. Hence shew that the graph lies entirely in the rectangle the equations of whose sides are $y=\pm 2/\sqrt{3}$, $x=\pm 2/\sqrt{3}$.

[The graph is an ellipse whose axes bisect the angles between the coordinate axes. See Ex. 18, Set XIV.]

8. If S_n denotes the sum of n terms of a geometric progression of common ratio x, shew that $(x-1)\dfrac{dS_n}{dx}=(n-1)S_n-nS_{n-1}$.

9. Find the gradient at any point P, (x, y), of the circle

$$(x-a)^2+y^2=k^2\{(x+a)^2+y^2\},$$

and shew that if A is the point $(-a, 0)$ and A' the point $(a, 0)$, the inclination of the tangent to AP is equal to the inclination of $A'P$ to Oy.

10. Prove that the curves whose equations are $2x=3y+ax^2y^2+bx^3y^4$, $3x+2y=cx^5+dx^3y$ intersect at right-angles at the origin.

11. Find the values of $\partial f/\partial x$, $\partial f/\partial y$ for the following dimetric functions $f(x, y)$:

(i) $x^2-2xy+y^2$; (ii) $x^3-axy^2+y^3$; (iii) $\sqrt{(x^2+y^2)}+\dfrac{x+\sqrt{y}}{x-\sqrt{y}}$.

12. If $f(x, y)=ax^2+2hxy+by^2$, shew that $x\partial f/\partial x+y\partial f/\partial y=2f$.

13. If $f(x, y)=\sum_{r=1}^{n} a_r x^{p_r} y^{q_r}$, where $p_r+q_r=m$, shew that

$$x\partial f/\partial x+y\partial f/\partial y=mf.$$

14. If $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}=1$, prove that $\left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial u}{\partial y}\right)^2=2\left(x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}\right)$.

15. If x, y are functions of t, so that $f(x, y)$ may be regarded as a function $\varphi(t)$ of t alone, shew that it follows from (9), Art. 82, that

$$\frac{d\varphi(t)}{dt}=\frac{\partial f(x, y)}{\partial x}\frac{dx}{dt}+\frac{\partial f(x, y)}{\partial y}\frac{dy}{dt}.$$

[If, in (9), $f(x, y)$, as a function of x alone, is denoted by $\psi(x)$, we have

$$\frac{d\psi(x)}{dx}=\frac{\partial f(x, y)}{\partial x}+\frac{\partial f(x, y)}{\partial y}\frac{dy}{dx}.$$

On multiplying each side by dx/dt, the required result follows. In these equations $\varphi(t)$, $\psi(x)$, $f(x, y)$ are usually replaced by f.]

EXAMPLES XIV

GEOMETRICAL PROBLEMS

1. Find the values of a, b in order that the straight line $y = ax + b$ may be a tangent to the parabola $y = x^2$ at the point for which $x = 1{\cdot}4$. [$2{\cdot}8$, $-1{\cdot}96$.]

2. Find the values of n for which the line $y = mx + n$ is a tangent to the circle $x^2 + y^2 = a^2$, m being given. [$\pm a\sqrt{(1 + m^2)}$.]

3. Find the relation between l, m, n which must be satisfied if the line $lx + my + n = 0$ is a tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$. [$n^2 = a^2l^2 + b^2m^2$.]

4. Find the equations of the tangent and normal at any point (h, k) of the ellipse $x^2/a^2 + y^2/b^2 = 1$, and at any point of the parabola $y^2 = 4ax$.
[The tangents are $hx/a^2 + ky/b^2 = 1$, $ky = 2a(h + x)$.]

5. Find the equation of the normal at any point t of the curve $x = f(t)$, $y = \varphi(t)$. [$\{x - f(t)\}f'(t) + \{y - \varphi(t)\}\varphi'(t) = 0$.]

6. Find the equation of the tangent to the curve $x^3 + y^3 = a^2(x - y)$ at the origin. [$y = x$.]

7. If the equation of a curve is

$$a_0x + a_1y + b_0x^2 + b_1xy + b_2x^2 + \dots + (k_0x^n + k_1x^{n-1}y + \dots + k_ny^n),$$

where a_0, a_1 are not both zero, shew that the equation of the tangent at the origin is $a_0x + a_1y = 0$.

8. The tangent is drawn to the curve $3x^2y - 2xy^2 = 1$ at the point $(1, 1)$. Find where it cuts the curve again. [$x = -1/20$.]

9. Find the condition that the astroid $x^{2/3} + y^{2/3} = c^{2/3}$ may touch the ellipse $x^2/a^2 + y^2/b^2 = 1$. [$c = a + b$.]

10. Find the subtangent at any point of (i) the semi-cubical parabola $y^2 = ax^3$, $(a > 0)$, (ii) the rectangular hyperbola $xy = c^2$. [$\frac{2}{3}x$, $|x|$.]

11. The tangent at any point of the semi-cubical parabola $y^2 = ax^3$ meets the axes Ox, Oy in T, U, respectively. Shew that the point whose coordinates are $x = OT$, $y = OU$ lies on the semi-cubical parabola $y^2 = \frac{27}{4}ax^3$.

12. Find the equations of the tangent and normal at any point of the curve $y = x/(a^2 + x^2)$. Shew that the ratio of the intercepts OT, OU is

$$(a^2 + x^2)^2/(x^2 - a^2).$$

13. Find the equation of the tangent at any point of the general conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.
Shew that the conditions that the conic may touch Ox at the origin are $g = 0$, $c = 0$, $f \neq 0$.

14. If the tangent at P on the curve $y^3 = 3ax^2 - x^3$ meets the curve again at Q, prove that $\tan\angle QOx + 2\tan\angle POx = 0$. Shew also that if the tangent at P is a normal at Q, P lies on the curve $4y(3a - x) = (2a - x)(16a - 5x)$.

15. The tangents to the Folium of Descartes $x^3 + y^3 = 3axy$ are drawn at the points, distinct from the origin, where the lines $y = \pm mx$ cut the curve. Prove that they intersect on the curve.

16. Prove that the condition that the normal to the curve $f(x, y) = 0$ may pass through the origin is $x\partial f/\partial y - y\partial f/\partial x = 0$. Deduce the equations of the principal axes of the conic $ax^2 + 2hxy + by^2 = 1$.

17. Find the form taken by the equation $x^3 + y^3 = 3axy$ when the axes are rotated through an angle of 45°. $\left[y^2 = x^2 \dfrac{3a\sqrt{2} - 2x}{3a\sqrt{2} + 6x}.\right]$

18. Find the form taken by the equation $x^2 + xy + y^2 = 1$ when the axes are rotated through an angle of 45°.

19. Find the form taken by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ when the axes are rotated through an angle θ given by $\tan 2\theta = 2h/(a - b)$.

20. Through any point (ξ, η) of the plane xOy pass two curves whose equations are $\lambda_1{}^2 p(x) + \lambda_1 q(x, y) + r(y) = 0$, $\lambda_2{}^2 p(x) + \lambda_2 q(x, y) + r(y) = 0$, where

$$p(x) = (1 - k^2)^2 x^2, \quad q(x, y) = 1 + k^2(x^2 + y^2)^2 + (1 + k^2)(y^2 - x^2), \quad r(y) = -y^2,$$

(k constant), and λ_1, λ_2 are the roots of the equation

$$\lambda^2 p(\xi) + \lambda q(\xi, \eta) + r(\eta) = 0.$$

Shew that the two curves intersect at right-angles at (ξ, η).

CHAPTER III

SUCCESSIVE DERIVATIVES
APPLICATION TO ALGEBRAIC FUNCTIONS

86. Introduction. Derived Functions. Notation for Higher Derivatives.

In the examples of the determination of derivatives given in Chap. II, the formulae obtained for the derivatives were valid in the ranges of continuity of the functions themselves, (except possibly at a finite number of points), and hence defined functions of the independent variable in those ranges. These functions are called the *derived functions* of the respective original functions. In general, the aggregate of the values of the derivative of a function, $f(x)$ say, each associated with the value of the independent variable x at which it occurs, defines a function of x and is called the *derived function of* $f(x)$. For the functions with which we are commonly concerned, the derived function will be defined in one or more continuous ranges. We are naturally led to inquire whether the derived function has also a derivative for any value of the independent variable. If this derivative exists, it is called the *second derivative*, or *second differential coefficient*, of the *original* function at that value. The derivative of the original function $f(x)$ is then distinguished, when necessary, as the *first* derivative of $f(x)$.

Again, the aggregate of the values of the second derivative of a function defines a function of the independent variable, and this is called the *second derived function* of the original function. In like manner, for the functions with which we are commonly concerned, the second derived function will be defined in one or more continuous ranges, and we may proceed to consider whether it has a derivative at any value of the independent variable. Such a derivative is called the *third derivative*, or *third differential coefficient*, of the original function, and the aggregate of its values forms the *third derived function*.

This process may be repeated, and, in general, for any positive integer n, the nth derivative, or the nth differential coefficient, of the original function is the first derivative of the $(n-1)$th derived function.

In the case of the algebraic functions considered in Chap. II, the successive derived functions are also algebraic, and in the simpler cases general formulae can be obtained for the nth derived functions, as will be shewn later in this Chapter.

We may illustrate at once by the function x^m, (m any real number). The successive derived functions are mx^{m-1}, $m(m-1)x^{m-2}$, ..., that of any order n being $m(m-1)\dots(m-n+1)x^{m-n}$ so long as m is not a positive integer. If m is a positive integer, the same formula gives the nth derived function if $n \leqslant m$. When $n=m$ the nth derived function is the constant $n!$, and for larger values of n the successive derived functions all reduce to the constant value 0, that is, they all vanish. Thus when m is a positive integer and $n \leqslant m$, we may write the nth derived function in the form $\dfrac{m!}{(m-n)!}x^{m-n}$.

The notion of successive derived functions was familiar to the founders of the Differential Calculus, especially in connection with Dynamics. The introduction of instantaneous velocity as the time-rate of increase of distance, that is, as the first derived function of the distance-time function, may be looked on as the first step in the mathematical theory of that subject. The discussion of the connection between the force acting and the motion produced required the introduction of the time-rate of increase of the velocity, to which, according to Galileo and Newton, the force is proportional. This new time-rate, which is called the acceleration of the point considered, is the first derived function of the velocity-time function and the second derived function of the distance-time function. In symbols, the velocity v is given in terms of the distance x and the time t by the equation $v=dx/dt$, and the acceleration α by the equation $\alpha=dv/dt$.

We shall shew, in the following Article, that the acceleration can also be expressed as a distance-rate in the form $\alpha=d(\frac{1}{2}v^2)/dx$.

We proceed to the detailed treatment of successive derivatives and derived functions. Suppose that y, or $f(x)$, is a function which possesses a (first) derivative dy/dx for all values of x in a certain range, so that the first derived function of y, which we denote by $g(x)$, say,[1] is defined in that range. This derived function may itself have a derivative, denoted by

$$\frac{dg(x)}{dx}, \quad \text{or} \quad \frac{d}{dx}g(x), \quad \dots\dots(1)$$

at some point x of its range, and this derivative is then the second

[1] The practice of writing $g(x)$ for dy/dx, even when there is no reference to a graph, has been referred to before; see footnote, p. 124, Art. 70.

derivative of the function y or $f(x)$. If we insert the value dy/dx of $g(x)$, the symbols for the second derivative take the forms

$$\frac{d\left(\dfrac{dy}{dx}\right)}{dx}, \quad \frac{d}{dx}\left(\frac{dy}{dx}\right). \qquad \text{.............................(2)}$$

If the second derivative exists at all points in a range, the same forms, regarded as functions of x, represent the second derived function of y in that range.

If, now, we proceed to consider the third derivative, the symbol corresponding to the first of the two forms given for the second derivative and obtained from it by replacing y by dy/dx is very unwieldy; that corresponding to the second is

$$\frac{d}{dx}\left\{\frac{d}{dx}\left(\frac{dy}{dx}\right)\right\}. \qquad \text{.............................(3)}$$

In like manner, the form

$$\frac{d}{dx}\left[\frac{d}{dx}\left\{\frac{d}{dx}\left(\frac{dy}{dx}\right)\right\}\right] \qquad \text{..........................(4)}$$

is arrived at for the fourth derivative.

The brackets are often omitted, or replaced by a dot or cross of separation, in these forms where no doubt can arise. A more condensed form, and one which is regularly employed, is obtained by writing $\dfrac{d}{dx}$ once only and indicating by a numeral how many times it occurs in the full form. Thus the successive derivatives are denoted by

$$\frac{d}{dx}y, \quad \left(\frac{d}{dx}\right)^2 y, \quad \dots, \quad \left(\frac{d}{dx}\right)^n y. \qquad \text{....................(5)}$$

The most common form of all is one which has arisen from a connection with the Calculus of Finite Differences. It can be shewn that if the nth derivative exists, it is equal to the focus, or limit, of the ratio $\Delta^n y/(\Delta x)^n$ when Δx converges to zero. Here $\Delta^n y$ is the nth difference [1] of the function y corresponding to the fixed increment Δx of x. If, then, we indicate the application of the limit-process by the substitution of d for Δ, exactly as in the case of the first derivative we arrive at the symbol $d^n y/(dx)^n$, or $\dfrac{d^n y}{(dx)^n}$. The current practice is

[1] See Art. 60. 2.

to omit the parentheses in the denominator, (not without loss of clarity), and to denote the successive derivatives by [1]

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3}, \quad \ldots, \quad \frac{d^ny}{dx^n}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

or, in the 'solidus' form, by

$$dy/dx, \quad d^2y/dx^2, \quad d^3y/dx^3, \quad \ldots, \quad d^ny/dx^n. \quad \ldots\ldots\ldots\ldots(7)$$

These are regarded as the standard forms.

Thus we write $\alpha = d^2x/dt^2$ for the acceleration α of a particle moving in a straight line, the displacement being x at time t.

When a more condensed notation is required, the symbols $y^{(1)}, y^{(2)}, \ldots, y^{(n)}$ are frequently employed.[2] There is here no indication of the independent variable. A corresponding notation, where $f(x)$ is substituted for y, has not this disadvantage. In this case the successive derivatives are $f^{(1)}(x), f^{(2)}(x), \ldots, f^{(n)}(x)$, where the first three are often conveniently replaced by $f'(x), f''(x), f'''(x)$.

The value of the nth derivative of y or $f(x)$ at the value x_1 of x is indicated by one or other of the symbols

$$\left(\frac{d^ny}{dx^n}\right)_{x_1}, \quad f^{(n)}(x_1), \quad y_1^{(n)}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

y_1 being the value of y corresponding to x_1.

Finally, it is desirable to call attention to the fact that in the notation $f^{(n)}(x)$, the independent variable with respect to which the successive derivatives are formed is always indicated by the expression in the parentheses following the symbol $f^{(n)}$, here x. If, for example, we write $f^{(n)}(x^2)$, this means the nth derivative of $f(x^2)$ with respect to the variable x^2.

The dimensions of higher derivatives are easily found. The second derivative is equal to the limit of $\left(\Delta \dfrac{dy}{dx}\right)\Big/\Delta x$, and if y is of dimensions n relative to x, dy/dx is of dimensions $n-1$. Hence, by Art. 64. 2, $\left(\Delta \dfrac{dy}{dx}\right)\Big/\Delta x$ is of dimensions $n-2$ and, as in the case of the first derivative, considered in Art. 67, the limit of this expression is of the same dimensions. Thus d^2y/dx^2 is of dimensions

[1] The notation (6) is due to Leibniz. The student should note carefully that the forms $\dfrac{d^2y}{d^2x}, \dfrac{d^3y}{d^3x}, \ldots, \dfrac{d^ny}{d^nx}$ are *not* used; they have not been suggested either by the meaning of the derivatives or by the condensation of more self-explanatory forms.

[2] Many writers employ the letter D for $\dfrac{d}{dx}$ and write $Dy, D^2y, \ldots, D^ny$ for the derivatives. We prefer not to use this notation in the elementary discussion.

$n-2$ relative to x. In particular, if x, y are of the same dimensions, d^2y/dx^2 is of dimensions -1.

Ex. 1. *The function* x^m. We now introduce the above notations for the successive derivatives, which have been considered above. Denoting the function by y or $f(x)$, we have

$$\frac{dy}{dx} \equiv y^{(1)} \equiv f^{(1)}(x) \equiv f'(x) = mx^{m-1}, \quad \text{.........................(9)}$$

$$\frac{d^2y}{dx^2} \equiv y^{(2)} \equiv f^{(2)}(x) \equiv f''(x) = m(m-1)x^{m-2}, \quad \text{.............(10)}$$

and the general result is expressed by

$$\frac{d^ny}{dx^n} \equiv y^{(n)} \equiv f^{(n)}(x) = m(m-1)\ldots(m-n+1)x^{m-n}, \quad \text{.....(11)}$$

where n may have any positive integral value unless m is a positive integer. If m is a positive integer,

$$\frac{d^nx^m}{dx^n} = \frac{m!}{(m-n)!}x^{m-n}, \quad (n \leqslant m), \quad \text{......................(12)}$$

$$\frac{d^nx^m}{dx^n} = 0, \quad (n > m). \quad \text{.......................................(13)}$$

The particular case of (12) when $n=m$, namely

$$\frac{d^mx^m}{dx^m} = m!, \quad \text{...(14)}$$

may be specially noticed.

Ex. 2. *The functions* $1/x$, $1/x^m$. The successive derivatives of these functions are included in the above result (11) but it is convenient to state the forms of their general derivatives separately. Putting $m = -1$ in (11), we get

$$\left(\frac{d}{dx}\right)^n \frac{1}{x} = (-1)^n \frac{n!}{x^{n+1}}, \quad \text{.............................(15)}$$

while if we write $-m$ for m in (11), we get

$$\left(\frac{d}{dx}\right)^n \frac{1}{x^m} = (-1)^n \frac{m(m+1)\ldots(m+n-1)}{x^{m+n}}. \quad \text{..............(16)}$$

When m is a positive integer, this takes the form

$$\left(\frac{d}{dx}\right)^n \frac{1}{x^m} = (-1)^n \frac{(m+n-1)!}{(m-1)!} \frac{1}{x^{m+n}}. \quad \text{..................(17)}$$

If we desire to proceed directly, without referring to the results of Ex. 1, we write, in the case of the function $1/x$,

$$\frac{dy}{dx} = -\frac{1}{x^2}, \quad \frac{d^2y}{dx^2} = \frac{2}{x^3}, \quad \frac{d^3y}{dx^3} = -\frac{3 \cdot 2}{x^4}, \quad \text{......................(18)}$$

the law of formation of the successive derivatives being obvious. Similarly for the function $1/x^m$.

Ex. 3. *The function* $A/(x-a)^m$, (*m a positive integer*). This is a simple extension of the case $1/x^m$, and according to (17) the general derivative is easily seen to be given by

$$\left(\frac{d}{dx}\right)^n \frac{A}{(x-a)^m} = (-1)^n \frac{(m+n-1)!}{(m-1)!} \frac{A}{(x-a)^{m+n}}. \quad \text{.............(19)}$$

With the aid of this result we are able to write down the nth derivative of a rational algebraic function of the form $f(x)/\varphi(x)$, where $f(x)$, $\varphi(x)$ are polynomials in x, $\varphi(x)$ being given as the product of *real linear factors*, some of which may be repeated. For it can be shewn that if $\varphi(x)$ is of the pth degree and contains r distinct linear factors, the typical one $x-a_k$ occurring as a power of index m_k, where $\sum_{k=1}^{r} m_k = p$, we may write

$$\frac{f(x)}{\varphi(x)} = \sum_{k=1}^{r} \left\{ \frac{A_{m_k}}{(x-a_k)^{m_k}} + \frac{A_{m_k-1}}{(x-a_k)^{m_k-1}} + \ldots + \frac{A_1}{x-a_k} \right\} + \psi(x), \quad \text{........(20)}$$

where the A's are constants and $\psi(x)$ is a polynomial which vanishes if the degree of $f(x)$ is less than that of $\varphi(x)$.

Without, however, appealing to the theory of elementary (partial) fractions,[1] the following Examples illustrate the intuitive method for the decomposition of simple rational functions of the kind here contemplated.

Ex. 4. *The function* $\dfrac{1}{x^2(x-1)}$. The identity

$$x-(x-1)=1 \quad \text{...................................(21)}$$

gives, on division by $x(x-1)$ and $x^2(x-1)$, in turn,

$$\frac{1}{x-1} - \frac{1}{x} = \frac{1}{x(x-1)}, \quad \frac{1}{x(x-1)} - \frac{1}{x^2} = \frac{1}{x^2(x-1)}, \quad \text{..............(22)}$$

and hence we have

$$\frac{1}{x^2(x-1)} = \frac{1}{x-1} - \frac{1}{x} - \frac{1}{x^2}. \quad \text{.............................(23)}$$

Employing the results (15), (17), (19), we thus have

$$\begin{aligned}\left(\frac{d}{dx}\right)^n \frac{1}{x^2(x-1)} &= (-1)^n \left\{ \frac{n!}{(x-1)^{n+1}} - \frac{n!}{x^{n+1}} - \frac{(n+1)!}{x^{n+2}} \right\} \\ &= (-1)^n n! \left\{ \frac{1}{(x-1)^{n+1}} - \frac{x+n+1}{x^{n+2}} \right\}. \quad \text{...............(24)}\end{aligned}$$

Ex. 5. Find the nth derivative of $x/(x^2-a^2)$. [Start with the identity $(x-a)+(x+a)=2x$, and divide by $2(x^2-a^2)$.]

87. A Useful Transformation of Second Derivatives.

We now prove the analytical theorem

$$\frac{d^2y}{dx^2} = \frac{d}{dy}\left\{\frac{1}{2}\left(\frac{dy}{dx}\right)^2\right\}, \quad \text{..........................(1)}$$

[1] This theory is given in detail in Art. 241, Chap. X.

which is of common occurrence and which corresponds, in particular, to the transformation of the acceleration of a moving point from a time-rate to a distance-rate, as mentioned on p. 178. The first step is to write

$$\frac{d^2y}{dx^2}=\frac{dg}{dx}, \quad\text{.................................(2)}$$

g standing for the first derived function dy/dx. Regarding g now as a function of y, which in turn is a function of x, the rule for the derivative of a function of a function gives

$$\frac{dg}{dx}=\frac{dg}{dy}\frac{dy}{dx}=g\frac{dg}{dy}=\frac{d}{dy}\left(\tfrac{1}{2}g^2\right), \quad\text{......................(3)}$$

which leads at once to the theorem as stated in the form (1). We have thus changed the variable with respect to which the second differentiation is performed from x to y.

In the kinematical case referred to, the transformation takes the form

$$\alpha=\frac{dv}{dt}=\frac{dv}{dx}\frac{dx}{dt}=v\frac{dv}{dx}=\frac{d}{dx}\left(\tfrac{1}{2}v^2\right), \quad\text{..........................(4)}$$

which expresses the acceleration as the distance-rate of increase of the half-square of the velocity.

Ex. 1. *The relation* $y^2=a+bx+cx^2$. We have $2y\dfrac{dy}{dx}=b+2cx$, and hence

$$\left(\frac{dy}{dx}\right)^2=\frac{b^2+4bcx+4c^2x^2}{4y^2}=\frac{b^2+4c(y^2-a)}{4y^2}=\frac{b^2-4ac}{4y^2}+c. \quad\text{..........(5)}$$

Therefore $$\frac{d^2y}{dx^2}=\frac{d}{dy}\left\{\frac{1}{2}\left(\frac{dy}{dx}\right)^2\right\}=-\frac{b^2-4ac}{4y^3}=-\frac{b^2-4ac}{4(a+bx+cx^2)^{3/2}}. \quad\text{...........(6)}$$

Ex. 2. Shew that $\dfrac{d^3y}{dx^3}=g\dfrac{dg}{dy}\dfrac{d}{dg}\left(g\dfrac{dg}{dy}\right)$. ....(7)

Ex. 3. Prove that if the velocity v is given by $v^2=V^2(1-x^2/a^2)$, (V, a constants), the acceleration is $-V^2x/a^2$.

88. Higher Derivatives in the Case of Implicit Functions.

The general treatment of the higher derivatives in this case belongs to a more advanced stage, but if the existence of the implicit function and its successive derivatives is assumed, the values of the latter can be conveniently obtained in succession by the process described in Arts. 81, 82.

Suppose, at first, that the implicit function y is defined by the equation

$$\sum_{r,s=1}^{m,n} A_{rs}f_r(x)\varphi_s(y)=0. \quad\text{..........................(1)}$$

On differentiating this equation with respect to x, we get, as in Art. 81,

$$\sum_{r,s=1}^{m,n} A_{rs} f_r'(x)\varphi_s(y) + \sum_{r,s=1}^{m,n} A_{rs} f_r(x)\varphi_s'(y)\frac{dy}{dx} = 0. \quad\text{.............(2)}$$

On differentiating a second time we get

$$\sum_{r,s=1}^{m,n} A_{rs} f_r''(x)\varphi_s(y) + 2\sum_{r,s=1}^{m,n} A_{rs} f_r'(x)\varphi_s'(y)\frac{dy}{dx}$$
$$+\sum_{r,s=1}^{m,n} A_{rs} f_r(x)\varphi_s''(y)\left(\frac{dy}{dx}\right)^2 + \sum_{r,s=1}^{m,n} A_{rs} f_r(x)\varphi_s'(y)\frac{d^2y}{dx^2} = 0, \quad\text{...(3)}$$

and this gives d^2y/dx^2. The process can be repeated any number of times to obtain higher derivatives.

If the relation connecting x and y has not the particular form assumed, the result stated in Art. 82 can always be used. Denoting the relation by $f(x, y) = 0$, a first differentiation with respect to x gives

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0. \quad\text{..............................(4)}$$

In order to make it clear that $\partial f/\partial x$, $\partial f/\partial y$ are functions of both x and y, we write (4) in the form

$$\varphi(x, y) + \psi(x, y)\frac{dy}{dx} = 0. \quad\text{.........................(5)}$$

A second differentiation then gives

$$\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{dy}{dx} + \left(\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}\right)\frac{dy}{dx} + \psi\frac{d^2y}{dx^2} = 0, \quad\text{.............(6)}$$

which determines d^2y/dx^2, and so on.

Since $\phi(x, y) = \dfrac{\partial f}{\partial x}$, we can write $\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right)$ or $\dfrac{\partial^2 f}{\partial x^2}$ instead of $\dfrac{\partial\phi}{\partial x}$ and, similarly, $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$ or $\dfrac{\partial^2 f}{\partial y\,\partial x}$ for $\dfrac{\partial\phi}{\partial y}$, $\dfrac{\partial^2 f}{\partial x\,\partial y}$ for $\dfrac{\partial\psi}{\partial x}$ and $\dfrac{\partial^2 f}{\partial y^2}$ for $\dfrac{\partial\psi}{\partial y}$. We can thus write (6) in the form

$$\frac{\partial^2 f}{\partial x^2} + \left(\frac{\partial^2 f}{\partial y\,\partial x} + \frac{\partial^2 f}{\partial x\,\partial y}\right)\frac{dy}{dx} + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2 + \frac{\partial f}{\partial y}\frac{d^2y}{dx^2} = 0. \quad\text{.........(7)}$$

It can be shewn that, under normal conditions,

$$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y}. \quad\text{............................(8)}$$

The student can verify immediately that this relation holds for functions of the type in (1).

Ex. 1. *The equation* $x^2+y^2=a^2$. The process of successive differentiation of the equation here leads to the equations

$$\left.\begin{array}{l} x+y\dfrac{dy}{dx}=0,\quad 1+\left(\dfrac{dy}{dx}\right)^2+y\dfrac{d^2y}{dx^2}=0,\quad 3\dfrac{dy}{dx}\dfrac{d^2y}{dx^2}+y\dfrac{d^3y}{dx^3}=0, \\ 3\left(\dfrac{d^2y}{dx^2}\right)^2+4\dfrac{dy}{dx}\dfrac{d^3y}{dx^3}+y\dfrac{d^4y}{dx^4}=0,\quad 10\dfrac{d^2y}{dx^2}\dfrac{d^3y}{dx^3}+5\dfrac{dy}{dx}\dfrac{d^4y}{dx^4}+y\dfrac{d^5y}{dx^5}=0, \end{array}\right\}\quad\text{......(9)}$$

whence we find, in succession,

$$\left.\begin{array}{l} \dfrac{dy}{dx}=-\dfrac{x}{y},\quad \dfrac{d^2y}{dx^2}=-\dfrac{a^2}{y^3},\quad \dfrac{d^3y}{dx^3}=-\dfrac{3a^2x}{y^5}, \\ \dfrac{d^4y}{dx^4}=-\dfrac{3a^2(4x^2+a^2)}{y^7},\quad \dfrac{d^5y}{dx^5}=-\dfrac{15a^2x(4x^2+3a^2)}{y^9}. \end{array}\right\}\quad\text{.........(10)}$$

Ex. 2. *The function* $y=1/(a^2+x^2)$. Writing this in the implicit form

$$(a^2+x^2)y-1=0, \quad\text{.................................(11)}$$

the application of the process described above leads to the equations

$$2xy+(a^2+x^2)\frac{dy}{dx}=0, \quad\text{........................(12)}$$

$$2y+4x\frac{dy}{dx}+(a^2+x^2)\frac{d^2y}{dx^2}=0, \quad\text{........................(13)}$$

$$6\frac{dy}{dx}+6x\frac{d^2y}{dx^2}+(a^2+x^2)\frac{d^3y}{dx^3}=0, \quad\text{........................(14)}$$

and so on. The forms of these equations suggest that, in the general case,

$$n(n-1)\frac{d^{n-2}y}{dx^{n-2}}+2nx\frac{d^{n-1}y}{dx^{n-1}}+(a^2+x^2)\frac{d^ny}{dx^n}=0, \quad\text{.................(15)}$$

and that this is the correct form is easily verified by the method of Induction. Thus, differentiating (15), we get

$$\{n(n-1)+2n\}\frac{d^{n-1}y}{dx^{n-1}}+(2nx+2x)\frac{d^ny}{dx^n}+(a^2+x^2)\frac{d^{n+1}y}{dx^{n+1}}=0, \quad\text{......(16)}$$

which is obtainable from (15) by changing n into $n+1$. Since this result is true for the case of $n=2$, we infer that it is true generally.

By actual substitution we find

$$\left.\begin{array}{l} \dfrac{dy}{dx}=-\dfrac{1}{(a^2+x^2)^2}2x,\quad \dfrac{d^2y}{dx^2}=\dfrac{2!}{(a^2+x^2)^3}(3x^2-a^2), \\ \dfrac{d^3y}{dx^3}=-\dfrac{3!}{(a^2+x^2)^4}(4x^3-4xa^2),\quad \dfrac{d^4y}{dx^4}=\dfrac{4!}{(a^2+x^2)^5}(5x^4-10x^2a^2+a^4). \end{array}\right\}\quad(17)$$

The general result of the same form, which can be readily verified by the method of Induction, is

$$\frac{d^ny}{dx^n}=(-1)^n\frac{n!}{(a^2+x^2)^{n+1}}\left\{(n+1)x^n-\binom{n+1}{3}x^{n-2}a^2+\binom{n+1}{5}x^{n-4}a^4-\dots\right\}. \quad\text{...(18)}$$

The successive derivatives of $1/(1+x^2)$ are given in a more compact form in Chap. VI, Art. 152. 2; those of $1/(a^2+x^2)$ are immediately deduced by putting x/a for x.

Ex. 3. *The function* $y = x/(a^2 + x^2)$. Writing this in the form

$$(a^2 + x^2)y - x = 0, \quad \text{.............................(19)}$$

and proceeding as in Ex. 2, we obtain the general results

$$n(n-1)\frac{d^{n-2}y}{dx^{n-2}} + 2nx\frac{d^{n-1}y}{dx^{n-1}} + (a^2 + x^2)\frac{d^ny}{dx^n} = 0, \ (n > 1), \quad \text{.........(20)}$$

$$\frac{d^ny}{dx^n} = (-1)^n \frac{n!}{(a^2 + x^2)^{n+1}}\left\{x^{n+1} - \binom{n+1}{2}x^{n-1}a^2 + \binom{n+1}{4}x^{n-3}a^4 - \dots\right\}. \quad \text{...(21)}$$

From the results of this and the preceding Example we are able to write down the nth derivative of an expression of the form

$$y = \frac{Ax + B}{x^2 + ax + b}, \quad (\tfrac{1}{4}a^2 < b). \quad \text{...........................(22)}$$

For $x^2 + ax + b = u^2 + k^2$, where $u = x + \frac{1}{2}a$, $k = \sqrt{(b - \frac{1}{4}a^2)}$, and hence

$$y = \frac{A(u - \frac{1}{2}a) + B}{u^2 + k^2} = A\frac{u}{u^2 + k^2} + (B - \tfrac{1}{2}aA)\frac{1}{u^2 + k^2}. \quad \text{............(23)}$$

Since $du/dx = 1$, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}, \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{du}\right) = \frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx} = \frac{d^2y}{du^2}, \quad \text{...........(24)}$$

and so on, and hence, if the nth derivative of y with respect to u is found, as in (18) and (21), the nth derivative of y with respect to x is obtained from it merely by replacing u by $x + \frac{1}{2}a$.

It is shewn in Art. 241 that if a rational function $f(x)/\varphi(x)$, containing an *unrepeated* quadratic factor of the form $x^2 + ax + b$, $(\frac{1}{4}a^2 < b)$, in the denominator $\varphi(x)$, is decomposed into elementary (partial) fractions, the component having that factor as denominator is of the form (22). With the aid of the result (19) of Ex. 3, Art. 86, we can thus write down the nth derivative of a rational function which has been decomposed into the sum of a polynomial and elementary fractions, provided that $\varphi(x)$ does not possess repeated quadratic factors.[1]

As in Art. 86, Exs. 4, 5, the intuitive method may be employed in simple cases to determine the elementary fractions.

Ex. 4. *The function* $y = \dfrac{1}{x(x^2 + x + 1)}$. If we start with the identity

$$(x^2 + x + 1) - x - x^2 = 1, \quad \text{.............................(25)}$$

division by $x(x^2 + x + 1)$ gives the required decomposition in the form

$$\frac{1}{x(x^2 + x + 1)} = \frac{1}{x} - \frac{1}{x^2 + x + 1} - \frac{x}{x^2 + x + 1}. \quad \text{..................(26)}$$

Writing $x^2 + x + 1 = (x + \frac{1}{2})^2 + \frac{3}{4} = u^2 + k^2$, where $u = x + \frac{1}{2}$, $k = \frac{1}{2}\sqrt{3}$, we have

$$y = \frac{1}{x} - \frac{1}{u^2 + k^2} - \frac{u - \frac{1}{2}}{u^2 + k^2} = \frac{1}{x} - \frac{1}{2}\frac{1}{u^2 + k^2} - \frac{u}{u^2 + k^2}. \quad \text{..............(27)}$$

[1] For the case of repeated quadratic factors, see Ex. 11, (iii), Set XVI, at the end of the present Chapter.

Since here $d^ny/dx^n = d^ny/du^n$, we have

$$\frac{d^ny}{dx^n} = (-1)^n \frac{n!}{x^{n+1}} - \frac{1}{2}(-1)^n \frac{n!}{(u^2+k^2)^{n+1}} \left\{(n+1)u^n - \binom{n+1}{3} u^{n-2}k^2 + \dots\right\}$$
$$- (-1)^n \frac{n!}{(u^2+k^2)^{n+1}} \left\{u^{n+1} - \binom{n+1}{2} u^{n-1}k^2 + \dots\right\}$$
$$= (-1)^n \frac{n!}{x^{n+1}} - (-1)^n \frac{n!}{(x^2+x+1)^{n+1}} \Big[(x+\tfrac{1}{2})^{n+1} - \tfrac{3}{4}\binom{n+1}{2}(x+\tfrac{1}{2})^{n-1} + \dots$$
$$+ \tfrac{1}{2}\left\{(n+1)(x+\tfrac{1}{2})^n - \tfrac{3}{4}\binom{n+1}{3}(x+\tfrac{1}{2})^{n-2} + \dots\right\}\Big]. \quad \dots(28)$$

89. Change of Independent Variable in Higher Derivatives.

We next investigate the effect of certain changes of independent variable on derivatives of order up to the third.

89. 1. *Function of a function.* Let $y = f(t)$, where $t = \varphi(x)$, so that

$$y = f\{\varphi(x)\} = \chi(x), \text{ say.} \quad \dots(1)$$

It is required to express the derivatives of y with respect to x in terms of the successive derivatives of y with respect to the intermediary variable t and those of t with respect to x. We have

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}, \quad \dots(2)$$

and, differentiating again, using the product-rule for the right-hand side,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dt}\right)\frac{dt}{dx} + \frac{dy}{dt}\frac{d}{dx}\left(\frac{dt}{dx}\right)$$
$$= \frac{d^2y}{dt^2}\left(\frac{dt}{dx}\right)^2 + \frac{dy}{dt}\frac{d^2t}{dx^2}. \quad \dots(3)$$

Similarly,

$$\frac{d^3y}{dx^3} = \frac{d^3y}{dt^3}\left(\frac{dt}{dx}\right)^3 + 3\frac{d^2y}{dt^2}\frac{dt}{dx}\frac{d^2t}{dx^2} + \frac{dy}{dt}\frac{d^3t}{dx^3}, \quad \dots(4)$$

and the process can be repeated.[1]

89. 2. *Parametric specification of x and y.* Suppose, now, that instead of the explicit form $t = \varphi(x)$ connecting the intermediary variable t with the independent variable x, t is given implicitly in terms of x by an equation of the form $x = \psi(t)$. Then we have the equations

$$x = \psi(t), \quad y = f(t), \quad \dots(5)$$

so that t may now be regarded as an independent variable or parameter. We proceed to determine the successive derivatives of y with respect to x in terms of the successive derivatives of y and x with respect to t.

The results of the preceding Sub-article are immediately applicable

[1] For the expression of the nth derivative, see Ex. 11, (i), Set XVI.

for this purpose on the assumption that the successive derivatives of y with respect to x exist. For if we interchange (the rôles of) x and t, the equations (2), (3), (4) determine dy/dx, d^2y/dx^2, d^3y/dx^3 in succession in terms of the derivatives of y and x with respect to t. It is, however, more satisfactory to start afresh from the formula

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}, \qquad (6)$$

demonstrated in Art. 79. The right-hand side of this equation is a function of t which can be differentiated repeatedly with respect to that variable so long as the successive derivatives of x and y with respect to t exist. It can therefore also be differentiated repeatedly with respect to x, by the rule for the derivative of a function of a function, because dt/dx exists. Thus we have

$$\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}=\frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right)\frac{1}{dx/dt}=\frac{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{d^2x}{dt^2}\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^3}, \qquad (7)$$

$$\frac{d^3y}{dx^3}=\frac{d}{dt}\left(\frac{d^2y}{dx^2}\right)\frac{1}{dx/dt}=\frac{\left(\frac{dx}{dt}\frac{d^3y}{dt^3}-\frac{d^3x}{dt^3}\frac{dy}{dt}\right)\frac{dx}{dt}-3\left(\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{d^2x}{dt^2}\frac{dy}{dt}\right)\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^5}, \qquad (8)$$

and so on.

89. 3. *Interchange of dependent and independent variable.* We are here concerned with the determination of the successive derivatives of x with respect to y upon the inversion of a relation of the form $y=\psi(x)$, so that y becomes the independent variable. This problem is a particular case of that of the last Sub-article obtained by taking the relation between y and t to be simply $y=t$ and then interchanging (the rôles of) x and y. It is, however, better to start with the relation

$$\frac{dx}{dy}=\frac{1}{dy/dx}, \qquad (9)$$

demonstrated in Art. 78. Differentiating with respect to y, we find

$$\frac{d^2x}{dy^2}=\frac{d}{dx}\left(\frac{dx}{dy}\right)\frac{dx}{dy}=\frac{d}{dx}\left(\frac{1}{dy/dx}\right)\frac{1}{dy/dx}=-\frac{d^2y/dx^2}{(dy/dx)^3}, \qquad (10)$$

$$\frac{d^3x}{dy^3}=-\frac{d}{dx}\left\{\frac{d^2y/dx^2}{(dy/dx)^3}\right\}\frac{1}{dy/dx}=-\frac{\frac{d^3y}{dx^3}\frac{dy}{dx}-3\left(\frac{d^2y}{dx^2}\right)^2}{\left(\frac{dy}{dx}\right)^5}, \qquad (11)$$

and so on. The rôles of x and y can, of course, be interchanged in these equations.

The equations (10), (11) can be put into more symmetrical forms. Thus, assuming dy/dx positive, (10) can be written

$$\frac{1}{(dy/dx)^{3/2}}\frac{d^2y}{dx^2}=-\frac{1}{(dx/dy)^{3/2}}\frac{d^2x}{dy^2}, \quad\text{.....................(12)}$$

while if dy/dx is negative, we must write

$$\frac{1}{(-dy/dx)^{3/2}}\frac{d^2y}{dx^2}=\frac{1}{(-dx/dy)^{3/2}}\frac{d^2x}{dy^2}. \quad\text{.....................(13)}$$

The symmetrical forms of (11) are now most easily obtained by differentiating the last two equations with respect to x.

Ex. 1. Find the transforms of dy/dx, d^2y/dx^2 when the independent variable x is expressed in terms of t by the relation $x=\surd(1-t^2)$.

We employ the results (6), (7), noticing that

$$\frac{dx}{dt}=-\frac{t}{(1-t^2)^{1/2}}, \quad \frac{d^2x}{dt^2}=-\frac{1}{(1-t^2)^{3/2}}. \quad\text{.....................(14)}$$

We find

$$\frac{dy}{dx}=-\frac{(1-t^2)^{1/2}}{t}\frac{dy}{dt}, \quad\text{..............................(15)}$$

$$\frac{d^2y}{dx^2}=\frac{-\dfrac{t}{(1-t^2)^{1/2}}\dfrac{d^2y}{dt^2}+\dfrac{1}{(1-t^2)^{3/2}}\dfrac{dy}{dt}}{-\dfrac{t^3}{(1-t^2)^{3/2}}}=\frac{1-t^2}{t^2}\frac{d^2y}{dt^2}-\frac{1}{t^3}\frac{dy}{dt}. \quad\text{...........(16)}$$

The student should verify that under this substitution, the transform of the expression $(1-x^2)\dfrac{d^2y}{dx^2}-(3x-1/x)\dfrac{dy}{dx}$ is $(1-t^2)\dfrac{d^2y}{dt^2}-(3t-1/t)\dfrac{dy}{dt}$.

90. Successive Derivatives of a Product. Leibniz's Theorem.

We now investigate an important theorem which gives the nth derivative of the product of two functions in terms of the successive derivatives, up to the same order, of the separate functions.

Let u, v be two functions of x and denote their product by y, so that $y=uv$. Each of the functions u, v is supposed to possess derivatives up to the order of the highest derivative of y which is to be found. Then we have, for the first derivative of y,

$$\frac{dy}{dx}=\frac{du}{dx}v+u\frac{dv}{dx}. \quad\text{..........................(1)}$$

When the second derivative of y is sought, each term on the right-hand side is, by supposition, the product of two differentiable functions. Accordingly we have

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{du}{dx}v\right)+\frac{d}{dx}\left(u\frac{dv}{dx}\right)=\left(\frac{d^2u}{dx^2}v+\frac{du}{dx}\frac{dv}{dx}\right)+\left(\frac{du}{dx}\frac{dv}{dx}+u\frac{d^2v}{dx^2}\right)$$

$$=\frac{d^2u}{dx^2}v+2\frac{du}{dx}\frac{dv}{dx}+u\frac{d^2v}{dx^2}, \quad\text{............(2)}$$

and, proceeding in the same way, we find

$$\frac{d^3y}{dx^3}=\frac{d^3u}{dx^3}v+3\frac{d^2u}{dx^2}\frac{dv}{dx}+3\frac{du}{dx}\frac{d^2v}{dx^2}+u\frac{d^3v}{dx^3}. \quad \text{.................(3)}$$

If on the right-hand side of equation (3) we replace each derivative of u, v by the power of u, v, respectively, whose index is equal to the order [1] of the derivative, the expression becomes the binomial expansion of $(u+v)^3$, and corresponding statements hold for equations (1), (2), the transformed expressions being $u+v$, $(u+v)^2$, respectively. We now prove, by Induction, that the corresponding statement is true for the general derivative.

We assume that, for a certain value of n,

$$\frac{d^ny}{dx^n}=\frac{d^nu}{dx^n}v+\binom{n}{1}\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx}+\ldots+\binom{n}{r}\frac{d^{n-r}u}{dx^{n-r}}\frac{d^rv}{dx^r}+\ldots+u\frac{d^nv}{dx^n}, \quad \ldots(4)$$

the corresponding transformed expression being $(u+v)^n$. On differentiating with respect to x, we get

$$\frac{d^{n+1}y}{dx^{n+1}}=\frac{d^{n+1}u}{dx^{n+1}}v+\left\{1+\binom{n}{1}\right\}\frac{d^nu}{dx^n}\frac{dv}{dx}+\ldots$$
$$+\left\{\binom{n}{r-1}+\binom{n}{r}\right\}\frac{d^{n-r+1}u}{dx^{n-r+1}}\frac{d^rv}{dx^r}+\ldots+u\frac{d^{n+1}v}{dx^{n+1}}. \quad \ldots(5)$$

Employing the known result $\binom{n}{r-1}+\binom{n}{r}=\binom{n+1}{r}$, (5) becomes

$$\frac{d^{n+1}y}{dx^{n+1}}=\frac{d^{n+1}u}{dx^{n+1}}v+\binom{n+1}{1}\frac{d^nu}{dx^n}\frac{dv}{dx}+\ldots$$
$$+\binom{n+1}{r}\frac{d^{n+1-r}u}{dx^{n+1-r}}\frac{d^rv}{dx^r}+\ldots+u\frac{d^{n+1}v}{dx^{n+1}}, \quad \text{..............(6)}$$

and this differs from (4) only by $n+1$ taking the place of n. Thus, if equation (4) holds for any one value of n, it holds for the next greater value. Since it is true for the values 1, 2, 3 of n, we infer it to be true for any positive integral value of n.

The result expressed by (4) is known as *Leibniz's Theorem*.

Ex. 1. *The function ux.* If we substitute x for v in (4), noticing that now $dv/dx=1$, $d^nv/dx^n=0$ for $n>1$, we get the result

$$\frac{d^n(ux)}{dx^n}=\frac{d^nu}{dx^n}x+n\frac{d^{n-1}u}{dx^{n-1}}. \quad \text{.............................(7)}$$

Ex. 2. Shew that

$$\frac{d^n(ux^2)}{dx^n}=\frac{d^nu}{dx^n}x^2+2n\frac{d^{n-1}u}{dx^{n-1}}x+n(n-1)\frac{d^{n-2}u}{dx^{n-2}}. \quad \text{.................(8)}$$

[1] The functions u, v themselves are here regarded as derivatives of zero order.

Ex. 3. *Continued differentiation of an equation.* Let the given equation be

$$2xy+(1+x^2)\frac{dy}{dx}=0. \quad \text{......(9)}$$

If we assume that the $(n+1)$th derivative of the function y exists, the application of Leibniz's theorem here gives

$$2x\frac{d^ny}{dx^n}+2n\frac{d^{n-1}y}{dx^{n-1}}+(1+x^2)\frac{d^{n+1}y}{dx^{n+1}}+2nx\frac{d^ny}{dx^n}+n(n-1)\frac{d^{n-1}y}{dx^{n-1}}=0,$$

i.e.

$$(1+x^2)\frac{d^{n+1}y}{dx^{n+1}}+2(n+1)x\frac{d^ny}{dx^n}+n(n+1)\frac{d^{n-1}y}{dx^{n-1}}=0. \quad \text{....(10)}$$

See also Ex. 2, p. 185.

Ex. 4. Find the nth derivative of $u(x-a)^r$, (r a positive integer, $n \geqslant r$).

As in Ex. 1, p. 181, we have

$$\frac{d^k}{dx^k}(x-a)^r=\frac{r!}{(r-k)!}(x-a)^{r-k}, \quad (k\leqslant r), \quad \text{but} \quad \frac{d^k}{dx^k}(x-a)^r=0, \quad (k>r). \quad \text{...(11)}$$

Thus the highest derivative of $(x-a)^r$ which occurs in the expression for the required nth derivative is the rth, and we have

$$\frac{d^ny}{dx^n}=\frac{d^nu}{dx^n}(x-a)^r+\ldots+\binom{n}{k}\frac{d^{n-k}u}{dx^{n-k}}\frac{r!}{(r-k)!}(x-a)^{r-k}+\ldots+\binom{n}{r}\frac{d^{n-r}u}{dx^{n-r}}r!$$

$$=\frac{d^nu}{dx^n}(x-a)^r+nr\frac{d^{n-1}u}{dx^{n-1}}(x-a)^{r-1}+\ldots+\frac{n!}{(n-k)!\,k!}\frac{r!}{(r-k)!}\frac{d^{n-k}u}{dx^{n-k}}(x-a)^{r-k}$$

$$+\ldots+\frac{n!}{(n-r+1)!}r\frac{d^{n-r+1}u}{dx^{n-r+1}}(x-a)+\frac{n!}{(n-r)!}\frac{d^{n-r}u}{dx^{n-r}}, \quad \text{......(12)}$$

the number of terms in the expression being $r+1$ and the general term written being the $(k+1)$th counting from the beginning.

It is convenient to write this result in the reverse order, thus

$$\frac{d^ny}{dx^n}=\frac{n!}{(n-r)!}\frac{d^{n-r}u}{dx^{n-r}}+\ldots+\frac{n!}{(n-r+k)!\,(r-k)!}\frac{r!}{k!}(x-a)^k\frac{d^{n-r+k}u}{dx^{n-r+k}}$$

$$+\ldots+(x-a)^r\frac{d^nu}{dx^n}, \quad \text{......(13)}$$

the general term now written being the $(k+1)$th counting from the beginning and coinciding with the $(r-k+1)$th term counting from the end in (12).

Ex. 5. Find the nth derivative of $(x^2-a^2)^r$, (r a positive integer).

We write $(x^2-a^2)^r=(x+a)^r(x-a)^r$. Observing that

$$\frac{d^{n-r+k}}{dx^{n-r+k}}(x+a)^r=\frac{r!}{(2r-n-k)!}(x+a)^{2r-n-k}, \quad (k\leqslant 2r-n), \quad \text{......(14)}$$

the general term in the above expression (13) becomes

$$\frac{n!}{(n-r+k)!\,(r-k)!}\frac{r!}{k!}\frac{r!}{(2r-n-k)!}(x-a)^k(x+a)^{2r-n-k}. \quad \text{......(15)}$$

The required derivative is obtained by summing all the terms obtained from (15) on allowing k to range from 0 to $2r-n$.

This Example is more simply treated by expanding $(x^2-a^2)^r$ before differentiating.

Ex. 6. Find the nth derivative of $\left(\frac{a-x}{x}\right)^r$, ($r$ a positive integer).

We write

$$\left(\frac{a-x}{x}\right)^r = (-1)^r (x-a)^r \frac{1}{x^r}. \qquad \text{.........................(16)}$$

Quoting from (17), Ex. 2, Art. 86, we have

$$\frac{d^{n-r+k}}{dx^{n-r+k}} \frac{1}{x^r} = (-1)^{n-r+k} \frac{(n+k-1)!}{(r-1)!} \frac{1}{x^{n+k}}, \qquad \text{..................(17)}$$

and hence, from (13), the required derivative is given by

$$\frac{d^n y}{dx^n} = r \sum_{k=0}^{r} (-1)^{n+k} \frac{(n+k-1)!\,n!}{(n-r+k)!\,(r-k)!\,k!} \frac{(x-a)^k}{x^{n+k}}. \qquad \text{............(18)}$$

This Example is more simply treated by expanding $(1-a/x)^r$ before differentiating.

Ex. 7. If $f(x)$ is a polynomial of degree n in x and a is any number, show that

$$f(x) = f(a) + (x-a) f^{(1)}(a) + \frac{1}{2!}(x-a)^2 f^{(2)}(a) + \dots + \frac{1}{n!}(x-a)^n f^{(n)}(a). \quad \text{...(19)}$$

In the polynomial $f(x)$ put $x = a + (x-a)$. The term x^r can then be expressed as a polynomial of degree r in $x-a$ by developing $\{a+(x-a)\}^r$ in powers of $x-a$ by the Binomial Theorem. Thus $f(x)$ can be expressed as a polynomial of degree n in $x-a$. Let this polynomial be $A_0 + (x-a)A_1 + \dots + \frac{(x-a)^r}{r!} A_r + \dots + \frac{(x-a)^n}{n!} A_n$. Then

$$f^{(r)}(x) = A_r + (x-a)A_{r+1} + \dots + \frac{(x-a)^{n-r}}{(n-r)!} A_n, \qquad \text{..............(20)}$$

and therefore, putting $x=a$, we get

$$f^{(r)}(a) = A_r, \quad (r = 1, 2, \dots, n). \qquad \text{........................(21)}$$

This gives the result stated.

Ex. 8. If in the preceding Example $f(a)$, $f^{(1)}(a)$, ..., $f^{(r-1)}(a)$ all vanish, $f(x)$ has $(x-a)^r$ as a factor.

[When this condition is satisfied the polynomial is said to have a *zero of order* r at $x=a$, and the equation $f(x)=0$ is said to have r roots equal to a or to have an r-tuple root a. In particular, if $f(a)=0, f^{(1)}(a)=0$, there is a *double root* a, and if, in addition, $f^{(2)}(a)=0$, there is a *triple root* a.]

EXAMPLES XV

ARTS. 86-88

1. If $y = \{x + \sqrt{(x^2-1)}\}^n$, prove that $(x^2-1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n^2 y$.

2. Shew that $\frac{d^2(u/v)}{dx^2} = \left(\frac{d^2u}{dx^2} v - u \frac{d^2v}{dx^2}\right)\frac{1}{v^2} - \frac{2}{v^3}\frac{dv}{dx}\left(\frac{du}{dx} v - u\frac{dv}{dx}\right)$.

3. If the curve $y = f(x)/f'(x)$ cuts the axis Ox at $x=a$ and $f'(a) \neq 0$, prove that its gradient there is 1.

4. If $f(x)=(x-x_1)(x-x_2)\dots(x-x_n)$, prove that

$$\frac{f'(x)}{f(x)}=\sum_{r=1}^{n}\frac{1}{x-x_r},\qquad \frac{f''(x)}{f(x)}=\sum_{r=1}^{n}\sum_{s=r+1}^{n}\frac{2}{(x-x_r)(x-x_s)}.$$

5. Find the nth derivative of $f(-x)$ with respect to x, the value of $f^{(n)}x$ being supposed known.

6. Shew that if $f(x)$ is an even function, $f^{(2n+1)}(0)=0$.

7. Find the nth derivative of each of the following functions:

(i) $(1-x)^m$; (ii) $a_0x^m+a_1x^{m-1}+\dots+a_m$; (iii) $(x+a)/(x+b)$;

(iv) $\dfrac{x^3}{x+a}$; (v) $\dfrac{x^2}{(x-3)(x-1)}$; (vi) $\dfrac{x^2+px+q}{(x-a)(x-b)}$; (vii) $\dfrac{1}{x(x-1)^2}$;

(viii) $\dfrac{1}{(x-a)(x-b)(x-c)}$; (ix) $\dfrac{x^5}{(x-1)^2(x+1)}$; (x) $\dfrac{x}{1+x+2x^2}$; (xi) $\dfrac{2x+3}{x^2+2}$.

8. If $y^2=(a-x)/x$, find d^2y/dx^2 by the method of Art. 87.

9. If $x=3y-4y^3$, shew that $9\left\{(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}\right\}+y=0$.

10. Prove that $\dfrac{d^4y}{dx^4}=g\dfrac{dg}{dy}\left\{\dfrac{d}{dg}\left(g\dfrac{dg}{dy}\right)\right\}^2+\left(g\dfrac{dg}{dy}\right)^2\dfrac{d^2}{dg^2}\left(g\dfrac{dg}{dy}\right)$.

11. If $ax^2+2hxy+by^2=c$, shew that $\dfrac{d^2y}{dx^2}=-\dfrac{c(ab-h^2)}{(hx+by)^3}$.

12. A particle moves in a straight line so that the square of its velocity is a linear function of the displacement at the instant. Shew that the acceleration is constant.

13. If the square of the velocity is a quadratic function of the displacement, shew that the acceleration varies as the distance from a fixed point on the line of motion.

14. If the time is a quadratic function of the displacement, shew that the acceleration is proportional to the cube of the velocity.

15. If $v^2=a+b/x$, shew that the acceleration varies inversely as the square of the distance from a fixed point on the line of motion.

16. If the square of the displacement is a quadratic function of the time, shew that the acceleration varies inversely as the cube of the displacement.

17. Shew that if $r^2=x^2+y^2+z^2$, the equation $\dfrac{\partial^2u}{\partial x^2}+\dfrac{\partial^2u}{\partial y^2}+\dfrac{\partial^2u}{\partial z^2}=0$ is satisfied when $u=1/r$.

18. If $f(x,y)=\sum_{r=1}^{n}\varphi_r(xy)\psi_r(y)$, where φ_r, ψ_r are differentiable functions of their arguments, prove that $\dfrac{\partial^2f}{\partial x\partial y}=\dfrac{\partial^2f}{\partial y\partial x}$.

19. If $u=x^mf(y/x)$, shew that (i) $\dfrac{\partial^2u}{\partial x\partial y}=\dfrac{\partial^2u}{\partial y\partial x}$, (ii) $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=mu$,
(iii) $x^2\dfrac{\partial^2u}{\partial x^2}+2xy\dfrac{\partial^2u}{\partial x\partial y}+y^2\dfrac{\partial^2u}{\partial y^2}=m(m-1)u$.

20. If $u=f(ax^2+2hxy+by^2)$, $v=\varphi(ax^2+2hxy+by^2)$ prove that

$$\frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y}\right).$$

21. If $u=x\varphi(y/x)+\psi(y/x)$, prove that $x^2\dfrac{\partial^2u}{\partial x^2}+2xy\dfrac{\partial^2u}{\partial x\partial y}+y^2\dfrac{\partial^2u}{\partial y^2}=0$.

22. If x, y are functions of t, so that $f(x, y)$ may be regarded as a function $\varphi(t)$ of t alone, shew that

$$\frac{d^2\varphi(t)}{dt^2}=\frac{\partial^2 f(x, y)}{\partial x^2}\left(\frac{dx}{dt}\right)^2+\left\{\frac{\partial^2 f(x, y)}{\partial x\,\partial y}+\frac{\partial^2 f(x, y)}{\partial y\,\partial x}\right\}\frac{dx}{dt}\frac{dy}{dt}$$
$$+\frac{\partial^2 f(x, y)}{\partial y^2}\left(\frac{dy}{dt}\right)^2+\frac{\partial f(x, y)}{\partial x}\frac{d^2x}{dt^2}+\frac{\partial f(x, y)}{\partial y}\frac{d^2y}{dt^2},$$

and, in particular, if $f(x, y)$ is regarded as a function $\psi(x)$ of x alone, that

$$\frac{d^2\psi(x)}{dx^2}=\frac{\partial^2 f(x, y)}{\partial x^2}+\left\{\frac{\partial^2 f(x, y)}{\partial x\,\partial y}+\frac{\partial^2 f(x, y)}{\partial y\,\partial x}\right\}\frac{dy}{dx}+\frac{\partial^2 f(x, y)}{\partial y^2}\left(\frac{dy}{dx}\right)^2+\frac{\partial f(x, y)}{\partial y}\frac{d^2y}{dx^2}.$$

[These results follow by differentiating the equations given in Ex. 15, p. 174, the first with respect to t, the second with respect to x. In the equations of the present Example, $\varphi(t)$, $\psi(x)$, $f(x, y)$ are usually replaced by f.]

EXAMPLES XVI

ARTS. 89, 90

1. If $\dfrac{d^2y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}+y=0$, shew that $x\dfrac{d^2x}{dy^2}-\left(\dfrac{dx}{dy}\right)^2-xy\left(\dfrac{dx}{dy}\right)^3=0$.

2. Prove that $\dfrac{d^2y/dx^2}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}\pm\dfrac{d^2x/dy^2}{\left\{1+\left(\dfrac{dx}{dy}\right)^2\right\}^{3/2}}=0$ according as dy/dx is positive or negative.

3. If $dy/dx>0$, shew that

$$\frac{1}{\sqrt{(dy/dx)}}\frac{d}{dx}\left\{\frac{d^2y}{dx^2}\Big/\left(\frac{dy}{dx}\right)^{3/2}\right\}=-\frac{1}{\sqrt{(dx/dy)}}\frac{d}{dy}\left\{\frac{d^2x}{dy^2}\Big/\left(\frac{dx}{dy}\right)^{3/2}\right\}.$$

4. If $\dfrac{d^2y}{dx^2}+\left(\dfrac{l}{x-a}+\dfrac{m}{x-b}+\dfrac{n}{x-c}\right)\dfrac{dy}{dx}=0$ where $l+m+n=2$, shew that

$$\frac{d^2y}{d\xi^2}+\left(\frac{l}{\xi-\alpha}+\frac{m}{\xi-\beta}+\frac{n}{\xi-\gamma}\right)\frac{dy}{d\xi}=0,$$

where $x=\dfrac{A\xi+B}{C\xi+D}$, $a=\dfrac{A\alpha+B}{C\alpha+D}$, etc., and $AD-BC=1$.

5. Employ Leibniz's Theorem to find the nth derivatives of the following functions :

(i) $(x-a)^2u$; (ii) x^3u ; (iii) $x^3/(1+x)^3$; (iv) $(ax+b)^p(cx+d)^q$.

6. Prove that the function $y=\dfrac{d^n}{dx^n}(a^2+x^2)^n$ satisfies the equation

$$\frac{d}{dx}\left\{\left(a^2+x^2\right)\frac{dy}{dx}\right\}-n(n+1)y=0.$$

7. If $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=2$, shew that

$$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}}-x(2n+1)\frac{d^{n+1}y}{dx^{n+1}}-n^2\frac{d^ny}{dx^n}=0.$$

8. If $y^{1/p}+y^{-1/p}=2x$, shew that

$$(x^2-1)\frac{d^{n+2}y}{dx^{n+2}}+(2n+1)x\frac{d^{n+1}y}{dx^{n+1}}+(n^2-p^2)\frac{d^ny}{dx^n}=0.$$

9. Prove that

$$v\frac{d^n u}{dx^n}=\frac{d^n}{dx^n}(uv)-\binom{n}{1}\frac{d^{n-1}}{dx^{n-1}}\left(u\frac{dv}{dx}\right)+\binom{n}{2}\frac{d^{n-2}}{dx^{n-2}}\left(u\frac{d^2v}{dx^2}\right)-\dots+(-1)^n u\frac{d^n v}{dx^n}.$$

10. Prove that $\left(\dfrac{d}{dx}\right)^n(uvw)=\sum A_{rst}\dfrac{d^r u}{dx^r}\dfrac{d^s v}{dx^s}\dfrac{d^t w}{dx^t}$, where $r+s+t=n$, A_{rst} is the coefficient of $u^r v^s w^t$ in the expansion of $(u+v+w)^n$ in powers of u, v, w, and d^0u/dx^0 is interpreted as u, and so on.

11. Prove by the method of Induction the following results :

(i) If $y=f(u)$ where $u=\varphi(x)$, then

$$\frac{d^n y}{dx^n}=X_{n1}\frac{dy}{du}+X_{n2}\frac{d^2y}{du^2}+\dots+X_{nn}\frac{d^n y}{du^n},$$

where $r!\,X_{nr}=\dfrac{d^n u^r}{dx^n}-\dbinom{r}{1}u\dfrac{d^n u^{r-1}}{dx^n}+\dbinom{r}{2}u^2\dfrac{d^n u^{r-2}}{dx^n}-\dots+(-1)^{r-1}\dbinom{r}{1}u^{r-1}\dfrac{d^n u}{dx^n}$;

(ii) $$(-1)^n\left(\frac{d}{dx}\right)^n f\left(\frac{1}{x}\right)=\frac{1}{x^{2n}}f^{(n)}\left(\frac{1}{x}\right)+\binom{n}{1}\frac{n-1}{x^{2n-1}}f^{(n-1)}\left(\frac{1}{x}\right)+\dots$$
$$+\binom{n}{r}\frac{(n-1)(n-2)\dots(n-r)}{x^{2n-r}}f^{(n-r)}\left(\frac{1}{x}\right)+\dots+\frac{n!}{x^{n+1}}f^{(1)}\left(\frac{1}{x}\right);$$

(iii) $$\frac{d^n f(x^2)}{dx^n}=\sum_{r=0}^{p}\frac{n(n-1)\dots(n-2r+1)}{r!}(2x)^{n-2r}f^{(n-r)}(x^2),$$

where p is equal to $\frac{1}{2}(n-1)$ if n is odd and to $\frac{1}{2}n$ if n is even ;

(iv) $$\frac{d^n f(\sqrt{x})}{dx^n}=\sum_{r=0}^{n-1}(-1)^r\frac{(n+r-1)(n+r-2)\dots(n-r)}{r!}\frac{f^{(n-r)}(\sqrt{x})}{(2\sqrt{x})^{n+r}}.$$

12. If $x=\xi/\sqrt{(1-\xi^2)}$, prove by the method of Induction that

$$\left(\frac{d}{dx}\right)^m\frac{1}{1+x^2}=\frac{(m+1)!}{(2m+1)(2m-1)\dots1}\cdot\frac{1}{(1+x^2)^{(m+1)/2}}\left(\frac{d}{d\xi}\right)^m(1-\xi^2)^{(2m+1)/2}.$$

13. If a polynomial $f(x)$ has $(x-a)^r$ as a factor, prove that $f^{(2)}(x)$ has $(x-a)^{r-2}$ as a factor.

14. Find the multiple roots of the equations :

(i) $x^5-3x^4-6x^3+10x^2+21x+9=0$; (ii) $x^5-x^4-4x^2+7x-3=0$;
(iii) $x^4-2x^3-11x^2+12x+36=0$.

[(i) -1 is a triple root and 3 a double root; (ii) 1 is a triple root; (iii) 3, -2 are double roots.]

CHAPTER IV

PROPERTIES OF FUNCTIONS ASSOCIATED WITH THE EXISTENCE OF DERIVATIVES. APPLICATION TO ALGEBRAIC FUNCTIONS

91. Introductory Remarks.

The present Chapter is concerned with some of the fundamental properties of functions which possess derivatives. The analytical theorems demonstrated are illustrated by applications to Geometry, more especially to the theory of plane curves. These applications must be looked on as merely incidental; further discussion of the properties of curves will be taken up later in Chaps. V, VIII.

92. Inference from the Sign of the Derivative at a Point. Sign of the Increment.

Let the function y of the independent variable x possess a definite derivative $(dy/dx)_{x_1}$ at x_1, so that the function $\Delta y/\Delta x$ is potentially continuous at x_1 and becomes continuous when completed by the value $(dy/dx)_{x_1}$ there. If, then, the derivative $(dy/dx)_{x_1}$ is different from zero, there exists a range, $(x_1 - h,\ x_1 + h)$ say, in which, according to Art. 54, Chap. I, the continuous completed function $(\Delta y/\Delta x)^*$ has the same sign as at x_1, that is, in which the function $\Delta y/\Delta x$, $(\Delta x \neq 0)$, has the same sign as $(dy/dx)_{x_1}$. In the case of a one-sided derivative, the same argument applies to a corresponding one-sided neighbourhood of x_1. Thus for all cases, in the appropriate neighbourhood of x_1, the increment Δy of y has the same sign as the increment Δx if $(dy/dx)_{x_1}$ is positive, and the opposite sign if $(dy/dx)_{x_1}$ is negative.

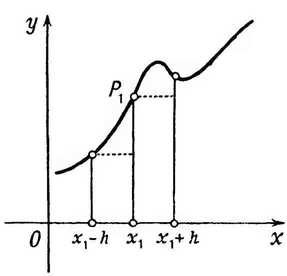


FIG. 53.

When $(dy/dx)_{x_1}$ is zero, no inference can be drawn in this way with regard to the sign of $\Delta y/\Delta x$. Further investigation is required in this case, which is considered below, Art. 98. 6.

The student should carefully notice that the result obtained does not imply that the function is *one-way* in the range $2h$. This range may be large enough to include an oscillation of the function, as shown in Fig. 53, and, moreover, this may be true however small h is taken; see Ex. 9, Art. 165, p. 395. It is, however, shewn below, Art. 98. 4, that if the sign of the derivative is positive (negative) *throughout* a range, the function is strictly up-way (down-way) throughout the range.

93. Extreme Values of a Function. Maxima and Minima.

One of the earliest applications of the Differential Calculus was to the determination of the greatest and least, or maximum and minimum, values of a function. We now proceed to discuss this question.

It is obviously convenient to have a term which will include both maximum and minimum; the word *extreme* is introduced for this purpose, and the *extreme values* of a function include both maximum and minimum values. A maximum may then be described as an *upper extreme* and a minimum as a *lower extreme.*

In the subsequent discussion it is necessary to distinguish between *absolute* and *local* extremes. A value is said to be an *absolute extreme* in a given range when it is the greatest (least) value of the function in that range. This value may, of course, occur any number of times in the range. Figures 54, 55, 56 illustrate various types of

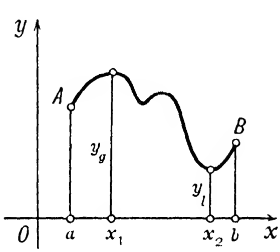


FIG. 54.

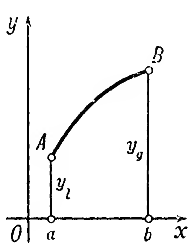


FIG. 55.

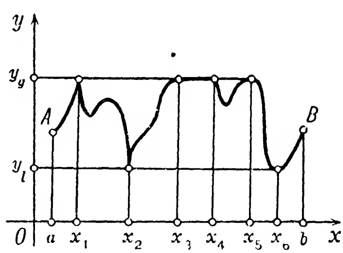


FIG. 56.

absolute extremes. It is clear from them that an absolute extreme may cease to be one when the range is altered.

A function is said to have a *local extreme value* at a point when this value is an absolute extreme for *some* two-sided neighbourhood, however small, of the point. This definition permits the repetition of the extreme value in such a range, and, as a special case, the function may have the same value throughout the range. We are thus led to distinguish two types of local extreme values. We describe the value of a function at a point as a *proper local extreme* if it is actually greater (less) than the values at all other

points of some neighbourhood of the point. Otherwise we speak of an *improper local extreme* at the point. These features are illustrated in Fig. 57, proper local minima occurring at x_1, x_5, proper local maxima at x_2, x_6, while the constant value of the function between x_3 and x_4 is an improper local extreme at all points in the range $[x_3, x_4]$.

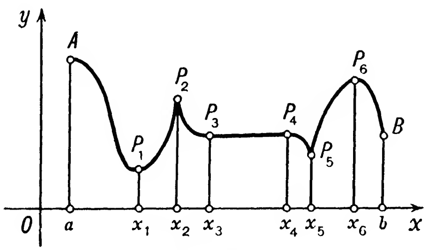


FIG. 57.

According to the above definition, the values of the function at the ends of a given range are not included among the local extreme values for that range even if they are local extremes for an *extended* range. These terminal values are considered in the following Article when discussing *terminal extremes*.

Up to the present no restrictions have been imposed on the nature of the function. We now proceed to consider the application of the Differential Calculus to the subject of extreme values, and therefore suppose that the function has a definite derivative throughout the range concerned, except possibly at special points.

94. Theorem of Zero Derivative at a Local Extreme Value. The Stationary Condition.

This theorem asserts that *if the function considered has a definite derivative at the value of the independent variable for which a local extreme occurs, the value of that derivative must be zero.*

Let the function y, or $f(x)$, have a local extreme value y_1 for the value x_1 of x. If we suppose that $(dy/dx)_{x_1}$ is positive, then, as in Art. 92, there exists a neighbourhood of x_1 within which $\Delta y/\Delta x$ is positive, $(\Delta x \neq 0)$, so that Δy is positive above and negative below x_1. The assumption thus leads to a change of sign of Δy, which, by the above statement, is not compatible with the existence of a local extreme. Similarly, the assumption of a negative derivative at x_1 leads to a contradiction, and we thus have the theorem in the form stated.

The geometrical equivalent of the theorem is that at a local extreme at which there is a definite tangent, this tangent is parallel to Ox, or 'horizontal.'

Ex. 1. *The function* ax^2+bx+c, $(a>0)$. As in Ex. 1, Art. 19, $f(x)$ has the minimum value $c-b^2/(4a)$ when $x=-b/(2a)$.

Now $f'(x)=2ax+b$, and when $x=-b/(2a)$, $f'(x)=0$. The theorem is therefore verified in this case.

Conversely, the theorem may be employed to determine the local extremes of a function which possesses a definite derivative at all points of a range, for all such extreme values occur for values of x which satisfy the equation

$$f'(x)=0. \qquad\qquad (1)$$

This, therefore, is a necessary condition for the existence of a local extreme value of the type just described. It is not, however, a sufficient condition, as may be seen by referring to Fig. 58, where there is a horizontal tangent at the point P_1, (x_1, y_1), which is not an extreme point on the curve.

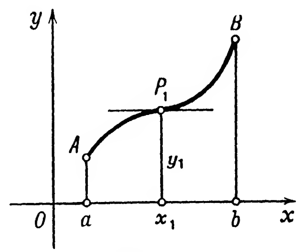


FIG. 58.

It is convenient to have a general name for the value of a function at a point at which the derivative vanishes, and since the rate of increase of the function with respect to the argument is then zero, we call such a value of the function a *stationary value*. The equation (1) is then referred to as the *stationary condition* for the function $f(x)$.

For the case illustrated in Fig. 58, there is a change in the sense of bending of the curve at the point where the stationary value occurs. The point may therefore be suitably described as a *point of contraflexure*,[1] (point of contrary flexure). In the present case the tangent is horizontal, but this is obviously not a necessary property of such points.

In the applications considered below the local extreme values which occur are, in general, stationary values, and are thus to be sought among the values of $f(x)$ corresponding to values of x which satisfy the equation (1). If an extreme such as that illustrated at P_2 or P_5 (Fig. 57) is likely to arise, special reference will be made to this possibility.

94. 1. *Change of variable.* If the relation between y and x is expressed with the aid of an intermediary variable, t say, by equations of the form $y=f(t)$, $t=\varphi(x)$, we have

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}, \qquad\qquad (2)$$

[1] Such a point is also called a *point of inflexion*.

and therefore $dy/dx=0$ if $dy/dt=0$ or if $dt/dx=0$. The stationary values of y with respect to x are thus found by considering in turn the stationary values of y with respect to t and of t with respect to x.

Ex. 2. *The function* $y=(a^2-x^2)^4-4c^6(a^2-x^2)$, $(a^2>c^2)$. If we write $a^2-x^2=t$, we get $y=t^4-4c^6t$, and therefore

$$\frac{dy}{dt}=4t^3-4c^6, \quad \frac{dt}{dx}=-2x. \qquad \text{.............................(3)}$$

Any stationary value of y with respect to t occurs when $4t^3-4c^6=0$, that is, when $t=c^2$, the corresponding values of x being given by $a^2-x^2=c^2$, or $x=\pm\sqrt{(a^2-c^2)}$. The stationary value of t with respect to x occurs when $x=0$. Therefore y is stationary with respect to x for the values 0, $\pm\sqrt{(a^2-c^2)}$ of x.

94. 2. *Note on the discrimination between stationary values.* From an intuitive point of view it is usually possible to distinguish between certain types of stationary values in the following way, which is satisfactory so far as algebraic functions are concerned. Referring to the preceding Figures, it appears that when the stationary value is a proper local maximum, the sign of the derivative $f'(x)$ is positive below and negative above the point, while a proper local minimum is characterized by the fact that $f'(x)$ changes sign from negative to positive as x increases through the value concerned. Again, for a stationary value such as illustrated in Fig. 58, there is no change of sign of $f'(x)$, which merely becomes zero at the value of x concerned.

While a discrimination between stationary values may sometimes be easily effected in this manner, the method is not the most systematic. We establish below further analytical criteria for the same purpose, the application of which usually leads to the required result in the most direct manner. In many cases, moreover, the nature of the stationary value can be correctly inferred from the conditions of the problem considered; the use of further criteria is then unnecessary, except as a check.

Ex. 3. *The function* $y=3x^4+4x^3+0{\cdot}5$. We have

$$\frac{dy}{dx}=12x^3+12x^2=12x^2(x+1), \qquad \text{..........................(4)}$$

which vanishes when $x=0$, $x=-1$. In the neighbourhood of $x=0$ the term $x+1$ is positive, so that dy/dx is then positive, except at $x=0$, where it vanishes. The value $x=0$ does not, therefore, give an extreme value of y. As x increases through the value -1, dy/dx changes sign from negative to positive, and the corresponding value of y, namely $0{\cdot}5$, is a minimum.

Portion of the graph of the function is shewn in Fig. 59. The point (0, 0·5) is a point of contraflexure with horizontal tangent.

94. 3. *Terminal extremes.* In the discussion of local extremes the end points of the range have been excluded. Let us now suppose that at such an end a one-sided derivative exists. Considering the lower end, x_1 say, let the derivative there be positive. Then there exists a one-sided neighbourhood of x_1 for which the increment Δy is positive, so that the value of the function at any point of it is greater than at x_1, and we say that the value $f(x_1)$ is a *terminal lower extreme* of the function. If the derivative is negative, the value $f(x_1)$ is a *terminal upper extreme.* If the value of the derivative is zero, we class the corresponding value of $f(x)$ among the stationary values of the function in the range. Exactly similar remarks apply to the upper end of the range.

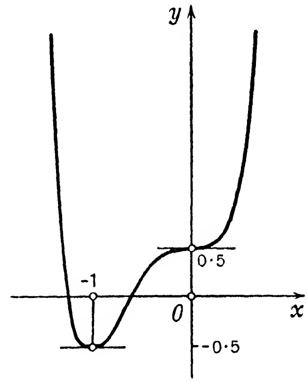


FIG. 59.
$y = 3x^4 + 4x^3 + 0{\cdot}5.$

94. 4. Since in the sequel the appearance of an improper local extreme value is of the nature of an exceptional case, it becomes unnecessary to maintain the explicit reference to ' proper ' extremes. This term is accordingly dispensed with when no confusion can arise, and the term ' local ' can also be dropped under similar conditions. The terms *maximum, minimum* will henceforth be employed as equivalent to *upper extreme, lower extreme,* with the use of the adjectives *local, absolute* and *terminal* where required to avoid possible confusion. Without qualification, therefore, a maximum need not be the greatest value of the function in the given range, nor a minimum the least.

If the *absolute extremes* of a function in a given range are required, it is necessary to compare the local extreme values found from the stationary condition $f'(x)=0$ with the terminal values and with the values, if any, at points at which there is not a definite value of the derivative. If a function has no local extreme values in a range of continuity, the greatest and least values must obviously occur at the ends of the range.

The above remarks are illustrated in Figs. 60, 61, 62. In Fig. 60, $dy/dx=0$ for the values a, x_1, b, the derivatives at a, b being one-sided. The value of y at x_1 is a (local) minimum and the terminal maxima at a, b are also stationary values of the function. The

absolute extremes in the range occur at x_1, b. In Fig. 61, dy/dx vanishes at $x=a$ and is positive at $x=b$. The terminal minimum at a is a stationary value and is, moreover, the least value of the function in the range ; the greatest value is the terminal maximum at b. Figure 62 illustrates the case of several maxima and minima in the given range. The greatest value of the function occurs at the upper end of the range and the least value is the local minimum at x_1, where the gradient is discontinuous.

94. 5. The complete discussion of the behaviour of a function in the neighbourhood of a stationary value, which is given in

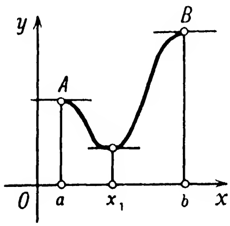


FIG. 60.

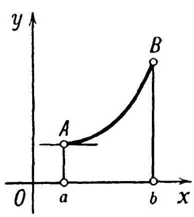


FIG. 61.

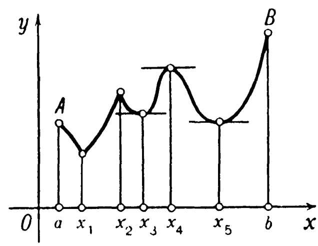


FIG. 62.

Arts. 105-108 below, requires the introduction of further analytical theorems, and we accordingly resume the general theory relating to the properties of functions.

95. Rolle's Theorem.

This is a theorem of great general importance which follows from the theorem of the preceding Article. It asserts that *if a function which is continuous in a closed range vanishes at the ends of the range and possesses a definite derivative at all points within it, then the derivative must vanish at some point within the range.* In symbolic form, if the function $f(x)$ is continuous in the range (x_1, x_2) and vanishes when $x=x_1$, $x=x_2$; and if $f'(x)$ exists throughout the range $[x_1, x_2]$; then $f'(x)=0$ for some value of x between x_1 and x_2.

The proof is immediate. If $f(x)$ is zero throughout the range (x_1, x_2), the derivative $f'(x)$ vanishes at each point of the range, and the theorem stated is therefore true in this case. If $f(x)$ is not zero throughout, it must, according to the statement of Art. 34, Chap. I, possess a finite extreme value (different from zero) within the range. At this extreme value there is a definite derivative, by assumption, and this derivative is zero by the theorem of the preceding Article. Rolle's Theorem therefore again holds and the proof is complete.

We can thus write $f'(\bar{x})=0$, where $\bar{x}$ is some value in the range

$[x_1, x_2]$; an equivalent form of statement[1] is $f'\{x_1+\theta(x_2-x_1)\}=0$, where $0<\theta<1$.

The geometrical equivalent of the theorem is that if the graph of the function meets Ox at x_1, x_2, then at some point $\bar{x}$ between these points the gradient is zero and the tangent parallel to Ox.

If the function $f(x)$ has equal values, different from zero, at x_1, x_2, we consider the function $f(x)-f(x_1)$ which vanishes at x_1, x_2 and has the derivative $f'(x)$. This derivative must therefore again vanish at some value of x between x_1 and x_2.

It should be noted that the theorem does not require the existence of a definite derivative, either one- or two-sided, at the end points x_1, x_2 of the range. The derivative may also become definite-infinite[2] (in the manner described in Art. 70) for a finite number of values of x in the range. Further, there may be more than one intermediate value of x at which the derivative vanishes.

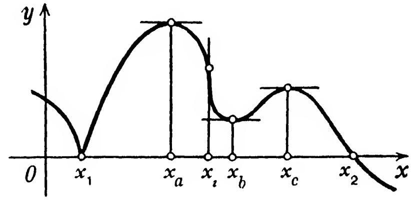


FIG. 63.

Some of these remarks are illustrated in Fig. 63. At x_ι the derivative is infinite; a definite derivative does not exist at x_1, while at x_a, x_b, x_c the derivative vanishes.

If the function is not continuous, or if $f'(x)$ does not exist for all values of x in $[x_1, x_2]$, the theorem is not necessarily true.

Ex. 1. *The function* $f(x)=(x-a)^m(x-b)^n$, $(m,\ n>0)$. Here $f(a)=0$, $f(b)=0$, and the conditions of the theorem are satisfied. We have

$$f'(x)=(x-a)^{m-1}(x-b)^{n-1}\{(m+n)x-mb-na\}, \quad \text{..............(1)}$$

and $f'(x)=0$ for $x=(mb+na)/(m+n)$, a value lying in the range $[a, b]$. It will be observed that if m, n are less than 1, the derivatives at $x=a$, $x=b$ are infinite.

Ex. 2. *Ultimate intersection of adjacent normals to a curve.* Let the normals at P_1, P_2 on a curve $y=f(x)$ intersect at $R_{1,2}$. It is required to find the position of the ultimate point of intersection, R^* say, as P_2 moves up to P_1 along the curve.

As in Art. 83. 2, the equation of the normal at (x, y) is

$$\xi-x+g(\eta-y)=0, \quad \text{...............................(2)}$$

where (ξ, η) are current coordinates and $g=f'(x)$. Since the normals at P_1, P_2 pass through the point $R_{1,2}$, of coordinates $(\xi_{1,2}, \eta_{1,2})$, say, the function $\varphi(x)$, defined by the equation

$$\varphi(x)=\xi_{1,2}-x+g(\eta_{1,2}-y), \quad \text{..........................(3)}$$

[1] Since any value of x between x_1 and x_2 can be expressed in the form $x_1+\theta(x_2-x_1)$, where θ lies between 0 and 1.

[2] The function is plainly not an extreme at a value of x for which the derivative is definite-infinite, so that the proof of the existence of a value of x for which the derivative vanishes is not affected.

vanishes when $x=x_1$, $x=x_2$. By Rolle's Theorem the derivative $\varphi'(x)$ vanishes for some value $\bar{x}$ between x_1 and x_2, and we have

$$-1+\overline{(dg/dx)}(\eta_{1,2}-\bar{y})-\bar{g}^2=0, \quad \text{.........................(4)}$$

the bars indicating values at $\bar{x}$.

This relation is true whatever the positions of P_1, P_2 on the curve. If P_2 moves up to P_1, so that $R_{1,2}$ moves up to R^*, of coordinates (ξ^*, η^*) say, then $\overline{(dg/dx)} \to (dg/dx)_{x_1}$, $\eta_{1,2} \to \eta^*$, $\bar{y} \to y_1$ and $\bar{g} \to g_1$. It follows that

$$\eta^*-y_1=\frac{1+g_1^2}{(dg/dx)_{x_1}}. \quad \text{.................................(5)}$$

Since equation (2) is satisfied when x, y have the values x_1, y_1 and ξ, η the values ξ^*, η^*, we find, with the aid of (5),

$$\xi^*-x_1=-\frac{g_1(1+g_1^2)}{(dg/dx)_{x_1}}. \quad \text{.............................(6)}$$

Equations (5), (6) give the required coordinates.

Simpler forms for these coordinates are obtained by shifting the origin to P_1 and taking the tangent and normal there as the axes of x and y, respectively. We then have $x_1=0=y_1=g_1$, and so

$$\xi^*=0, \quad \eta^*=\frac{1}{(dg/dx)_0}. \quad \text{.............................(7)}$$

See also Ex. 2, Art. 97.

The above results are employed in Art. 135 when dealing with the subject of *Curvature of a Plane Curve.*

95. 1. *Application of Rolle's Theorem to the determination of the distribution of the roots of an equation.* If $f(x)$, $f'(x)$ are both continuous functions of x, Rolle's Theorem is often useful in the determination of the number and distribution of the roots of the equation

$$f(x)=0. \quad \text{.................................(8)}$$

For if x_1, x_2 are any two real roots of this equation, we have $f'(\bar{x})=0$, where $\bar{x}$ lies in the range $[x_1, x_2]$. It follows that at most one real root of (8) lies between any two consecutive roots of the equation

$$f'(x)=0. \quad \text{.................................(9)}$$

The roots of (9) therefore separate those of (8), and when the former can be determined, the number and general distribution of the latter follow.

It should be noticed that a root of equation (8) may coincide with a root of equation (9). This case is often conveniently thought of as a limiting one, in which two roots of equation (8), with a root of equation (9) between them, move up to coincidence. Such a root is called a *double root* of equation (8). If the equation $f''(x)=0$ is also satisfied by the same value of x, the root is called a *triple root*, and so on; compare Ex. 8, Art. 90.

Suppose, for example, that the roots of (9) in ascending order are x_1', x_2', x_3', x_4'. Then the (real) roots of (8), if any, will be distributed so that not more than one occurs in any one of the intervals $[-\infty, x_1')$, (x_1', x_2'), (x_2', x_3'), (x_3', x_4'), $(x_4', \infty]$. To determine whether a root actually occurs in any one interval, we insert the values of the ends of the intervals in $f(x)$ and make use of the property that if a function is positive at one end and negative at the other, it is zero somewhere between. If in the present case we find

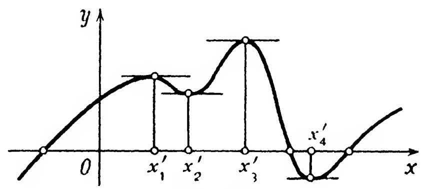


FIG. 64.

$$f(-\infty)<0, \quad f(x_1')>0, \quad f(x_2')>0, \quad f(x_3')>0, \quad f(x_4')<0, \quad f(\infty)>0,$$

we conclude that (8) has three real roots, one in each of the intervals $[-\infty, x_1']$, $[x_3', x_4']$, $[x_4', \infty]$. The geometrical representation of this case may be as shewn in Fig. 64.

The above procedure is specially useful in the case of polynomial equations since the derivative equation (9) is of lower degree than the original equation. If the roots of equation (9) cannot be readily found, their distribution may be obtained by an investigation of the roots of the equation $f''(x)=0$, and so on. From the information so obtained, the general distribution of the roots of the original equation may be deduced.

Ex. 3. *The equation* $3x^4+4x^3+0{\cdot}5=0$. Quoting from Ex. 3, p. 200, $f'(x)$ vanishes when $x=-1$, $x=0$. The roots, if any, of the given equation are thus to be sought in the intervals $[-\infty, -1)$, $(-1, 0)$, $(0, \infty]$. We find $f(-\infty)>0, f(-1)<0, f(0)>0, f(\infty)>0$. There are therefore two roots, one less than -1, the other between -1 and 0. See Fig. 59, p. 201.

Ex. 4. *The general cubic equation*

$$x^3+3ax^2+3bx+c=0. \qquad \text{.............................(10)}$$

We can write this in the form $(x+a)^3+3(b-a^2)x+c-a^3=0$, and hence in the form $(x+a)^3+3(b-a^2)(x+a)+c-3ab+2a^3=0$. Thus, putting ξ for $x+a$, the equation to be solved becomes

$$f(\xi)=\xi^3+3H\xi+G=0, \qquad \text{..........................(11)}$$

where $$H=b-a^2, \quad G=c-3ab+2a^3. \qquad \text{.......................(12)}$$

There is at least one real root of (11) since $f(\xi)\to-\infty$ when $\xi\to-\infty$ and $f(\xi)\to+\infty$ when $\xi\to+\infty$.

There is only one real root of (11) if the equation $f'(\xi)=0$, that is, $\xi^2+H=0$, has not real roots. Thus if $H>0$ there is only one real root of (11).

If $H=0$, there is only one real root (unless $G=0$), since the equation (11) is then $\xi^3=-G$.

If $H<0$, the equation $\xi^2+H=0$ has the roots $-\sqrt{(-H)}$, $\sqrt{(-H)}$, which we denote by α, β, respectively, so that $\alpha+\beta=0$ and $\alpha\beta=H$. Then

$$\left.\begin{array}{c} f(\alpha)=\alpha^3+3H\alpha+G=2H\alpha+G, \quad f(\beta)=2H\beta+G, \\ f(\alpha)+f(\beta)=2G, \quad f(\alpha)f(\beta)=4H^3+G^2. \end{array}\right\} \quad \text{.............}(13)$$

If $f(\alpha)$, $f(\beta)$ have the same sign, that is, if $f(\alpha)f(\beta)$ is positive, there is only one real root of (11) and it lies between $-\infty$ and α, or between β and ∞, according as the common sign is positive or negative. Thus there is only one real root if $4H^3+G^2>0$ and it lies between $-\infty$ and $-\sqrt{(-H)}$, or between $\sqrt{(-H)}$ and ∞, according as G is positive or negative.

If $f(\alpha)$, $f(\beta)$ have opposite signs, that is, if $f(\alpha)f(\beta)$ is negative, there are three real roots, one in each of the ranges $[-\infty, \alpha]$, $[\alpha, \beta]$, $[\beta, \infty]$. Thus there are three real roots if $4H^3+G^2<0$, one in each of the ranges stated.

If $f(\xi)$, $f'(\xi)$ vanish together, there is a double root of (11). Since $f(\alpha)f(\beta)=4H^3+G^2$, the condition for a double root is $4H^3+G^2=0$. The double root is $-\sqrt{(-H)}$ or $\sqrt{(-H)}$ according as G is positive or negative.

If $f(\xi)$, $f'(\xi)$, $f''(\xi)$ vanish together, there is a triple root. Since $f''(\xi)=6\xi$, the conditions for a triple root are $H=0$, $G=0$.

Returning to the original equation (10), we can now state :

(i) There is one real root only if $b>a^2$.

(ii) If $b<a^2$, there is one real root or three according as

$$c^2-6abc-3a^2b^2+4a^3c+4b^3 \gtrless 0. \quad \text{........................}(14)$$

In the first case the root lies between $-\infty$ and $-\sqrt{(a^2-b)}$ or between $\sqrt{(a^2-b)}$ and ∞ according as $c-3ab+2a^3 \gtrless 0$. In the second case there is a root in each of the ranges $[-\infty, -\sqrt{(a^2-b)}]$, $[-\sqrt{(a^2-b)}, \sqrt{(a^2-b)}]$, $[\sqrt{(a^2-b)}, \infty]$.

(iii) If $b<a^2$ and $c^2-6abc-3a^2b^2+4a^3c+4b^3=0$, there is a double root.

(iv) If, finally, $b=a^2$ and $c-3ab+2a^3=0$, (so that the expression on the left-hand side of (14) vanishes), there is a triple root.

Expressions for the real roots of the general cubic equation in terms of transcendental functions are given in Ex. 7, Art. 173.

Ex. 5. Shew that if the equation (10) has a double root, it is $\frac{1}{2}(ab-c)/(b-a^2)$ and that if the same equation has a triple root, it is $-a$.

Ex. 6. Shew that if $H>0$ in (11), the root is positive or negative according as G is negative or positive.

Ex. 7. *The equation* $x^4-2x^3-2x+1=0$. Here we have

$$f'(x)=4x^3-6x^2-2, \quad f''(x)=12x^2-12x. \quad \text{..................}(15)$$

As the roots of the equation $f'(x)=0$ are not obvious, we employ the equation $f''(x)=0$ to determine their distribution. This equation is satisfied when $x=0$, $x=1$, and the roots of $f'(x)=0$ are therefore to be sought in the intervals $[-\infty, 0)$, $(0, 1)$, $(1, \infty]$ of x. It is easily shewn that there is only one root, which lies in the third interval. Inspection shews that it lies between 1 and 1·5.

We conclude that the given equation has at most two real roots. If they both exist, one will be less and the other greater than about 1·5. We

find, in fact, that $f(0)=1$, $f(1)=-2$, so that there is a root between 0 and 1. Its value, correct to five places of decimals, is investigated in Ex. 1, Art. 118, p. 251, and is 0·43542. There is also a second root, which is the reciprocal of the one found, namely 2·29663, approximately. This reciprocal property is seen from the fact that the given equation is equivalent to $1/x^4 - 2/x^3 - 2/x + 1 = 0$, $(x \neq 0)$.

96. The Intermediate First Derivative Theorem.

This theorem is an extension of Rolle's Theorem to the case where the values y_1, y_2 of the function y at the ends x_1, x_2 of the range of x are unequal. It states that *if a function which is continuous in a closed range possesses a definite derivative at all points within the range, the average rate of increase of the function for that range is equal to the derivative at some point within the range.*

Before giving a formal demonstration, we illustrate the theorem graphically. If P_1, P_2 (Fig. 65) are the points (x_1, y_1), (x_2, y_2) on the graph of the function, referred to the axes Ox, Oy, and if the curve is referred to new axes P_1x', P_1y', along and perpendicular to P_1P_2 as shewn, the ordinates of the curve relative to these axes vanish at the ends P_1, P_2 of the chord. The geometrical equivalent of Rolle's Theorem can therefore be applied and it asserts that there is some point $\bar{P}$ on the curve between P_1 and P_2 at which the tangent is parallel to P_1x', that is, at which the gradient with respect to P_1x' vanishes.

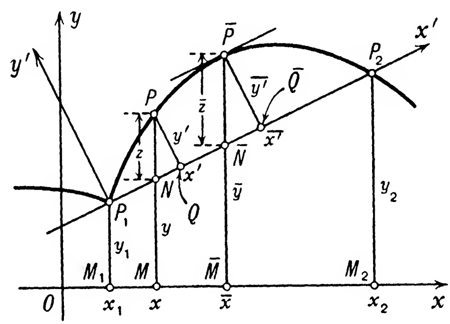


FIG. 65.

This is plainly equivalent to the analytical statement

$$\left(\frac{dy'}{dx'}\right)_{\bar{x}'} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where $P_1\bar{Q} = \bar{x}'$. Going back to the original axes, we may thus state that the gradient at $\bar{P}$ relative to Ox is equal to that of the chord P_1P_2. This latter is equal to $(y_2 - y_1)/(x_2 - x_1)$, and we thus get the equivalent analytical statement

$$\left(\frac{dy}{dx}\right)_{\bar{x}} = \frac{y_2 - y_1}{x_2 - x_1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where $\bar{x}$ is the abscissa of $\bar{P}$ relative to the axes Ox, Oy. This is a direct symbolic statement of the theorem to be demonstrated.

We now proceed to a formal proof. We introduce a new dependent variable, z say, defined by the equation

$$z = y + ax + b, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where a, b are constants to be chosen so that z vanishes at the ends of the range (x_1, x_2); thus

$$0 = y_1 + ax_1 + b, \quad 0 = y_2 + ax_2 + b. \qquad (4)$$

These give

$$a = -\frac{y_2 - y_1}{x_2 - x_1}, \quad b = \frac{x_1 y_2 - x_2 y_1}{x_2 - x_1}, \qquad (5)$$

and hence (3) becomes

$$z = y - \frac{y_2 - y_1}{x_2 - x_1} x + \frac{x_1 y_2 - x_2 y_1}{x_2 - x_1}. \qquad (6)$$

This function satisfies the conditions of Rolle's Theorem and its derivative dz/dx must therefore vanish for some point, $\bar{x}$ say, within the range. We have, from (3),

$$\frac{dz}{dx} = \frac{dy}{dx} + a, \qquad (7)$$

and the result $(dz/dx)_{\bar{x}} = 0$ thus leads to

$$\left(\frac{dy}{dx}\right)_{\bar{x}} = -a = \frac{y_2 - y_1}{x_2 - x_1}, \qquad (8)$$

which demonstrates the theorem.

It is easily shewn that in Fig. 65 the distance NP, (z), is given by equation (6), and since NP is always in a constant ratio to QP, a maximum value of QP corresponds to a maximum value of NP. The advantage of working with NP instead of QP is that we can employ x as the independent variable and so avoid the analytical transformation of equation (1).

As in the case of Rolle's Theorem, the derivative of the given function y need not exist at the end points x_1, x_2 and it may become definite-infinite (in the sense described in Art. 70) for certain values of x within the range; it is plain, however, that any such value of x cannot fulfil the condition to be satisfied by the value $\bar{x}$.

The theorem demonstrated is called the *Intermediate First Derivative Theorem*, (abbreviated to $I.D_1.T.$), the name being suggested by the appearance of the value of the first derivative at some intermediate point of the range considered. The adjective 'first' is employed because we subsequently proceed to the development of similar theorems in which the intermediate values of derivatives of higher order appear. The theorem is also sometimes called the *First Mean Value Theorem* (of the Differential Calculus), or the *Mean Value Theorem*, simply.

97. Equivalent Forms of the Intermediate First Derivative Theorem. Increment of a Function in a Range.

The theorem is expressed in a variety of symbolic forms all of which are in common use and which are obtained from the above equation (8) when written in the form

$$y_2 - y_1 = (x_2 - x_1)\left(\frac{dy}{dx}\right)_{\bar{x}}, \quad \text{.......................(1)}$$

or

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(\bar{x}). \quad \text{..........................(2)}$$

Instead of using $\bar{x}$ to denote the intermediate value of x, we may, as in Art. 95, employ the symbol $x_1 + \theta(x_2 - x_1)$, where $0 < \theta < 1$. The equation (2) then becomes

$$f(x_2) - f(x_1) = (x_2 - x_1)f'\{x_1 + \theta(x_2 - x_1)\}. \quad \text{.................(3)}$$

Introducing the increment notation, we have the forms

$$f(x_1 + \Delta x) - f(x_1) = \Delta x f'(\bar{x}) \quad \text{............................(4)}$$

$$= \Delta x f'(x_1 + \theta\Delta x), \quad \text{.................(5)}$$

and we may also write this in the form

$$\Delta f(x) = \Delta x f'(x_1 + \theta\Delta x). \quad \text{.................(6)}$$

Each of the above forms expresses the increment of the function in the range as the product of the range and some intermediate value of the derivative. A slightly different complexion is given to the theorem when we write (4), for example, in the form

$$f(x_1 + \Delta x) = f(x_1) + \Delta x f'(\bar{x}), \quad \text{.....................(7)}$$

which gives the value of the function $f(x)$ at the value $x_1 + \Delta x$ in terms of the value $f(x_1)$ and some intermediate value of the derivative. The value x_1 may be appropriately described as the *starting value* and $x_1 + \Delta x$ as the *end value*, and the result, in common with all the preceding equivalent formulae, is true whether the increment Δx is positive or negative.

A variant of (7) which is in common use is obtained by taking the value a as starting point and $a + h$ as end point, where h may be positive or negative. We then have

$$f(a + h) = f(a) + hf'(\bar{x}) \quad \text{............................(8)}$$

$$= f(a) + hf'(a + \theta h), \quad \text{.....................(9)}$$

where $\bar{x}$ lies in $[a, a + h]$ and $0 < \theta < 1$. A special case is that in which the starting point is the value 0, (9) then taking the form

$$f(h) = f(0) + hf'(\theta h). \quad \text{........................(10)}$$

97. 1. *Variable end point.* The values x_1, x_2 can be taken anywhere in a range throughout which $f(x)$ has a definite derivative. If, then, x_1 is regarded as a fixed starting point, the above formulae apply to any end point x in the range, and therefore, replacing x_2 (or $x_1+\Delta x$) by x, we have, corresponding to (7),

$$f(x)=f(x_1)+(x-x_1)f'(\bar{x}) \quad \text{.......................................(11)}$$

$$=f(x_1)+(x-x_1)f'\{x_1+\theta(x-x_1)\}, \quad (0<\theta<1). \quad \text{.......(12)}$$

The magnitude of the proper fraction θ will, of course, depend on the magnitude of the range $x-x_1$ as well as of x_1, that is, it is a function of x and x_1. The dependence of θ on x is often indicated by writing it in the functional form $\theta(x)$, the starting point x_1 being supposed fixed.

If the starting point is 0, we have the forms

$$f(x)=f(0)+xf'(\bar{x}) \quad \text{.....................................(13)}$$

$$=f(0)+xf'(\theta x), \quad (0<\theta<1). \quad \text{.................(14)}$$

Ex. 1. *The function* $y=ax^2+bx+c$. Here $dy/dx=2ax+b$. If the points (x_1, y_1), (x_2, y_2) are on the curve, we have

$$\frac{y_2-y_1}{x_2-x_1}=a(x_1+x_2)+b. \quad \text{..............................(15)}$$

Equating this to the value of the derivative at the intermediate point $\bar{x}$, we get

$$2a\bar{x}+b=a(x_1+x_2)+b, \quad \text{...................................(16)}$$

so that
$$\bar{x}=\tfrac{1}{2}(x_1+x_2)=x_1+\tfrac{1}{2}(x_2-x_1). \quad \text{.......................(17)}$$

In this case we conclude that the value of θ is 1/2.

Ex. 2. *Ultimate intersection of adjacent normals to a curve.* This problem has already been considered in Ex. 2, Art. 95, p. 203. We here give an alternative demonstration, employing the $I.D_1.T.$ Let P_1, P be adjacent points on the curve, and suppose, for simplicity, that the curve is referred to axes P_1x, P_1y along the tangent and normal at P_1, its equation then being $y=f(x)$. The normal at P, (x, y), is

$$\xi-x+g(\eta-y)=0, \quad \text{................................(18)}$$

and this meets the normal at P_1, that is the axis P_1y, where $\xi=0$, $\eta=y+x/g$.

Now, by the $I.D_1.T.$,

$$g(x)=g(0)+xg'(\bar{x}), \quad \text{...............................(19)}$$

and since here $g(0)=0$, we can write

$$\eta=y+1/g'(\bar{x})=y+1/f''(\bar{x}). \quad \text{..........................(20)}$$

As P moves up to P_1 along the curve, $y\to 0$ and $f''(\bar{x})\to f''(0)$. The required coordinates are thus given by

$$\xi^*=0, \quad \eta^*=1/f''(0), \quad \text{..............................(21)}$$

as before.

98. Analytical Deductions from the Intermediate First Derivative Theorem.

98. 1. If $f'(x)$ is zero throughout the range[1] (x_1, x_2), then $f(x)$ is constant in the range.

For the result

$$f(x) = f(x_1) + (x - x_1)f'(\bar{x}), \quad \text{.......................(1)}$$

where x is in (x_1, x_2) and $\bar{x}$ in $[x_1, x]$, now gives $f(x) = f(x_1)$ for all values of x in (x_1, x_2). If the value of the function is equal to C at x_1, then we have

$$f(x) = C \quad \text{.................................(2)}$$

throughout the range.

This is the converse of the theorem, proved in Ex. 1, Art. 66, p. 120, that the derivative of a constant is zero.

98. 2. If $f'(x)$ has a constant value, c say, throughout (x_1, x_2), then $f(x)$ differs from the function cx in the range by a constant merely.

For we have, from (1),

$$\begin{aligned} f(x) &= f(x_1) + (x - x_1)c \\ &= cx + \{f(x_1) - cx_1\}. \quad \text{.........................(3)} \end{aligned}$$

If the value of the function is C at the point x_1, its value in the range (x_1, x_2) is determined by the equation

$$f(x) = c(x - x_1) + C. \quad \text{...........................(4)}$$

98. 3. If two functions $f(x)$, $g(x)$ have equal derivatives at all points of the range (x_1, x_2), they differ by a constant merely in the range.

For the derivative of the function $f(x) - g(x)$ is zero throughout (x_1, x_2), and therefore, by the above Sub-article 1, its value is constant throughout the range.

It follows that if we know one function, $\varphi(x)$ say, whose derivative is equal to $f'(x)$ throughout the range, all other functions having the same derivative differ from $\varphi(x)$ by a constant merely in the range. The geometrical equivalent of this result is that if two curves in a given range have the same gradient for each value of the abscissa, the corresponding ordinates differ by a constant; see Fig. 66.

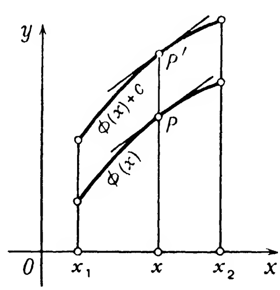


FIG. 66.

[1] Here the condition imposed on $f'(x)$ includes the condition of continuity of $f(x)$ in the closed range.

Ex. 1. Since the derivative of the function $\frac{1}{2}x^2$ is x, the most general function whose derivative is x is $\frac{1}{2}x^2+C$, where C is arbitrary.

Ex. 2. Since the derivative of $\frac{1}{3}x^3$ is x^2, the most general function whose derivative is x^2 is $\frac{1}{3}x^3+C$, where C is arbitrary. We conclude that the most general function whose derivative is $\frac{1}{2}x^2+a$, (a a given constant), is

$$\frac{1}{3\times 2}x^3+ax+C, \quad \text{or} \quad \frac{1}{3!}x^3+ax+C,$$

where C is arbitrary.

Ex. 3. The most general function whose derivative is ax^2+bx+c, (a, b, c given constants), is $\frac{1}{3}ax^3+\frac{1}{2}bx^2+cx+C$, ($C$ arbitrary), while the most general function whose derivative is

$$\frac{1}{3!}ax^3+\frac{1}{2!}bx^2+cx+d \text{ is } \frac{1}{4!}ax^4+\frac{1}{3!}bx^3+\frac{1}{2!}cx^2+dx+C,$$

(C arbitrary).

Ex. 4. The most general function whose derivative is the polynomial $\sum\limits_{r=0}^{n} a_r x^r$ is the polynomial $\sum\limits_{r=0}^{n} \frac{1}{r+1}a_r x^{r+1}+C$, ($C$ arbitrary).

Ex. 5. The most general function whose derivative is the function $f'(x)-m$, (m constant), is $f(x)-mx+C$, (C arbitrary).

Ex. 6. The most general function whose derivative is the function

$$f'(x)-f'(0)-mx$$

is $f(x)-xf'(0)-\frac{1}{2}mx^2+C$, ($C$ arbitrary).

The results of this and the preceding Examples are employed in the following Article in connection with the investigation of *polynomial inequalities associated with a function.*

98. 4. If $f'(x)$ exists (being finite or definite-infinite) and is positive at all points in the range $[x_1, x_2]$, where x_2 is supposed greater than x_1, then $f(x)$ is strictly up-way throughout the range (x_1, x_2), in which it is supposed continuous.

For if x, x' are any two points of the range, we have

$$f(x')=f(x)+(x'-x)f'(\bar{x}), \quad \dots\dots\dots\dots(5)$$

where $\bar{x}$ lies within the range $[x, x']$. If this range is positive,[1] then, since $f'(\bar{x})$ is positive, we must have

$$f(x')>f(x), \quad (x_1 \leqslant x < x' \leqslant x_2), \quad \dots\dots\dots\dots(6)$$

which proves the theorem.

Similarly, if $f'(x)$ is negative throughout, we prove that

$$f(x')<f(x), \quad (x_1 \leqslant x < x' \leqslant x_2), \quad \dots\dots\dots\dots(7)$$

so that the function is strictly down-way throughout (x_1, x_2).

[1] We have described (Art. 22. 2) a range (a, b) as positive when the difference $b-a$ between its upper and lower terminals is positive.

The modifications of the above results when the ranges introduced are negative are obvious.

98. 5. The same conclusions as in the preceding Sub-article are reached if the derivative vanishes at points of the range, provided that it does not change sign in the range and that it does not vanish *throughout* any sub-range.

Thus, assuming $f'(x)$ positive when not zero, (5) gives

$$f(x') \geqslant f(x), \quad (x < x'), \quad \text{.........................(8)}$$

so that the function cannot decrease with increase of the argument. We now shew that the sign of equality can be deleted. Suppose, for the sake of argument, that $f(x')=f(x)$. Then if the function varies at all in the range (x, x'), there must be a sub-range within it for which the function decreases with increase of the argument, and this is inconsistent with the above conclusion. If the function does not vary, the derivative is constantly zero in the range (x, x'), and this is contrary to the initial assumption. Therefore, under the conditions stated we have

$$f(x') > f(x), \quad (x < x'). \quad \text{.........................(9)}$$

If, now, the range (x_1, x_2) is positive, this result holds for any two values x, x' in it such that $x_1 \leqslant x < x' \leqslant x_2$, and the function $f(x)$ is strictly up-way, as before.

If $f'(x)$ is negative when not zero, we prove in the same way that $f(x)$ is strictly down-way in the (positive) range (x_1, x_2).

The modifications in these results when the ranges introduced are negative are obvious.

Cor. I. If the function $f(x)$ vanishes for the value 0 of x, which is taken as the starting point of the positive range $(0, x_2)$, so that $f(x_1)=f(0)=0$, we have $f(x)>0$ if $f'(x)$ is positive when not zero, and $f(x)<0$ if $f'(x)$ is negative when not zero.

If the range $(0, x_2)$ is negative, the above inequality signs are merely reversed.

Cor. II. If the derivative does not change sign in the positive range $[x_1, x_2]$ and is positive throughout some sub-range, then the value of the function at x_2 is greater than the value at x_1.

For the function never decreases in any part of the range and it increases in the sub-range mentioned. We thus have $f(x_2)>f(x_1)$, $(x_1<x_2)$.

Similarly, if the derivative is negative when not zero, we have $f(x_2)<f(x_1)$, $(x_1<x_2)$.

If the range is negative, the inequalities connecting $f(x_2)$ and $f(x_1)$ are merely reversed.

Ex. 7. Prove that the arithmetic mean of two positive numbers is greater than their geometric mean.

Let x_1, x_2 be the two numbers, where we suppose $x_2 > x_1$. The function $f(x)$ defined by the equation

$$f(x) = \tfrac{1}{2}(x_1 + x) - \sqrt{(x_1 x)}, \quad (x_1 \leqslant x \leqslant x_2), \qquad \text{.................. (10)}$$

vanishes when $x = x_1$ and its derivative is given by

$$f'(x) = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{(x_1/x)}, \qquad \text{.................................. (11)}$$

which is positive if $x > x_1$. Hence we have $f(x) > f(x_1)$, so that $f(x) > 0$.

In particular, we have $f(x_2) > 0$, that is,

$$\tfrac{1}{2}(x_1 + x_2) - \sqrt{(x_1 x_2)} > 0, \qquad \text{.............................. (12)}$$

which proves the proposition.

Ex. 8. Prove that the polynomial $2x^3 + 2x^2 - 10x + 6$ is positive if x is greater than 1.

Denoting the polynomial by $f(x)$, we have

$$\left.\begin{array}{l} f(1) = 0, \quad f'(x) = 6x^2 + 4x - 10, \quad f'(1) = 0, \\ \qquad\qquad f''(x) = 12x + 4, \quad f''(x) > 0 \text{ if } x > 1. \end{array}\right\} \qquad \text{.............. (13)}$$

Since the function $f'(x)$ vanishes when $x = 1$ and has a positive derivative when $x > 1$, it is positive when $x > 1$, according to the above *Cor.* I. By repeating the argument on the function $f(x)$, the desired result is obtained.

98. 6. *Application to maxima and minima.* The results of the last two Sub-articles are those to which reference has been made in Art. 92. By means of them we can justify the rules obtained in an intuitive manner in Art. 94. 2 for the discrimination between certain types of stationary values. Thus, let $f'(x_1) = 0$ and suppose that $f'(x) > 0$ within a certain range which extends up to x_1 (from below); then for $x < x_1$, $f(x)$ is strictly up-way and increases to the value $f(x_1)$. If $f'(x) < 0$ to the right of x_1, $f(x)$ is strictly down-way for $x > x_1$, decreasing from the value $f(x_1)$, and we have a maximum at x_1, but if $f'(x) > 0$ to the right of x_1, the value $f(x_1)$ is a stationary value of the type illustrated in Fig. 67. If $f'(x)$ remains zero throughout a range which extends from x_1 to the right, the value $f(x_1)$ is an improper local extreme. The alternative possibilities require no discussion.

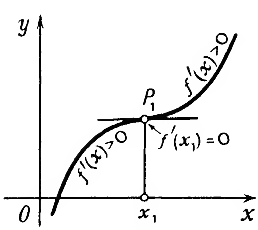


FIG. 67.

Attention must again be called to the fact that in deciding the nature of a function in the neighbourhood of a stationary value in

the above way, the sign of the derivative is required *throughout* a range within which the point concerned lies. It is shewn below, following the introduction of the theorems referred to previously and which involve intermediate values of derivatives of higher order than the first, that the condition as to the sign of $f'(x)$ throughout such a range can be replaced by a condition which involves the sign of the second or a higher derivative *at the point itself*.

99. Polynomial Inequalities associated with a Function.

In the present Article we establish a system of polynomial inequalities associated with a given function. These inequalities are closely connected with the Intermediate Higher Derivative Theorems already referred to and, in fact, directly suggest them in a restricted form. The formal proof of the general theorem is given in the following Article.

The system of inequalities is demonstrated by a step-by-step process which is suggested immediately by the theorems of the preceding Article, in particular, *Cor.* I, Sub-article 5.

99. 1. *Polynomials of the first degree.* The given function $f(x)$ is assumed to be continuous in a range $(0, x_2)$ and to possess a *finite* derivative $f'(x)$ at each point of the range; in other words, the derivative is to have finite upper and lower bounds in the range. If m is any number less than the lower bound and M any number greater than the upper bound, then in the range we have

$$f'(x) - m > 0, \qquad \text{(1)}$$

$$f'(x) - M < 0. \qquad \text{(2)}$$

We consider these inequalities in turn.

In order to apply the theorem of the corollary mentioned to the first inequality (1), we proceed to construct a function whose derivative is $f'(x) - m$ and which vanishes when $x = 0$. The most general function having this derivative is, as in Ex. 5, p. 212, $f(x) - mx + C$, where C is arbitrary. This function vanishes when $x = 0$ if C is chosen so that $f(0) + C = 0$, and the particular function sought is therefore

$$f(x) - f(0) - mx. \qquad \text{(3)}$$

By the theorem referred to, this function is positive if x is positive, that is, if the range $(0, x_2)$ is positive. We therefore have the inequality

$$f(x) - f(0) - mx > 0, \quad (0 < x \leqslant x_2). \qquad \text{(4)}$$

In a corresponding manner, from (2) we establish the inequality

$$f(x)-f(0)-Mx<0, \quad (0<x\leqslant x_2). \quad \text{..........(5)}$$

We may write (4) and (5) in the forms

$$\left.\begin{aligned} f(x)&>f(0)+mx, \\ f(x)&<f(0)+Mx, \end{aligned}\right\} \quad (0<x\leqslant x_2), \quad \text{..........(6)}$$

which are the inequalities to be demonstrated.

As applied to the whole (positive) range $(0, x_2)$, these become

$$\left.\begin{aligned} f(x_2)&>f(0)+mx_2, \\ f(x_2)&<f(0)+Mx_2, \end{aligned}\right\} \quad \text{..........(7)}$$

so that we may write

$$f(x_2)=f(0)+\mu x_2, \quad \text{..........(8)}$$

where μ is some number between m and M, however near these numbers are to the respective lower and upper bounds of $f'(x)$ in the range. It follows that μ cannot lie outside the range whose ends are these lower and upper bounds.[1]

If the derivative $f'(x)$ is *continuous* in the range $(0, x_2)$, μ must clearly be a value of the derivative for some value of x in the range. We may thus write

$$\mu=f'(\theta x_2), \quad (0\leqslant\theta\leqslant 1), \quad \text{..........(9)}$$

and equation (8) then becomes

$$f(x_2)=f(0)+x_2 f'(\theta x_2). \quad \text{..........(10)}$$

The results (8), (10) are established in a corresponding way when x_2 is negative.

If, now, instead of a fixed end point x_2 we introduce a variable end point x, (10) takes the form

$$f(x)=f(0)+xf'(\theta x), \quad (0\leqslant\theta\leqslant 1), \quad \text{..........(11)}$$

which is, of course, included in (is a special case of) the Intermediate First Derivative Theorem.[2]

99. 2. *Polynomials of the second degree.* We use the preceding demonstration as a model but start with the second derivative $f''(x)$ of the given function $f(x)$. This function and its first derivative $f'(x)$ are assumed continuous in the range $(0, x_2)$, and the second

[1] The numbers m, M, μ refer throughout to the whole range $(0, x_2)$. It is plain that in the inequalities (6) the numbers m, M could be defined with respect to the bounds of the derivative for the range $(0, x)$ and an equation corresponding to (8) written down for that range.

[2] It will be recalled that in this theorem the proof does not require the continuity of $f'(x)$, and the number θ actually lies between 0 and 1.

derivative is supposed to be *finite* throughout. If m, M are numbers respectively less and greater than the lower and upper bounds of $f''(x)$ in the range, then in the range we have

$$f''(x) - m > 0, \qquad \text{(12)}$$

$$f''(x) - M < 0. \qquad \text{(13)}$$

Suppose, at first, that the range $(0, x_2)$ is positive. As above, the function

$$f'(x) - f'(0) - mx \qquad \text{(14)}$$

has the positive derivative $f''(x) - m$ in the range and it vanishes when $x = 0$. It is therefore positive in the range. We now construct a function whose derivative is the function (14) and which vanishes when $x = 0$. The most general function having the stated derivative is, as in Ex. 6, p. 212, $f(x) - xf'(0) - \frac{1}{2}mx^2 + C$, where C is arbitrary, and this function vanishes if C is chosen so that $f(0) + C = 0$. Thus the particular function to be constructed is

$$f(x) - f(0) - xf'(0) - \tfrac{1}{2}mx^2, \qquad \text{(15)}$$

and this, by the preceding argument, is positive in the range. We therefore have the inequality

$$f(x) - f(0) - xf'(0) - \tfrac{1}{2}mx^2 > 0. \qquad \text{(16)}$$

In a corresponding way, from (13) we establish the inequality

$$f(x) - f(0) - xf'(0) - \tfrac{1}{2}Mx^2 < 0. \qquad \text{(17)}$$

We may write (16) and (17) in the forms

$$\left.\begin{aligned} f(x) &> f(0) + xf'(0) + \tfrac{1}{2}mx^2, \\ f(x) &< f(0) + xf'(0) + \tfrac{1}{2}Mx^2, \end{aligned}\right\} \quad (0 < x \leqslant x_2), \qquad \text{(18)}$$

which are the inequalities to be demonstrated.

As applied to the whole (positive) range $(0, x_2)$, these become

$$\left.\begin{aligned} f(x_2) &> f(0) + x_2 f'(0) + \tfrac{1}{2}mx_2^2, \\ f(x_2) &< f(0) + x_2 f'(0) + \tfrac{1}{2}Mx_2^2, \end{aligned}\right\} \qquad \text{(19)}$$

so that we may write

$$f(x_2) = f(0) + x_2 f'(0) + \tfrac{1}{2}\mu x_2^2, \qquad \text{(20)}$$

where μ is some number in the range $[m, M]$, however near m, M are to the respective lower and upper bounds of $f''(x)$ in the range $(0, x_2)$. It follows that μ cannot lie outside the range whose ends are these lower and upper bounds.

If the second derivative $f''(x)$ is *continuous* in the range $(0, x_2)$,

then μ must clearly be a value of the derivative for some value of x in the range. We may thus write

$$\mu = f''(\theta x_2), \quad (0 \leqslant \theta \leqslant 1), \qquad \text{.......(21)}$$

and (20) then becomes

$$f(x_2) = f(0) + x_2 f'(0) + \tfrac{1}{2}x_2^2 f''(\theta x_2). \qquad \text{.......(22)}$$

The results (20), (22) are established in a corresponding way when x_2 is negative.

For a variable end point x we have the form

$$f(x) = f(0) + xf'(0) + \tfrac{1}{2}x^2 f''(\theta x), \quad (0 \leqslant \theta \leqslant 1), \qquad \text{.......(23)}$$

a result which we shew below is included in the Intermediate Second Derivative Theorem.

99. 3. *Polynomials of the nth degree.* We are now prepared to write down the corresponding polynomials of the nth degree in x. We assume that the given function $f(x)$ is continuous in the range $(0, x_2)$ and that it possesses derivatives up to the nth order, the first $n-1$ of these being continuous and the nth, $f^{(n)}(x)$, being finite (but not necessarily continuous) at all points of the range. We introduce two numbers m, M, which are respectively less than the lower bound and greater than the upper bound of $f^{(n)}(x)$ in the range, so that we have

$$f^{(n)}(x) - m > 0, \qquad \text{.......(24)}$$

$$f^{(n)}(x) - M < 0, \qquad \text{.......(25)}$$

throughout the range, which we suppose at first to be positive.

Consider the inequality (24). We construct the function

$$f^{(n-1)}(x) - f^{(n-1)}(0) - mx, \qquad \text{.......(26)}$$

which vanishes when $x=0$ and has the positive derivative $f^{(n)}(x) - m$ in the range. It is therefore positive in the range. Continuing in this way, the successive positive functions introduced are

$$f^{(n-2)}(x) - f^{(n-2)}(0) - xf^{(n-1)}(0) - \frac{x^2}{2!}m,$$

$$f^{(n-3)}(x) - f^{(n-3)}(0) - xf^{(n-2)}(0) - \frac{x^2}{2!}f^{(n-1)}(0) - \frac{x^3}{3!}m,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f^{(1)}(x) - f^{(1)}(0) - xf^{(2)}(0) - \frac{x^2}{2!}f^{(3)}(0) - \ldots - \frac{x^{n-2}}{(n-2)!}f^{(n-1)}(0) - \frac{x^{n-1}}{(n-1)!}m,$$

$$f(x) - f(0) - xf^{(1)}(0) - \frac{x^2}{2!}f^{(2)}(0) - \frac{x^3}{3!}f^{(3)}(0) - \ldots - \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) - \frac{x^n}{n!}m,$$

where $0 < x \leqslant x_2$. We thus have the inequality

$$f(x) > f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n}{n!}m. \quad \ldots(27)$$

In a corresponding way, from (25) we reach the inequality

$$f(x) < f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n}{n!}M. \quad ..(28)$$

These latter are the inequalities to be demonstrated.

We conclude, as before, that

$$f(x_2) = f(0) + x_2 f^{(1)}(0) + \frac{x_2^{\,2}}{2!}f^{(2)}(0) + \ldots + \frac{x_2^{\,n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x_2^{\,n}}{n!}\mu, \quad ..(29)$$

where μ is some number in the range whose end points are the lower and upper bounds of the derivative $f^{(n)}(x)$ in the range $(0, x_2)$.

When $f^{(n)}(x)$ is *continuous*, we may write

$$\mu = f^{(n)}(\theta x_2), \quad (0 \leqslant \theta \leqslant 1), \quad \ldots\ldots\ldots\ldots(30)$$

and (29) becomes

$$f(x_2) = f(0) + x_2 f^{(1)}(0) + \frac{x_2^{\,2}}{2!}f^{(2)}(0) + \ldots$$
$$+ \frac{x_2^{\,n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x_2^{\,n}}{n!}f^{(n)}(\theta x_2). \quad \ldots(31)$$

The results (29), (31) are established in a corresponding manner when x_2 is negative.

Introducing x as variable end point, we have the form

$$f(x) = f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n}{n!}f^{(n)}(\theta x), \quad (32)$$

which, as we shall shew in the following Article, is included in the General Intermediate Derivative Theorem.[1]

99. 4. *Starting point different from zero.* Throughout the whole of the present Article we have worked only with a range $(0, x_2)$. When the range is (x_1, x_2), the fundamental inequalities corresponding to (27) and (28) are obtained by replacing 0 by x_1 and x by $x - x_1$. Thus, if $x_1 < x \leqslant x_2$, we get

$$\left.\begin{aligned} f(x) &> f(x_1) + (x - x_1)f^{(1)}(x_1) + \frac{(x - x_1)^2}{2!}f^{(2)}(x_1) + \ldots + \frac{(x - x_1)^n}{n!}m, \\ f(x) &< f(x_1) + (x - x_1)f^{(1)}(x_1) + \frac{(x - x_1)^2}{2!}f^{(2)}(x_1) + \ldots + \frac{(x - x_1)^n}{n!}M, \end{aligned}\right\} \ldots(33)$$

while, corresponding to (29), we have

$$f(x_2) = f(x_1) + (x_2 - x_1)f^{(1)}(x_1) + \frac{(x_2 - x_1)^2}{2!}f^{(2)}(x_1) + \ldots + \frac{(x_2 - x_1)^n}{n!}\mu. \quad \ldots(34)$$

[1] The symbolic statement of this theorem is the same as (32) except that θ now lies between 0 and 1 and the continuity of $f^{(n)}(x)$ is not required.

If $f^{(n)}(x)$ is continuous throughout (x_1, x_2), we may further write

$$\mu = f^{(n)}\{x_1 + \theta(x_2 - x_1)\}, \quad (0 \leqslant \theta \leqslant 1). \quad \text{......(35)}$$

100. The General Intermediate Derivative Theorem.

We now establish the *Intermediate nth Derivative Theorem*, (usually abbreviated to *I.D$_n$.T.*), which involves an *intermediate* value of the nth derivative $f^{(n)}(x)$ of a function $f(x)$ in a given range $(0, x_2)$.[1] The function $f(x)$ is assumed to be continuous throughout the range $(0, x_2)$, its successive derivatives up to the order $n-1$ inclusive are assumed to be continuous in the range $(0, x_2]$ and the nth derivative is supposed to exist (being finite or definite-infinite) for all values of x in the range $[0, x_2]$. The proof here given applies when n is any positive integer.

The result (29) of the preceding Article suggests the consideration of a function, $\varphi(x)$ say, defined by the equation

$$\varphi(x) = f(x) - \{f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \dots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n}{n!}\mu\} \quad (1)$$

in the range $(0, x_2)$, where μ is chosen so that $\varphi(x)$ vanishes when $x = x_2$, that is, we are to have

$$f(x_2) - \{f(0) + x_2 f^{(1)}(0) + \dots + \frac{x_2^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x_2^n}{n!}\mu\} = 0. \quad \text{.....(2)}$$

The function $\varphi(x)$ is continuous in the range $(0, x_2)$, since $f(x)$ and the other terms in the expression defining it are so. The same expression shews that $\varphi(x)$ has a definite derivative at each point of the range $(0, x_2)$. Rolle's Theorem is therefore applicable, and since $\varphi(x)$ vanishes at the ends of the range $(0, x_2)$, its derivative $\varphi^{(1)}(x)$ vanishes for some intermediate value, $\bar{x}_1$ say, in the range $[0, x_2]$, and we conclude that

$$f^{(1)}(\bar{x}_1) - \{f^{(1)}(0) + \bar{x}_1 f^{(2)}(0) + \dots + \frac{\bar{x}_1^{n-2}}{(n-2)!}f^{(n-1)}(0) + \frac{\bar{x}_1^{n-1}}{(n-1)!}\mu\} = 0. \quad \text{...(3)}$$

Consider next the function $\psi(x)$, defined by the equation

$$\psi(x) = f^{(1)}(x) - \{f^{(1)}(0) + xf^{(2)}(0) + \dots + \frac{x^{n-2}}{(n-2)!}f^{(n-1)}(0) + \frac{x^{n-1}}{(n-1)!}\mu\}. \quad \text{.(4)}$$

This is continuous in the range $(0, \bar{x}_1)$; it vanishes when $x = 0$ and also, in virtue of (3), when $x = \bar{x}_1$. Further, it has a definite derivative throughout the range $(0, \bar{x}_1)$, and therefore, by Rolle's Theorem,

[1] The corresponding result for a range (x_1, x_2) is easily deduced and is given a little later.

its derivative vanishes for some value, $\overline{x_2}$ say, in the range $[0, x_1]$. We conclude that

$$f^{(2)}(\overline{x_2}) - \{f^{(2)}(0) + \overline{x_2}f^{(3)}(0) + \ldots + \frac{\overline{x_2}^{n-3}}{(n-3)!}f^{(n-1)}(0) + \frac{\overline{x_2}^{n-2}}{(n-2)!}\mu\} = 0. \quad \ldots(5)$$

Proceeding in this way, we are led finally to consider the function $\chi(x)$ defined by the equation

$$\chi(x) = f^{(n-1)}(x) - \{f^{(n-1)}(0) + x\mu\}. \quad \ldots\ldots\ldots\ldots(6)$$

This is continuous in the range $(0, \overline{x_{n-1}})$, where x_{n-1} lies in the range $[0, x_2]$; it vanishes when $x=0$ and when $x = x_{n-1}$, and its derivative exists at all points of the range $[0, \overline{x_{n-1}})$. Applying Rolle's Theorem, we obtain the result $f^{(n)}(\overline{x_n}) - \mu = 0$, or

$$\mu = f^{(n)}(\overline{x_n}) = f^{(n)}(\bar{x}), \text{ say}, \quad \ldots\ldots\ldots\ldots(7)$$

where $\bar{x}$ lies in $[0, \overline{x_{n-1}}]$ and is therefore an intermediate value in the range $[0, x_2]$.

On substituting the value $f^{(n)}(\bar{x})$ of μ in (2) and transposing the terms in braces, we get, as the symbolic expression of the Intermediate nth Derivative Theorem for the function $f(x)$ in the range $(0, x_2)$,

$$f(x_2) = f(0) + x_2 f^{(1)}(0) + \frac{x_2^2}{2!}f^{(2)}(0) + \ldots + \frac{x_2^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x_2^n}{n!}f^{(n)}(\bar{x}), \quad (8)$$

where $\bar{x}$ lies in $[0, x_2]$. We may also write $\bar{x} = \theta x_2$, where now $0 < \theta < 1$, the final term on the right-hand side of (8) becoming

$$\frac{x_2^n}{n!}f^{(n)}(\theta x_2). \quad \ldots\ldots\ldots\ldots(9)$$

For a range (x_1, x_2), the theorem takes the form

$$f(x_2) = f(x_1) + (x_2 - x_1)f^{(1)}(x_1) + \frac{(x_2 - x_1)^2}{2!}f^{(2)}(x_1) + \ldots$$
$$+ \frac{(x_2 - x_1)^{n-1}}{(n-1)!}f^{(n-1)}(x_1) + \frac{(x_2 - x_1)^n}{n!}f^{(n)}(\bar{x}), \quad \ldots(10)$$

while if the starting point is a and the end point $a+h$, we have the form

$$f(a+h) = f(a) + hf^{(1)}(a) + \frac{h^2}{2!}f^{(2)}(a) + \ldots$$
$$+ \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h), \quad \ldots(11)$$

where $0 < \theta < 1$. The form (11) is commonly employed.

A variable end point can now be introduced. The values x_1, x_2 may be anywhere in a range throughout which $f^{(n)}(x)$ exists, and

therefore, keeping x_1 as a fixed starting point, the formula (10) holds for any end point x in such a range, giving

$$f(x)=f(x_1)+(x-x_1)f^{(1)}(x_1)+\dots$$
$$+\frac{(x-x_1)^{n-1}}{(n-1)!}f^{(n-1)}(x_1)+\frac{(x-x_1)^n}{n!}f^{(n)}(\bar{x}), \quad \dots(12)$$

where now $\bar{x}$, an intermediate value of the variable in the range $[x_1, x]$, is a function of x. When the starting point is the value 0 of x, this takes the form

$$f(x)=f(0)+xf^{(1)}(0)+\dots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+\frac{x^n}{n!}f^{(n)}(\bar{x}), \quad \dots .(13)$$

where $\bar{x}$ lies in $[0, x]$, and we can also write the final term on the right-hand side in the form $\frac{x^n}{n!}f^{(n)}(\theta x)$, where θ lies in the range $[0, 1]$ and is a function of x.

100. 1. *The Intermediate Second Derivative Theorem.* When $n=2$ we get the particular case of the *Intermediate Second Derivative Theorem*, (abbreviated to $I.D_2.T.$), in one or other of the following forms according to the range of application :

(i) Range $(0, x)$.

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(\theta x), \quad (0<\theta<1). \quad \dots\dots\dots\dots(14)$$

(ii) Range (x_1, x).

$$f(x)=f(x_1)+(x-x_1)f'(x_1)+\frac{(x-x_1)^2}{2!}f''(\bar{x}). \quad \dots\dots\dots\dots\dots(15)$$

(iii) Range $(a, a+h)$.

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a+\theta h), \quad (0<\theta<1). \dots\dots\dots(16)$$

In each case the range may be positive or negative.

Ex. 1. Shew that if $f''(x)$ is continuous in the range $(x, x+2h)$, then

$$\lim_{h\to 0}\frac{f(x+2h)-2f(x+h)+f(x)}{h^2}=f''(x). \quad \dots\dots\dots\dots\dots\dots\dots\dots(17)$$

We have, by the $I.D_2.T.$,

$$f(x+2h)-2f(x+h)+f(x)=f(x)+2hf'(x)+2h^2f''(\bar{x})$$
$$-2\{f(x)+hf'(x)+\tfrac{1}{2}h^2f''(\bar{x}^*)\}+f(x)$$
$$=h^2\{2f''(\bar{x})-f''(\bar{x}^*)\}, \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(18)$$

where $\bar{x}$, $\bar{x}^*$ are intermediate values of x in the respective ranges $[x, x+2h]$, $[x, x+h]$. If $h\to 0$, then $\bar{x}\to x$ and $\bar{x}^*\to x$ so that $f''(\bar{x})\to f''(x)$ and $f''(\bar{x}^*)\to f''(x)$. On dividing each side of (18) by h^2 and proceeding to the limit, the stated result is thus obtained.

The result may be put into the form [1]

$$\lim_{\Delta x \to 0} \frac{\Delta^2 f(x)}{(\Delta x)^2} = f''(x), \quad \text{......(19)}$$

where $\Delta^2 f(x)$ is the *second difference* of $f(x)$, (Art. 60. 2).

101. Taylor Polynomials.

In the expression of the Intermediate nth Derivative Theorem as applied to a function $f(x)$ in a range $(0, x)$, namely,

$$f(x) = f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n}{n!}f^{(n)}(\bar{x}), \quad ..(1)$$

the first n terms on the right-hand side form a polynomial in x of degree $n-1$, the coefficients being definite constants. We may call such a polynomial the *Taylor* [2] *polynomial* of degree $n-1$ for the function $f(x)$ considered. This polynomial is characterized by the fact that at the starting point, $(x=0)$, its value and the value of its successive derivatives up to the $(n-1)$th are equal respectively to those of the function $f(x)$. The excess of the function over its Taylor polynomial of degree $n-1$ is called the *Taylor remainder after n terms*, (or the *Taylor remainder of order n*), and is usually denoted by $R_n(x)$, so that

$$R_n(x) = \frac{x^n}{n!}f^{(n)}(\bar{x}). \quad \text{......(2)}$$

Other expressions for the remainder after n terms are investigated below, Art. 128, and in Chap. XVIII a definite formula, in which no undetermined number such as $\bar{x}$ appears, is found for this remainder.

The expressions $\frac{x^n}{n!}f^{(n)}(\bar{x})$, $\frac{x^n}{n!}\mu$ are often useful as determining a range in which the remainder after n terms lies. Such a range may be called a *remainder range*. The expression on the right-hand side of (2) is, rather loosely, called the *Lagrange formula* [3] for the remainder after n terms.

From the *practical* standpoint, the important fact is that in many cases the Taylor polynomials afford a convenient means of evaluat-

[1] The statement in Art. 86 with respect to the limit of $\Delta^n y/(\Delta x^n)$ is consequently now proved for the case when $n=2$.

[2] After Brook Taylor, who first investigated (in 1715) this mode of expression for a function.

[3] Historically and rationally the use of the term ' formula ' is no doubt correct, but from the present elementary point of view, a ' formula ' for anything is usually taken to be something that *determines* it (and not merely assigns a range for it, as here).

ing functions, that is, they are approximation formulae. In such cases the expressions for the remainder ranges are used to find the conditions under which the remainder is small enough to be neglected to the order of accuracy required. Many examples of this process occur in subsequent discussions. If the degree of the Taylor polynomial employed is fixed, the closeness of the approximation is limited and the error committed can be diminished indefinitely only by indefinite diminution of x.

From the *theoretical* standpoint, interest centres in the question whether the approximation by a Taylor polynomial can be made as close as we please by increasing the degree of the polynomial, when x is fixed. When this can be done, the Taylor polynomials of a function lead up to the *Taylor Infinite Series,* which defines the function exactly. This question is taken up in Art. 126, below.

If the starting point is x_1, the corresponding Taylor polynomial of degree $n-1$ is a polynomial in $x-x_1$, namely,

$$f(x_1)+(x-x_1)f^{(1)}(x_1)+\frac{(x-x_1)^2}{2!}f^{(2)}(x_1)+\ldots+\frac{(x-x_1)^{n-1}}{(n-1)!}f^{(n-1)}(x_1), \quad \ldots(3)$$

and the Taylor remainder, or Lagrange formula for the remainder after n terms, is now given by

$$R_n(x)=\frac{(x-x_1)^n}{n!}f^{(n)}(\bar{x}), \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

where $\bar{x}$ lies within the range (x_1, x_2).

102. Special Applications of the Intermediate Derivative Theorems.

We now consider some special applications of the Intermediate Derivative Theorems (and the associated Taylor polynomials) to the investigation of certain properties of functions and their graphs. Five divisions of the discussion are made, as follows :

I. *Approximate Evaluation of Functions. Small Corrections.* (Arts. 103, 104.)

II. *Maxima and Minima.* (Arts. 105-108.)

III. *Form of a Curve at a Point. Contact of Curves.* (Arts. 109-116.)

IV. *Newton's Method for Approximating to a Root of an Equation.* (Arts. 117-120.)

V. *Theory of Proportional Parts. Interpolation.* (Arts. 121-125.)

The treatment in each division applies to any function $f(x)$ which satisfies certain broad conditions, but in this Chapter illustrative

examples can be given only for algebraic functions. Illustrations by means of transcendental functions are given in Chaps. VI, VII as the various functions are introduced.

A further algebraic illustration of the $I.D_n.T.$ is given in Art. 127 of this Chapter where the Binomial Theorem for the expansion of $(1+x)^m$ is investigated. Finally it may be mentioned that the subject of the following Chapter, *Curvature of Plane Curves*, is also based on the Intermediate Derivative Theorems.

I. APPROXIMATE EVALUATION OF FUNCTIONS. SMALL CORRECTIONS.

103. Approximate Evaluation of Functions.

We here consider the application of the Taylor polynomials of the first and second degree to the approximate evaluation of functions. The employment of higher polynomials is illustrated later when dealing with particular functions. As remarked above, when the degree of the polynomial is fixed, the closeness of approximation attained is governed by the magnitude of the chosen increment of the independent variable. Thus the discussion involves an estimation of the error committed with the use of a chosen increment, and also the limitation of the increment necessary to secure a given degree of accuracy.

103. 1. *Polynomial of the first degree.* We begin by considering the application of the Taylor polynomial of the first degree. Let the value of the given function $f(x)$ be known when $x=x_1$, and let Δx denote an increment of x from x_1. The polynomial in question is here

$$f(x_1)+\Delta x f'(x_1). \quad \text{.............................(1)}$$

Now, assuming that the $I.D_2.T.$ is applicable to the function in the range $(x_1, x_1+\Delta x)$, we have

$$f(x_1+\Delta x)=f(x_1)+\Delta x f'(x_1)+\tfrac{1}{2}(\Delta x)^2 f''(\bar{x}), \quad \text{.............(2)}$$

where $\bar{x}$ lies in the range $[x_1, x_1+\Delta x]$. The error involved in the proposed approximation is therefore the remainder term in (2), namely $\frac{1}{2}(\Delta x)^2 f''(\bar{x})$. If we suppose that $|f''(\bar{x})|$ is not greater than a certain positive number M in any neighbourhood of x_1 that can possibly come into consideration in this discussion, a remainder range is $(-\frac{1}{2}(\Delta x)^2 M, \frac{1}{2}(\Delta x)^2 M)$, and the magnitude of the error committed cannot exceed $\frac{1}{2}(\Delta x)^2 M$.

To secure a prescribed degree of accuracy which limits the magni-

tude of the error to an assigned positive number ε, it is sufficient to make Δx satisfy the inequality $\frac{1}{2}(\Delta x)^2 M < \varepsilon$, that is, to make

$$|\Delta x| < \sqrt{(2\varepsilon/M)}. \qquad \text{.........................(3)}$$

If the neighbourhood of x_1 as so determined is smaller than that to which the maximum value M of $|f''(x)|$ applies, we may tentatively replace M by the maximum value of $|f''(x)|$ for a smaller range, and this may lead to a larger range for $|\Delta x|$. In practice such a procedure is not usually necessary, the order of magnitude of $|\Delta x|$ being sufficiently well known to make the first choice of M satisfactory.

Since in the approximate evaluation of functions we are usually concerned with values of ε which are small compared with 1, the magnitude of Δx will also usually be small compared with 1. In such a case it is convenient to denote this difference by δx, as on previous occasions when a limitation on the magnitude of the increment is implied, so that, loosely speaking, δx is a ' small ' number.

The increment of the dependent variable will, in general, be small when that of the independent variable is small, and the same symbol δ is used to denote such an increment. Thus we write δy, or $\delta f(x)$, to denote the increment of y, or $f(x)$, corresponding to the increment δx.

With this notation we have the approximation formula

$$f(x_1+\delta x) \approx f(x_1) + \delta x f'(x_1), \qquad \text{.......................(4)}$$

and the increment of the function is given approximately by

$$\delta f(x) \approx \delta x f'(x_1). \qquad \text{.............................(5)}$$

The magnitude of the error involved in each of these approximations is $\frac{1}{2}(\delta x)^2 |f''(\bar{x})|$, which, being ultimately in the ratio $|\frac{1}{2}f''(x_1)|$ to $(\delta x)^2$, is of the second order of smallness, (Art. 59. 1), if δx is small.

It may be observed that, according to the $I.D_1.T.$, we have

$$f(x_1+\delta x) = f(x_1) + \delta x f'(x_1+\theta\delta x), \quad (0<\theta<1), \qquad \text{..........(6)}$$

and a comparison of (4) and (6) shews that the approximation process is equivalent to ignoring the variation of $f'(x)$ in the range $(x_1, x_1+\delta x)$, the value taken being that at x_1. Many instances of ignoration of the variation of the first or a higher derivative in a small range will arise subsequently.

It should be noted that in a range of x in which the formula (4) is applied, the function $f(x)$ is replaced by a linear expression in x. For if we write x for $x_1+\delta x$, we have $f(x) \approx f(x_1) + (x-x_1)f'(x_1)$. We shall have other examples in the sequel of the replacement of a

function by a linear expression. In all such cases it is said that the function is taken to follow a *linear law.*

Hitherto we have considered the *absolute*[1] error involved in the approximate evaluation of the function or of the increment of the function. If, for any purpose, the *fractional* error of the increment is required, we must consider the ratio of the term $\frac{1}{2}(\delta x)^2 f''(\bar{x})$ to the term $\delta x f'(x_1)$. If the magnitude of this error, namely

$$\tfrac{1}{2}|\delta x| \,.\, |f''(\bar{x})| / |f'(x_1)|,$$

is to be less than an assigned positive number ε, we must now have

$$|\delta x| < 2\varepsilon |f'(x_1)|/M, \quad\text{.........................(7)}$$

where M is used in the same sense as above.

103. 2. *Geometrical interpretation.* A geometrical equivalent of the approximation (4) may be obtained as follows. Let AB (Fig. 68) be the graph of the function $f(x)$ in a certain range, and let P_1, P have the respective abscissae x_1, $x_1+\delta x$. The ordinates M_1P_1, MP are given by

$$M_1P_1 = f(x_1), \quad MP = f(x_1+\delta x). \quad\text{..(8)}$$

The tangent P_1R, of slope ψ_1, and the line P_1L drawn parallel to Ox, meet MP in R, L, respectively. From the Figure we have

$$LR/P_1L = \tan\psi_1 = f'(x_1),$$

so that

$$LR = P_1L\tan\psi_1 = \delta x f'(x_1). \quad\text{....(9)}$$

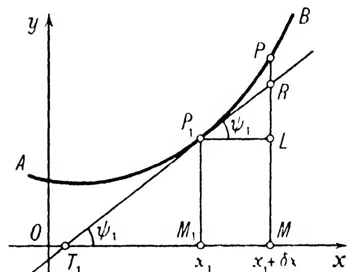


FIG. 68.

Now $$MP = ML + LR + RP = M_1P_1 + LR + RP, \quad\text{...........(10)}$$

and we may thus write

$$f(x_1+\delta x) = f(x_1) + \delta x f'(x_1) + RP. \quad\text{.................(11)}$$

Comparing this result with (2), we see that the algebraic length RP is given by

$$RP = \tfrac{1}{2}(\delta x)^2 f''(\bar{x}). \quad\text{.........................(12)}$$

The approximation (4) is thus equivalent geometrically to replacing the ordinate MP of the curve by the ordinate MR of the tangent,

[1] In this connection the term 'absolute error' is used for the algebraic excess of the approximation over the real value, as contrasted with the 'fractional' or 'relative error,' which is the ratio of the absolute error to the real value. It is usually sufficient, as in the text, to compare the error with the approximate value obtained for the increment.

that is, to ignoring the part RP. As we have seen, this part is of the second order of smallness if P_1L, that is, M_1M, is small.

The fractional error of the increment is the ratio of RP to LR, and this can be made as small as we please by sufficiently diminishing the magnitude of M_1M.

103. 3. *Polynomial of the second degree.* If we now go on to consider the second degree Taylor polynomial as an approximation formula for the value of $f(x_1+\delta x)$, the procedure is exactly similar to the above. Thus we write

$$f(x_1+\delta x)\approx f(x_1)+\delta x f'(x_1)+\tfrac{1}{2}(\delta x)^2 f''(x_1), \quad\text{............(13)}$$

which is equivalent to ignoring the variation of the second derivative in the range $(x_1, x_1+\delta x)$. Assuming that the $I.D_3.T.$ is applicable to this range, the error involved here is the remainder after three terms, namely $\frac{1}{6}(\delta x)^3 f'''(\bar{x})$, where $\bar{x}$ lies in the range $[x_1, x_1+\delta x]$.

If we assume that $|f'''(x)|$ is not greater than a certain number M^* in any neighbourhood of x_1 that can possibly arise in this discussion, a remainder range is $(-\frac{1}{6}(\delta x)^3M^*, \frac{1}{6}(\delta x)^3M^*)$. The absolute error will therefore be less than an assigned ε if $\frac{1}{6}|\delta x|^3M^*<\varepsilon$, that is, if

$$|\delta x|<(6\varepsilon/M^*)^{1/3}, \quad\text{........................(14)}$$

which determines a suitable range for $|\delta x|$.

Ex. 1. Investigate the restrictions on the magnitude of h in order that the value of $\surd(x_1+h)$ may be given correct to a stated degree of accuracy by the Taylor polynomial of the first degree, the value of $\surd x_1$ being supposed known.

Here $f(x)=\surd x$, $(x>0)$, so that

$$f'(x)=\frac{1}{2x^{1/2}}, \quad f''(x)=-\frac{1}{4x^{3/2}}. \quad\text{........................(15)}$$

The polynomial approximation in question is therefore

$$\surd(x_1+h)\approx\surd x_1.\left(1+\frac{h}{2x_1}\right), \quad\text{............................(16)}$$

and the absolute (excess) error E introduced is $h^2/\{8(\bar{x})^{3/2}\}$.

Suppose, at first, that $h>0$. Then $\bar{x}>x_1$ and we have

$$E<h^2/(8x_1^{3/2})$$
$$<\varepsilon \text{ (assigned) if } h<\surd(8\varepsilon x_1^{3/2}). \quad\text{........................(17)}$$

If $h<0$, a lower barrier to the value of $\bar{x}$ is now x_1+h. In the applications considered below, $|h|$ is never greater than $\frac{1}{10}x_1$, and therefore we have

$$E<h^2/\{8(\tfrac{9}{10}x_1)^{3/2}\}$$
$$<\varepsilon \text{ (assigned) if } -h<\surd\{8\varepsilon(\tfrac{9}{10}x_1)^{3/2}\}. \quad\text{...................(18)}$$

It follows that if we take

$$|h|<\surd\{8\varepsilon(\tfrac{9}{10}x_1)^{3/2}\}, \quad\text{..............................(19)}$$

the absolute error committed will, in either case, be less than ε.

To take a simple numerical illustration, let $x_1=4$, $\varepsilon=10^{-4}$. Then we are to have

$$|h|<\sqrt{\{8\times 10^{-4}\times(\tfrac{36}{10})^{3/2}\}}\approx 0\cdot 072. \quad\text{.....................(20)}$$

We can thus calculate $\sqrt{(4\cdot 072)}$ and $\sqrt{(3\cdot 928)}$ with an error of less than 1 in the fourth place of decimals by means of the approximation (16). The values found are

$$\left.\begin{aligned}\sqrt{(4\cdot 072)}&\approx 2+\tfrac{1}{4}\times 0\cdot 072=2\cdot 0180,\\ \sqrt{(3\cdot 928)}&\approx 2-\tfrac{1}{4}\times 0\cdot 072=1\cdot 9820.\end{aligned}\right\} \quad\text{.....................(21)}$$

The fractional error in the value of $\sqrt{(x_1+h)}$, as determined by (16), is

$$\frac{h^2/\{8(\bar{x})^{3/2}\}}{\sqrt{(x_1+h)}}. \quad\text{.................................(22)}$$

If $h>0$, this is less than $h^2/(8x_1^2)$, while if $h<0$ (and of magnitude less than $\frac{1}{10}x_1$), it is less than $h^2/\{8(\frac{9}{10}x_1)^2\}$.

The results obtained shew that for a given value of $|h|$, (less than $\frac{1}{10}x_1$), the absolute error in approximating to $\sqrt{(x_1+h)}$ by means of the formula (16) is always less than $h^2/\{8(\frac{9}{10}x_1)^{3/2}\}$, or $0\cdot 1464h^2/x_1^{3/2}$, and the fractional error always less than $h^2/\{8(\frac{9}{10}x_1)^2\}$, or $0\cdot 1543h^2/x_1^2$. In practice it is only necessary to consider the approximation

$$\sqrt{(1+h)}\approx 1+\tfrac{1}{2}h \quad\text{.................................(23)}$$

in this connection, for we can write (16) in the form

$$\sqrt{x_1}\,.\,\sqrt{(1+h/x_1)}\approx\sqrt{x_1}\,.\,(1+\tfrac{1}{2}h/x_1), \quad\text{.....................(24)}$$

which reduces to (23) on dividing out by $\sqrt{x_1}$ and changing the meaning of h. The absolute error involved in the approximation (23) is less than $0\cdot 1464h^2$ and the fractional error less than $0\cdot 1543h^2$ provided $|h|<\frac{1}{10}$.

Ex. 2. *Approximate evaluation of* $\sqrt{N}$, (*N a positive integer*). We start with the identity

$$(N+a^2)^2-(N-a^2)^2=4Na^2, \quad\text{.........................(25)}$$

which gives

$$N=\left(\frac{N+a^2}{2a}\right)^2\left\{1-\left(\frac{N-a^2}{N+a^2}\right)^2\right\}. \quad\text{.........................(26)}$$

If a is the positive integer whose square is nearest to N, the term

$$\{(N-a^2)/(N+a^2)\}^2$$

is usually small compared with 1, and, corresponding to (23), we may employ the approximation

$$\sqrt{N}\approx\frac{N+a^2}{2a}\left\{1-\frac{1}{2}\left(\frac{N-a^2}{N+a^2}\right)^2\right\}. \quad\text{.........................(27)}$$

For example, taking $N=10$, the appropriate value of a is 3, and (27) gives

$$\sqrt{10}\approx\frac{19}{6}\left(1-\frac{1}{2}\times\frac{1}{19^2}\right)=\frac{19}{6}-\frac{1}{12\times 19}\approx 3\cdot 162281. \quad\text{..............(28)}$$

The investigation of Ex. 1 above shews that the error introduced in writing $1-\frac{1}{2}\times\frac{1}{19^2}$ for $\sqrt{\left(1-\frac{1}{19^2}\right)}$ does not exceed $0\cdot 1464\times\frac{1}{19^4}$, so that the absolute error in the above calculation of $\sqrt{10}$ does not exceed $\frac{19}{6}\times 0\cdot 1464\times\frac{1}{19^4}$, or about $3\cdot 6\times 10^{-6}$. The value of $\sqrt{10}$ correct to six places of decimals is actually $3\cdot 162278$.

Ex. 3. *Approximate evaluation of* $(x_1+h)^{1/m}$. Here $f(x)=x^{1/m}$, $(x>0)$, and we have

$$f'(x)=\frac{1}{m}x^{\frac{1}{m}-1},\quad f''(x)=\frac{1}{m}\left(\frac{1}{m}-1\right)x^{\frac{1}{m}-2}. \qquad (29)$$

The Taylor polynomial approximation of the first degree is therefore expressed by

$$(x_1+h)^{\frac{1}{m}}\approx x_1^{\frac{1}{m}}+h\frac{1}{m}x_1^{\frac{1}{m}-1}, \qquad (30)$$

the absolute error E being given by

$$E=\frac{|1-m|}{2m^2}h^2(\bar{x})^{\frac{1}{m}-2}. \qquad (31)$$

(i) If m is negative, or positive and greater than 1/2, we may write (31) in the form

$$E=\frac{|1-m|}{2m^2}\frac{h^2}{(\bar{x})^{2-\frac{1}{m}}}. \qquad (32)$$

If $h>0$, so that $\bar{x}>x_1$, this gives

$$E<\frac{|1-m|}{2m^2}\frac{h^2}{x_1^{2-\frac{1}{m}}}, \qquad (33)$$

while if $h<0$, ($|h|$ being less than $\frac{1}{10}x_1$, which will commonly be the case in applications), we have

$$E<\frac{|1-m|}{2m^2}\frac{h^2}{(\frac{9}{10}x_1)^{2-\frac{1}{m}}}. \qquad (34)$$

With the aid of the results (33), (34), we can determine a suitable range for h in order to secure a given degree of accuracy with the use of (30) and for the values of m indicated.

(ii) Suppose next that m lies in the range $[0, \frac{1}{2}]$. If $0<h<\frac{1}{10}x_1$, we find from (31) that

$$E<\frac{|1-m|}{2m^2}h^2(\tfrac{11}{10}x_1)^{\frac{1}{m}-2}, \qquad (35)$$

while if $h<0$, we find

$$E<\frac{|1-m|}{2m^2}h^2x_1^{\frac{1}{m}-2}. \qquad (36)$$

These expressions allow us to determine a suitable range for h in order to secure a stated accuracy for the values of m indicated.

From (30) we have, on dividing out by $x_1^{1/m}$ and changing the meaning of h,

$$(1+h)^{1/m}\approx 1+\frac{1}{m}h, \qquad (37)$$

which is a form frequently employed.

Ex. 4. *Approximate evaluation of* $1/\sqrt{(x_1+h)}$. Here $m=-2$ and (30) gives

$$\frac{1}{\sqrt{(x_1+h)}}\approx\frac{1}{\sqrt{x_1}}\left(1-\frac{1}{2}\cdot\frac{h}{x_1}\right), \qquad (38)$$

while $E=\frac{3}{8}h^2/(\bar{x})^{5/2}$. If $h>0$, we have $E<\frac{3}{8}h^2/x_1^{5/2}$, so that E is less than an

assigned ε if $h < \sqrt{(\frac{8}{3}\varepsilon x_1{}^{5/2})}$. If $h < 0$ (and in magnitude less than $\frac{1}{10}x_1$), we have $E < \frac{3}{8}h^2/(\frac{9}{10}x_1)^{5/2}$, and now E is less than an assigned ε if $-h < \sqrt{\{\frac{8}{3}\varepsilon(\frac{9}{10}x_1)^{5/2}\}}$.

If, for example, $x_1 = 1$, $\varepsilon = 10^{-4}$, the above restrictions to be imposed on h are, in turn,

$$h < 0{\cdot}0163, \quad -h < 0{\cdot}0138.$$

Hence, if $|h| < 0{\cdot}0138$, the error committed in using the approximation

$$\frac{1}{\sqrt{(1+h)}} \approx 1 - \tfrac{1}{2}h \quad \text{.....................(39)}$$

is less than 1 in the fourth place of decimals.

104. Calculation of Small Corrections.

The theory of the preceding Article is constantly applied in Science in the discussion of the effect of small changes in the measures of physical quantities on the values of other quantities which are determined mathematically from them. If a definite small change is in question, the new value of a dependent quantity could, of course, be found by the same process as that applied to find the old value, but simple approximation formulae can usually be employed to great advantage and without affecting the degree of accuracy of the results. The mathematical discussion is not much altered if it is required to find the greatest possible error in the value of a dependent quantity when an upper barrier to the error of an observed quantity is assigned. So long as the rate of change of the dependent quantity with respect to the observed quantity has the same sign within the range of any error to be considered, the change of the dependent quantity cannot be greater than that due to the assigned upper barrier, and the latter change can be calculated by the appropriate approximation formula. Usually it is sufficient to employ the linear formula based on the assumption of a uniform rate of change.

To put these remarks into symbolic form, let y be the measure of the dependent quantity and let $y = f(x)$ be the relation connecting it with the measure x of an observed quantity. Then, according to the preceding Article, if δx is the assigned change, or the assigned upper barrier to the change, of x from the value x_1, the corresponding change of y is given approximately by

$$\delta y \approx \delta x f'(x_1). \quad \text{.....................(1)}$$

The magnitude of the error in δy committed in using this formula is $\frac{1}{2}(\delta x)^2\,|f''(\bar{x})|$, which, for purposes of estimation, may be taken as

$$\tfrac{1}{2}(\delta x)^2\,|f''(x_1)|. \quad \text{.....................(2)}$$

The fractional change of y from the value y_1, corresponding to the fractional change $\delta x/x_1$ of x, is given approximately by

$$\frac{\delta y}{y_1} \approx \frac{\delta x}{x_1} \frac{x_1 f'(x_1)}{f(x_1)} \quad \text{.............................(3)}$$

Ex. 1. The measured value of the radius of a circle is subject to an error of p per cent. at most. Find the possible percentage error in the calculated value of the area.

We have $A = \pi r^2$, and if $|\delta r|$ is the greatest error in the measurement of r, then

$$|\delta A| \approx \left|\frac{dA}{dr}\delta r\right| = 2\pi r\,|\delta r|, \quad \text{.............................(4)}$$

which gives approximately the maximum error in A. From this we have

$$\frac{\delta A}{A} \approx \frac{2\pi r}{\pi r^2}\delta r = 2\frac{\delta r}{r}, \quad \text{.................................(5)}$$

and hence, if $100|\delta r|/r < p$, then $100|\delta A|/A < 2p$, or the percentage error in A does not exceed $2p$.

104. 1. *More than one variable.* When the quantity y depends on more than one variable and the measure of each variable is subject to error, the total correction to be applied to the calculated value of y is to be found by calculating the small corrections due to each of the variables on the supposition that the remaining variables are correct, and adding. The formal justification of this rule requires the theory of the derivatives of polymetric functions, and cannot be given here. In certain simple cases, however, there is no practical difficulty in verifying its correctness, as in the following Examples.

Ex. 2. The lengths a, b, c of the edges of a cuboid are increased by δa, δb, δc, respectively. Find an approximate formula for the increase in the volume V of the cuboid.

We have $V = abc$. Supposing b, c unaltered, the increase of V due to the change δa is $bc\delta a$. Similarly, $ca\delta b$ and $ab\delta c$ are the increments of V due to the changes of b and c, each taken separately. The total change of V is therefore given approximately by

$$\delta V \approx bc\delta a + ca\delta b + ab\delta c. \quad \text{..............................(6)}$$

This result is easily verified. For, accurately,

$$\delta V = (a+\delta a)(b+\delta b)(c+\delta c) - abc, \quad \text{.......................(7)}$$

and this leads to (6) when the first product on the right-hand side is developed and terms containing products of two or more of the small quantities δa, δb, δc are neglected.

Ex. 3. The period τ of a simple pendulum of length l is given by the formula $\tau = 2\pi\sqrt{(l/g)}$, where g is the acceleration due to gravity. If l, g are increased by δl, δg, find the increase in the calculated value of τ.

The increase in τ due to the change in l alone is approximately $\frac{d\tau}{dl}\delta l$, or $\frac{\pi}{l}\sqrt{\frac{l}{g}}\,.\,\delta l$. That due to the change in g alone is $\frac{d\tau}{dg}\delta g$, or $-\frac{\pi}{g}\sqrt{\frac{l}{g}}\,.\,\delta g$. The total increase is thus given by

$$\delta\tau \approx \frac{\pi}{l}\sqrt{\frac{l}{g}}\,.\,\delta l - \frac{\pi}{g}\sqrt{\frac{l}{g}}\,.\,\delta g, \qquad (8)$$

or by

$$\frac{\delta\tau}{\tau} \approx \frac{1}{2}\frac{\delta l}{l} - \frac{1}{2}\frac{\delta g}{g}. \qquad (9)$$

Thus the percentage increment of τ is one-half the excess of the percentage increment of l over the percentage increment of g.

If we are merely given that l and g are subject to errors of magnitudes $|\delta l|$, $|\delta g|$, we must allow for the possibility of δl and δg having opposite signs in equations (8), (9). Thus τ will be subject to an error of magnitude $|\delta\tau|$, given by

$$\frac{|\delta\tau|}{\tau} \approx \frac{1}{2}\frac{|\delta l|}{l} + \frac{1}{2}\frac{|\delta g|}{g}. \qquad (10)$$

To verify the approximation (8), we observe that, accurately,

$$\delta\tau = 2\pi\sqrt{\frac{l+\delta l}{g+\delta g}} - 2\pi\sqrt{\frac{l}{g}} = \tau\sqrt{\frac{1+\delta l/l}{1+\delta g/g}} - \tau$$

$$\approx \tau\left(1+\frac{1}{2}\frac{\delta l}{l}\right)\left(1-\frac{1}{2}\frac{\delta g}{g}\right) - \tau, \qquad (11)$$

as in Exs. 1, 4 of the preceding Article. If we now develop the product in the last expression and neglect the term containing the product $\delta l\,\delta g$, we arrive at (9).

II. MAXIMA AND MINIMA.

105. Introductory Remarks.

We now resume the subject of maxima and minima from Art. 98. 6 and take up the systematic discussion of the discrimination between the stationary values of a function $f(x)$ in a range. This, as we shall see, involves the determination of the sign of the second or a higher derivative at the value considered of the independent variable. The function $f(x)$ is assumed to be continuous in the range and to possess a definite first derivative throughout, except possibly at isolated points; these are excluded from the following discussion. Suppose that there is a stationary value of $f(x)$ at the value x_1 of x, so that $f'(x_1)=0$. Then for the determination of the nature of the stationary value we have the following criteria according to the values of the successive derivatives of $f(x)$ at x_1.

106. Second Derivative Criterion.

We assume that there is a neighbourhood of x_1 throughout which the $I.D_2.T.$ is applicable. If $x_1+\Delta x$ is a point of this range, we have

$$f(x_1+\Delta x)=f(x_1)+\tfrac{1}{2}(\Delta x)^2 f''(x_1+\theta\Delta x), \quad (0<\theta<1), \text{........}(1)$$

the term $\Delta x f'(x_1)$ vanishing since $f'(x_1)=0$.

Suppose, now, that $f''(x)$ is *continuous* at x_1 and, moreover, that its value there is not zero. By Art. 22. 3, there is then a neighbourhood of x_1 throughout which $f''(x)$ has the same sign as $f''(x_1)$. If $x_1+\Delta x$ is taken to be a point of this neighbourhood, $x_1+\theta\Delta x$ is a point of it and $f''(x_1+\theta\Delta x)$ therefore has the same sign as $f''(x_1)$. If this sign is positive, the second term on the right-hand side of (1) is positive for either sign of Δx, and hence $f(x_1+\Delta x)>f(x_1)$. There is thus a proper local minimum value $f(x_1)$ of the function when $x=x_1$. Similarly, if the sign of $f''(x_1)$ is negative, the value $f(x_1)$ is a proper local maximum. If, however, $f''(x_1)=0$, no conclusion can be drawn as to the sign of $f''(x)$ in any neighbourhood of x_1 and we require to employ higher derivatives for the discrimination; see below.

Hence, if

(i) $f'(x_1)=0$, $f''(x_1)>0$, then $f(x_1)$ is a local minimum,

(ii) $f'(x_1)=0$, $f''(x_1)<0$, then $f(x_1)$ is a local maximum.

(iii) $f'(x_1)=0$, $f''(x_1)=0$, then further discussion is required.

Ex. 1. *The function* $x+a^2/x$, $(x\neq 0,\ a>0)$. We have

$$f'(x)=1-a^2/x^2, \quad f''(x)=2a^2/x^3. \text{........................}(2)$$

The first derivative vanishes when $x=a$, $x=-a$; at the former value $f''(x)>0$ and at the latter $f''(x)<0$. There is therefore a local minimum of the function when $x=a$, namely $2a$, and a local maximum $-2a$ when $x=-a$.

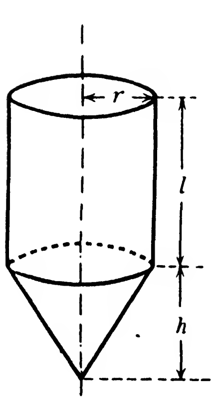


FIG. 69.

Ex. 2. A circular cylindrical tank, open at the top, has a conical bottom. It is required to find the least area of sheet metal for a given volume of tank.

This problem is most conveniently considered in two steps. First we suppose that in addition to a *given* volume V of tank there is also a *given* radius r of cylindrical portion, and we determine the ratio of the length l (Fig. 69) of cylindrical portion to the height h of cone which makes the total area of surface a minimum. In this way we express l, h in terms of r, V, and the relations obtained must hold whatever the values of r and V. We then suppose that V only is given. The area of surface is expressed in terms of the single variable r and its minimum value investigated.

(i) V, r *given.* The area of surface A is given by

$$A = 2\pi rl + \pi r\sqrt{(r^2+h^2)}, \quad\text{.....(3)}$$

and the volume by

$$V = \pi r^2 l + \tfrac{1}{3}\pi r^2 h = \pi r^2(l + \tfrac{1}{3}h). \quad\text{.....(4)}$$

Eliminating l between these equations we obtain

$$A = \pi r\left\{\frac{2V}{\pi r^2} - \tfrac{2}{3}h + \sqrt{(r^2+h^2)}\right\}, \quad\text{.....(5)}$$

expressing A as a function of h alone. We now obtain

$$\frac{dA}{dh} = \pi r\left\{-\frac{2}{3} + \frac{h}{(r^2+h^2)^{1/2}}\right\}, \quad \frac{d^2A}{dh^2} = \frac{\pi r^3}{(r^2+h^2)^{3/2}}. \quad\text{.....(6)}$$

The first derivative vanishes when $5h^2 = 4r^2$, or $h = 2r/\sqrt{5} \approx 0{\cdot}894r$, and since the second derivative is positive, the corresponding value of A is a minimum. Hence, whatever the given value of r, the least area of surface is obtained for the above value of h, and, from (4), l is then given by

$$l = \frac{V}{\pi r^2} - \frac{2\sqrt{5}}{15}r. \quad\text{.....(7)}$$

(ii) V *only given.* We now consider variations of r. For the above values of h, l, (3) gives

$$A = 2\pi r\left(\frac{V}{\pi r^2} - \frac{2\sqrt{5}}{15}r\right) + \pi r\sqrt{(r^2 + \tfrac{4}{5}r^2)} = \frac{2V}{r} + \frac{\pi\sqrt{5}}{3}r^2. \quad\text{.....(8)}$$

The derivative dA/dr vanishes when $-\dfrac{2V}{r^2} + \dfrac{2\pi\sqrt{5}}{3}r = 0$, that is, when

$$r^3 = \frac{3V}{\pi\sqrt{5}} = \frac{3\sqrt{5}}{5\pi}V. \quad\text{.....(9)}$$

There is thus only one value of r for which A is stationary, and this corresponds to a minimum, the second derivative of A being positive.

We now find

$$h^3 = \frac{24}{25\pi}V, \quad l^3 = \frac{3}{25\pi}V, \quad\text{.....(10)}$$

and the minimum area is $(9\pi\sqrt{5}\,.\,V^2)^{1/3}$, or $3{\cdot}984V^{2/3}$.

The most economical proportions are thus expressed by

$$h = 2r/\sqrt{5} = 0{\cdot}894r, \quad l = r/\sqrt{5} = 0{\cdot}447r, \quad\text{.....(11)}$$

the total height then being $1{\cdot}341r$.

106. 1. *Case of implicit functions.* The treatment of the extreme values of a function y given implicitly by an equation of the form

$$f(x, y) = 0 \quad\text{.....(12)}$$

proceeds as above as soon as the derivatives of y with respect to x which are required are determined. The process of 'differentiating an equation' can, in simple cases, be carried out successively for this purpose, if we assume the existence of the function y and its derivatives. From (12) we get, on differentiation, an equation

involving x, y, dy/dx, and this in turn yields an equation involving x, y, dy/dx and d^2y/dx^2. Putting $dy/dx=0$ in the former of these, we get an equation which, with (12), determines the stationary values, and the signs of d^2y/dx^2 at these values can be obtained from the latter of the two equations mentioned.

Introducing partial derivatives, we have, as in Art. 88,

$$\left.\begin{aligned} \frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0, \\ \frac{\partial^2 f}{\partial x^2}+\left(\frac{\partial^2 f}{\partial y\,\partial x}+\frac{\partial^2 f}{\partial x\,\partial y}\right)\frac{dy}{dx}+\frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2+\frac{\partial f}{\partial y}\frac{d^2y}{dx^2}=0, \end{aligned}\right\} \quad \text{........(13)}$$

and on putting $dy/dx=0$ in each of these, we get the two equations mentioned above in the forms

$$\frac{\partial f}{\partial x}=0, \quad \frac{\partial^2 f}{\partial x^2}+\frac{\partial f}{\partial y}\frac{d^2y}{dx^2}=0. \quad \text{.....................(14)}$$

We here assume that $\partial f/\partial y$ does not vanish at a stationary value.

Ex. 3. *The equation* $x^2+xy+y^2-1=0$. ....(15)

Here we have, on differentiating,

$$2x+y+(x+2y)\frac{dy}{dx}=0, \quad 2+2\frac{dy}{dx}+(x+2y)\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2=0. \text{ ....(16)}$$

When $dy/dx=0$ we have

$$2x+y=0, \quad \frac{d^2y}{dx^2}=-\frac{2}{x+2y}=\frac{2}{3x}. \text{ ....(17)}$$

Putting $y=-2x$ in (15), we find $x^2-2x^2+4x^2-1=0$, so that $x=\pm 1/\sqrt{3}$, and hence $y=\mp 2/\sqrt{3}$. These give the stationary values. For the upper signs, d^2y/dx^2 is positive and the value is a minimum; for the lower, d^2y/dx^2 is negative and the value is a maximum.

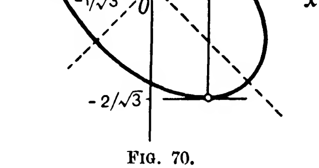


FIG. 70.

The equation (15) represents an ellipse of semi-axes $\sqrt{2}$, $\sqrt{\frac{2}{3}}$. The axes of the ellipse bisect the angles between the coordinate axes, (Fig. 70). The above solutions give the positions of the points of the ellipse which are farthest from Ox.

106. 2. *Change of independent variable.* The discussion of the maximum and minimum values of a given function can frequently be simplified by the introduction of an auxiliary variable which may be treated either as an intermediary variable (case of a function of a function), or as a new independent variable (case of parametric specification).

In the first case, let $y=f(t)$, $t=\varphi(x)$, so that $y=f\{\varphi(x)\}$. Then

$$\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}, \quad \frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}\left(\frac{dt}{dx}\right)^2+\frac{dy}{dt}\frac{d^2t}{dx^2}. \quad \text{.................(18)}$$

If we suppose that $dt/dx \neq 0$, the stationary values of y with respect to x correspond to the stationary values of y with respect to t. When $dy/dt=0$, the sign of d^2y/dx^2 is the same as that of d^2y/dt^2. Hence if y is a maximum (minimum) with respect to t it is also a maximum (minimum) with respect to x. Suppose, next, that $dt/dx=0$ and $dy/dt \neq 0$. The nature of each of the corresponding stationary values of y is now determined from the sign of the product of dy/dt and d^2t/dx^2.

In the case of parametric specification by equations of the form $y=f(t)$, $x=\psi(t)$, say, we have

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}, \quad \frac{d^2y}{dx^2}=\frac{\dfrac{d^2y}{dt^2}}{\left(\dfrac{dx}{dt}\right)^2}-\frac{\dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{\left(\dfrac{dx}{dt}\right)^3}. \quad \text{.................(19)}$$

We must here suppose that $dx/dt \neq 0$, the stationary values of y with respect to x corresponding to those with respect to t. The second equation shews that if y is a maximum (minimum) with respect to t, it is also a maximum (minimum) with respect to x.

Ex. 4. *The function* $y=(a^2-x^2)^4-4c^6(a^2-x^2)$, $(a^2>c^2)$. Putting $t=a^2-x^2$, we have $y=t^4-4c^6t$, and hence

$$\frac{dt}{dx}=-2x, \quad \frac{d^2t}{dx^2}=-2, \quad \frac{dy}{dt}=4t^3-4c^6, \quad \frac{d^2y}{dt^2}=12t^2. \quad \text{...........(20)}$$

The stationary values [1] of y with respect to x occur when $dy/dt=0$, that is, when $t=c^2$ and $x=\pm\sqrt{(a^2-c^2)}$; and when $dt/dx=0$, that is, when $x=0$ and $t=a^2$. Since $d^2y/dt^2>0$, the values $x=\pm\sqrt{(a^2-c^2)}$ correspond to minimum values of y. Further we have

$$\frac{dy}{dt}\frac{d^2t}{dx^2}=-2(4t^3-4c^6), \quad \text{...........................(21)}$$

and when $t=a^2$, the sign of this expression is negative. The corresponding value of y is therefore a maximum.

106. 3. *Absolute extreme in a range.* The following Example illustrates the determination of an absolute extreme value of a function in a given range.

Ex. 5. If the function $\sqrt{(1+x^2)}$ is replaced by the approximation $0{\cdot}93+0{\cdot}43x$ in the range $(0, 1)$ of x, find the greatest fractional error committed.

[1] These have already been investigated in Art. 94. 1, Ex. 2.

We may take the fractional error E to be given by

$$E=\frac{\sqrt{(1+x^2)}-(0{\cdot}93+0{\cdot}43x)}{\sqrt{(1+x^2)}}=1-\frac{0{\cdot}93+0{\cdot}43x}{\sqrt{(1+x^2)}}. \quad \text{..........(22)}$$

The stationary condition $dE/dx=0$ gives

$$\sqrt{(1+x^2)}(0{\cdot}43)-(0{\cdot}93+0{\cdot}43x)\frac{x}{\sqrt{(1+x^2)}}=0, \quad \text{..............(23)}$$

so that $0{\cdot}43-0{\cdot}93x=0$, or $x=0{\cdot}46$, approximately. It is easily shewn that for this value of x, $d^2E/dx^2>0$, so that the corresponding value of E is a local minimum. This value is approximately $-0{\cdot}03$.

At the lower end of the range ($x=0$) we have $E=1-0{\cdot}93=0{\cdot}07$, while at the upper end ($x=1$) we have $E=1-1{\cdot}36/\sqrt{2}\approx 0{\cdot}04$. The greatest fractional error is therefore about 7 per cent., and it occurs at the lower end of the range.

107. Third Derivative Criterion.

We here consider the case in which $f''(x_1)=0$ as well as $f'(x_1)=0$, and suppose that there is a neighbourhood of x_1 throughout which the $I.D_3.T.$ is applicable. If $x_1+\Delta x$ is in this range, we have

$$f(x_1+\Delta x)=f(x_1)+\tfrac{1}{6}(\Delta x)^3 f'''(x_1+\theta\Delta x), \quad (0<\theta<1). \text{........(1)}$$

Suppose now that $f'''(x)$ is *continuous* and different from zero at x_1. Then there is a neighbourhood of x_1 throughout which $f'''(x)$ has the same sign as $f'''(x_1)$. If $x_1+\Delta x$ is taken to be a point of this neighbourhood, the sign of $f'''(x_1+\theta\Delta x)$ is the same as that of $f'''(x_1)$. The increment $f(x_1+\Delta x)-f(x_1)$ has then the sign of $(\Delta x)^3 f'''(x_1)$. Since this sign changes with that of Δx, the stationary value $f(x_1)$ is neither a maximum nor a minimum. If $f'''(x_1)$ is positive (negative) the function $f(x)$ is strictly up-way (down-way) in the neighbourhood of x_1 referred to.

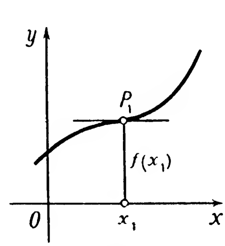


FIG. 71.

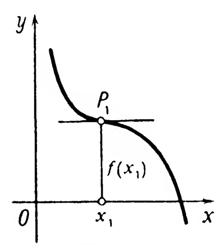


FIG. 72.

Interpreted graphically, the point P_1 (Figs. 71, 72) is a point of contraflexure with horizontal tangent. In Fig. 71, the third derivative is positive at P_1, and in Fig. 72, negative.

If $f'''(x_1)=0$, we must proceed to a fourth derivative criterion. This is included in the general criterion given below.

Ex. 1. *The function* x^3. We have

$$f'(x)=3x^2, \quad f''(x)=6x, \quad f'''(x)=6, \quad \text{....................(2)}$$

and hence, for the value 0 of x,

$$f'(0)=0, \quad f''(0)=0, \quad f'''(0)=6. \quad \text{......................(3)}$$

The stationary value at $x=0$ is therefore not an extreme and the function is strictly up-way in the neighbourhood of that point.

108. General Form of Criterion.

We now consider the case in which the first $n-1$ derivatives vanish at $x=x_1$ and suppose that there is a neighbourhood of x_1 throughout which the $I.D_n.T.$ is applicable. If $x_1+\Delta x$ is in this range, we have

$$f(x_1+\Delta x)=f(x_1)+\frac{1}{n!}(\Delta x)^n f^{(n)}(x_1+\theta\Delta x), \quad (0<\theta<1). \quad \text{......(1)}$$

If $f^{(n)}(x)$ is continuous and differs from zero at x_1, we conclude, as before, that for sufficiently small values of Δx, the sign of $f^{(n)}(x_1+\theta\Delta x)$ is the same as that of $f^{(n)}(x_1)$. Hence the sign of the increment $f(x_1+\Delta x)-f(x_1)$ will, for such values of Δx, be the same as that of $(\Delta x)^n f^{(n)}(x_1)$.

If n is *even*, this latter expression has the same sign as $f^{(n)}(x_1)$. If this sign is positive, there is a minimum value of $f(x)$ at x_1, and if negative, a maximum.

If n is *odd*, the sign of the expression changes with that of Δx. The stationary value at x_1 is therefore neither a maximum nor a minimum. This corresponds graphically to a point of contraflexure with horizontal tangent. The ordinates of the graph continually increase if $f^{(n)}(x_1)$ is positive, and continually decrease if $f^{(n)}(x_1)$ is negative.

Ex. 1. *The function* x^4. Here we have

$$f^{(1)}(x)=4x^3, \quad f^{(2)}(x)=12x^2, \quad f^{(3)}(x)=24x, \quad f^{(4)}(x)=24. \quad \text{..........(2)}$$

Since $f^{(1)}(0)=0=f^{(2)}(0)=f^{(3)}(0)$, $f^{(4)}(0)>0$, there is a minimum value 0 when $x=0$.

III. FORM OF A CURVE RELATIVE TO THE TANGENT AT A POINT OF IT. CONTACT OF TWO CURVES.

109. Inferences from the Sign of the Second Derivative.

Let P_1, (x_1, y_1), be a point on the curve $y=f(x)$ at which there is a definite tangent, which is at first supposed not parallel to Oy, so that $f'(x_1)$ is finite.[1] If the $I.D_2.T.$ is applicable to a range $(x_1, x_1+\Delta x)$, we have

$$f(x_1+\Delta x)=f(x_1)+\Delta x f'(x_1)+\tfrac{1}{2}(\Delta x)^2 f''(x_1+\theta\Delta x). \quad \text{.........(1)}$$

Suppose now that $f''(x)$ is continuous at x_1, its value there being

[1] The case in which the tangent is parallel to Oy is considered in Art. 114. 1.

different from zero. Then for sufficiently small values of Δx, the sign of $f''(x_1+\theta\Delta x)$ will be the same as that of $f''(x_1)$. We conclude that if this sign is positive (negative), the difference

$$f(x_1+\Delta x)-\{f(x_1)+\Delta x f'(x_1)\} \quad \text{.....................(2)}$$

is positive (negative).

Referring to Fig. 73, and recalling the results established in Art. 103. 2, we have $f(x_1+\Delta x)=MP$, $f(x_1)=M_1P_1$, $\Delta x f'(x_1)=LR$, and therefore, with the aid of (1) we have

$$RP=\tfrac{1}{2}(\Delta x)^2 f''(x_1+\theta\Delta x). \quad \text{...(3)}$$

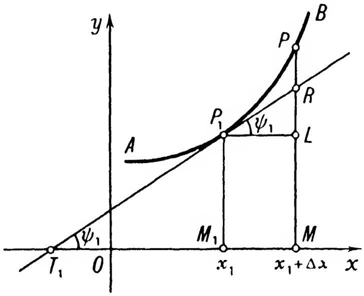


FIG. 73.

The distance RP therefore corresponds to the difference (2). If $f''(x_1)>0$, it follows that $MP>MR$, while if $f''(x_1)<0$, then $MP<MR$, whether M lies to the right or left of M_1.

110. Sides of a Line.

It is now found convenient to define the positive and negative sides of a straight line with respect to a given direction. To do this, we consider the two half-planes bounded by the line. If a point moves in the given direction and crosses[1] the line, it passes from one half-plane to the other. In the half-plane which it enters it is said to be on the *positive* side of the line with respect to the given direction, while in the other half-plane it is said to be on the *negative* side of the line. The half-planes themselves can clearly also be distinguished as positive and negative with respect to the given direction. A configuration lies on the positive side of a line, or in the positive half-plane, if all its points do so.

With these definitions, the theorem of the preceding Article can be stated in the form that if $f''(x_1)$ is positive (negative), there is a neighbourhood of the point M_1, (x_1), for which the curve lies on the positive (negative) side of the tangent at P_1 with respect to the direction Oy.

Owing to the practice of drawing Oy in the direction described as ' upwards ' on a sheet (or board), it is customary to say that in the neighbourhood of P_1 the curve lies *above* (*below*) the tangent at P_1 if $f''(x_1)$ is positive (negative).

[1] If the given direction is parallel to the line, the distinction to be defined does not exist.

111. Relative Position of Arc and Chord.

We now prove that in a range for which $f''(x)$ is positive (negative), any portion of the curve lies below (above) the chord joining its ends.

An informal demonstration may be given on the following lines. If $f''(x)$ is always positive, the gradient of the curve is strictly up-way with respect to x. Thus if we draw the tangents at the ends, (x_1, y_1), (x_2, y_2) say, of an arc, (where $x_2 > x_1$), and a set of intermediate tangents, the open polygon formed by these tangents and terminated at the ends of the arc lies below the chord joining these ends. If $f''(x)$ is negative in the neighbourhood, the polygon lies above the chord. Since the points of the curve can be made to approximate as nearly as we please to such a polygon by increasing the number of its sides, the required result follows. We now proceed to a formal proof.

Taking (x_1, y_1), (x_2, y_2) as the coordinates of the ends of an arc of the curve and (x, y) as the coordinates of an intermediate point, the theorem is proved by shewing that the gradient of the *chord* joining the points (x_1, y_1), (x, y) increases (decreases) continually as x increases from x_1 to x_2 if $f''(x)$ is positive (negative) throughout (x_1, x_2).

The gradient of this chord is $\{f(x) - f(x_1)\}/(x - x_1)$, where $x_1 < x < x_2$, and its rate of increase with respect to x is

$$\frac{(x - x_1)f'(x) - f(x) + f(x_1)}{(x - x_1)^2}. \quad \text{.......................(1)}$$

Now

$$f(x_1) = f\{x - (x - x_1)\} = f(x) - (x - x_1)f'(x) + \tfrac{1}{2}(x - x_1)^2 f''(\bar{x}), \quad \text{....(2)}$$

where $\bar{x}$ lies in $[x_1, x]$. The rate (1) can therefore be written $\tfrac{1}{2}f''(\bar{x})$, which is positive (negative) if $f''(x)$ is positive (negative) in (x_1, x_2). Thus all the points such as P, (x, y), lie below (above) the chord joining (x_1, y_1) and (x_2, y_2) if $f''(x)$ is positive (negative); see Fig. 74.

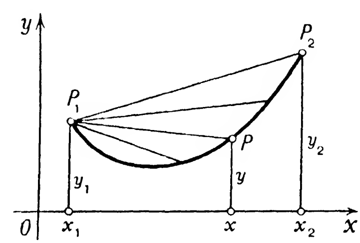


FIG. 74.

112. Concavity and Convexity of a Curve.

The properties proved above are those associated with the notion of *concavity* (or *convexity*) of a curve. We describe a curve as concave or convex, with respect to an assigned direction, *at a point* P_1 *on it*, according as its deflection in the neighbourhood of P_1 from the tangent at that point is on the

positive or negative side of the tangent with respect to the assigned direction. Referring to Fig. 75, the curve is concave at P_1 and convex at P_2 with respect to the direction of the axis Oy. For the reason already explained, a curve which is, at a point on it, concave (convex) with respect to the direction Oy is described as *concave* (*convex*) *upwards* at the point. The above argument shews that if there is a continuous second derivative at the point, the curve is concave (convex) upwards there if the sign of this second derivative is positive (negative).

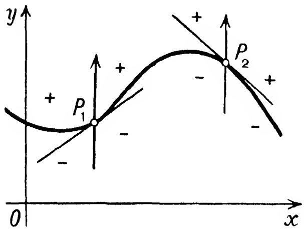


FIG. 75.

The notion of concavity with respect to an assigned direction is also applied to a curve in a finite range. Thus we describe a curve as concave (convex) with respect to the assigned direction *in a range* if it is concave (convex) with respect to this direction at all points of the range. Taking the direction of the y-axis as that to which the concavity is referred, it follows that if there is a continuous second derivative throughout the range, the curve is concave (convex) upwards in the range if the second derivative is positive (negative) throughout.

Ex. 1. Examine the form of the curve $y=x^3-2x^2+2x-1$ in the neighbourhood of $x=1$.

We have $f'(x)=3x^2-4x+2$, $f''(x)=6x-4$. When $x=1$, the sign of the second derivative is positive and the curve is therefore concave upwards. Also $f(1)=0$, $f'(1)=1$, so that the ordinate to the point is 0 and the gradient there 1, that is, $\tan\psi=1$, $\psi=\frac{1}{4}\pi$ (Fig. 76).

112. 1. *Case in which $f''(x_1)$ is definite-infinite.* In the above discussion we have assumed that $f''(x)$ is continuous and therefore finite at x_1. The following Example illustrates the determination of the form of a curve relative to the tangent at a point where the second derivative becomes definite-infinite.

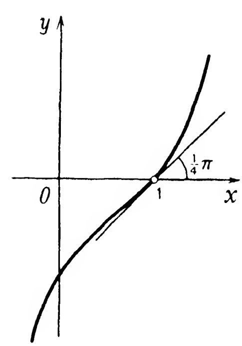


FIG. 76.

Ex. 1. *The equation* $y-y_1=g_1(x-x_1)+k(x-x_1)^m$. Here k, m are disposable constants and g_1 is the gradient at x_1. Differentiating, we get

$$g-g_1=km(x-x_1)^{m-1}. \qquad (1)$$

Now dg/dx, or d^2y/dx^2, will become definite-infinite at x_1 if the graph of the function $g(x)$ is continuous and has a vertical tangent (and a point of

contraflexure) at x_1. For the sake of definiteness consider the case in which $m=4/3$, $km=1$, so that $g-g_1=(x-x_1)^{1/3}$. This gives

$$x-x_1=(g-g_1)^3. \quad \text{..................(2)}$$

The graph of $g(x)$ is then of the form shewn in Fig. 77, (where $x_1=1$, $g_1=1$). From (2) we have

$$\frac{dg}{dx}=\frac{1}{3(x-x_1)^{2/3}}, \quad \text{..................(3)}$$

so that dg/dx is positive and finite except at x_1, where it is definite-infinite.

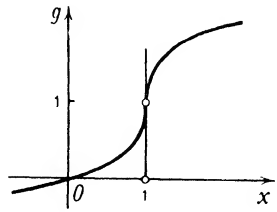


FIG. 77.

It follows that the graph of the function

$$y=y_1+g_1(x-x_1)+\tfrac{3}{4}(x-x_1)^{4/3} \quad \text{........(4)}$$

is concave upwards in the neighbourhood of x_1, as shewn in Fig. 78, (where $x_1=1$, $y_1=1$), but at this point it has a (positive) definite-infinite value of the second derivative.

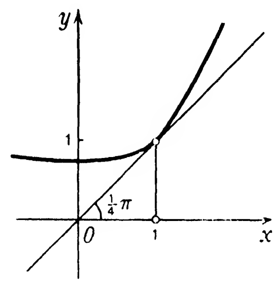


FIG. 78

113. Use of the Third Derivative.

Suppose, next, that $f''(x_1)=0$, so that, assuming the $I.D_3.T.$ applicable to the range $(x_1, x_1+\Delta x)$,

$$f(x_1+\Delta x)=f(x_1)+\Delta x f'(x_1)+\tfrac{1}{6}(\Delta x)^3 f'''(x_1+\theta\Delta x), \quad (0<\theta<1). \text{...(1)}$$

If $f'''(x)$ is continuous at x_1 and $f'''(x_1)\neq 0$, we conclude, as before, that for suitably restricted values of Δx, the difference

$$f(x_1+\Delta x)-\{f(x_1)+\Delta x f'(x_1)\} \quad \text{.....................(2)}$$

has the sign of $(\Delta x)^3 f'''(x_1)$, which changes when that of Δx changes.

Referring to Fig. 73, p. 240, we now have the result

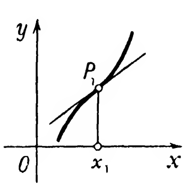
$$RP=\tfrac{1}{6}(\Delta x)^3 f'''(x_1+\theta\Delta x). \quad \text{.......................(3)}$$

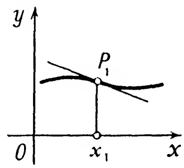
It follows that for all points M of a certain neighbourhood of M_1 within which the sign of $f'''(x)$ is constant, the sign of RP is that of the product of the (algebraic) increment M_1M and $f'''(x_1)$. If $f'''(x_1)>0$, then $MP>MR$ to the right of M_1 and $MP<MR$ to the left. If $f'''(x_1)<0$, then $MP<MR$ to the right of M_1 and $MP>MR$ to the left. In each case the curve crosses its tangent at P_1 and there is a change in the sense of bending as we pass through P_1. This point is therefore a *point of contraflexure* on the curve (Art. 94), and the above argument shews that if the second derivative is continuous at such a point, its value there must be zero.

The abscissae of the points of contraflexure of a curve are thus to be sought among the values of x which satisfy the equation

$$f''(x)=0. \quad \ldots\ldots(4)$$

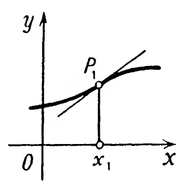
Employing the language of the preceding Articles, we say that the form of the curve relative to the tangent at P_1 changes from convex upwards (on the left) to concave upwards (on the right) if $f'''(x_1)$ is positive, and from concave to convex upwards if $f'''(x_1)$ is negative. These cases are illustrated in Figs. 79-82.

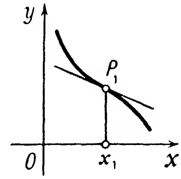




$f''(x_1)=0, \quad f'''(x_1)>0.$

FIG. 79. FIG. 80.





$f''(x_1)=0, \quad f'''(x_1)<0.$

FIG. 81. FIG. 82.

114. General Criterion.

If $f^{(3)}(x_1)=0$, the form of the curve may be decided by appealing to the sign of the fourth derivative at x_1, and so on. If the first of the derivatives of $f(x)$, after $f^{(1)}(x)$, which does not vanish at x_1 is the nth, we have, on assuming that there is a neighbourhood of x_1 within which the $I.D_n.T.$ is applicable,

$$f(x_1+\Delta x)=f(x_1)+\Delta x f^{(1)}(x_1)+\frac{1}{n!}(\Delta x)^n f^{(n)}(x_1+\theta\Delta x), \quad (0<\theta<1). \ldots(1)$$

We conclude that

$$RP=\frac{1}{n!}(\Delta x)^n f^{(n)}(x_1+\theta\Delta x). \quad \ldots\ldots(2)$$

Assuming $f^{(n)}(x)$ continuous at x_1, then for a certain neighbourhood of x_1 the sign of $f^{(n)}(x_1+\theta\Delta x)$ is that of $f^{(n)}(x_1)$, and for points in such a neighbourhood the sign of RP is that of $(\Delta x)^n f^{(n)}(x_1)$. If n is even, this sign is that of $f^{(n)}(x_1)$, and the curve lies above or below its tangent at P_1, (that is, is concave or convex upwards at P_1), according as this sign is positive or negative. If n is odd, the curve crosses its tangent at P_1, which is therefore a point of contraflexure.

Ex. 1. Examine the nature of the graphs of the following functions at $x=1$: (i) x^3-3x^2+2x+1, (ii) $x^4-4x^3+6x^2-3x+3$.

(i) We have

$$f^{(1)}(x)=3x^2-6x+2, \quad f^{(2)}(x)=6x-6, \quad f^{(3)}(x)=6. \quad \ldots\ldots(3)$$

Therefore $f(1)=1, f^{(1)}(1)=-1, f^{(2)}(1)=0, f^{(3)}(1)>0$. The point (1, 1) is thus a point of contraflexure at which the gradient is -1.

(ii) Here we have

$$\left.\begin{aligned} f^{(1)}(x) &= 4x^3 - 12x^2 + 12x - 3, \quad f^{(2)}(x) = 12x^2 - 24x + 12, \\ & f^{(3)}(x) = 24x - 24, \quad f^{(4)}(x) = 24. \end{aligned}\right\} \quad \dots(4)$$

These give $f(1) = 3$, $f^{(1)}(1) = 1$, $f^{(2)}(1) = 0$, $f^{(3)}(1) = 0$, $f^{(4)}(1) > 0$. The curve is concave upwards in the neighbourhood of $(1, 3)$, the gradient there being 1.

114. 1. *Case in which the tangent is parallel to Oy.* The simplest procedure in this case is to treat y as the independent variable, the equation of the curve being supposed expressed in the form $x = f(y)$. The discussion is thus brought back to that of the maximum and minimum values of x with respect to y, for we now have $f^{(1)}(y_1) = 0$. Assuming a continuous second derivative $f^{(2)}(y)$ at P_1, we may state that there is a neighbourhood of P_1 in which the curve is concave or convex with respect to the direction Ox according as $f^{(2)}(y_1)$ is positive or negative. If, however, $f^{(2)}(y_1) = 0$, $f^{(3)}(y_1) \neq 0$, there will be a point of contraflexure at P_1, and so on.

115. Contact of Two Curves.

We now consider a simple extension of the above discussion to the investigation of the position of a curve relative to another curve in the neighbourhood of a point of meeting.

Let AB, CD (Fig. 83) be the curves $y = f(x)$, $y = \varphi(x)$, which meet at P_1, (x_1, y_1), so that

$$f(x_1) - \varphi(x_1) = 0. \quad \dots\dots\dots\dots(1)$$

Let P, P' be neighbouring points on the curves, having coordinates $(x, f(x))$, $(x, \varphi(x))$, where $x = x_1 + \Delta x$. Then we have

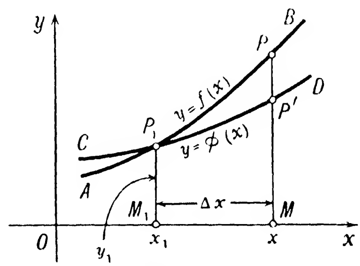


FIG. 83.

$$P'P = f(x) - \varphi(x) = f(x_1 + \Delta x) - \varphi(x_1 + \Delta x). \quad \dots\dots\dots\dots(2)$$

Suppose, now, that there is a neighbourhood of x_1 in which the $I.D_n.T.$ is applicable for each of the functions $f(x)$, $\varphi(x)$, and hence for the difference $f(x) - \varphi(x)$. If $x_1 + \Delta x$ is in this neighbourhood, then

$$\begin{aligned} f(x_1 + \Delta x) - \varphi(x_1 + \Delta x) &= \{f(x_1) - \varphi(x_1)\} + \Delta x\{f^{(1)}(x_1) - \varphi^{(1)}(x_1)\} \\ &\quad + \frac{1}{2!}(\Delta x)^2\{f^{(2)}(x_1) - \varphi^{(2)}(x_1)\} + \dots \\ &\quad + \frac{1}{n!}(\Delta x)^n\{f^{(n)}(\bar{x}) - \varphi^{(n)}(\bar{x})\}, \quad \dots\dots\dots(3) \end{aligned}$$

where $\bar{x}$ lies in $[x_1, x_1+\Delta x]$. The first term in braces on the right-hand side vanishes, according to (1), since the curves intersect at P_1.

(i) Suppose, at first, that $f^{(1)}(x_1) \neq \varphi^{(1)}(x_1)$. The curves do not then touch at P_1, but cross at an angle. Taking $n=1$, (3) gives

$$P'P = \Delta x\{f^{(1)}(\bar{x}) - \varphi^{(1)}(\bar{x})\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

so that, assuming continuity of the first derivatives at x_1, $P'P$ is ultimately in the ratio of $\{f^{(1)}(x_1) - \varphi^{(1)}(x_1)\}$ to Δx and is thus of the first order of smallness if $|\Delta x|$ is small. The separation of the curves as we proceed from P_1 is ultimately proportional to Δx.

It follows from (4) that in a certain neighbourhood of x_1, the algebraic distance $P'P$ has the same sign as $\Delta x\{f^{(1)}(x_1) - \varphi^{(1)}(x_1)\}$. If $f^{(1)}(x_1) > \varphi^{(1)}(x_1)$, as in the Figure, P lies above P' when these points are to the right of P_1, and below when to the left. If $f^{(1)}(x_1) < \varphi^{(1)}(x_1)$, these statements are reversed.

(ii) Suppose, next, that $f^{(1)}(x_1) = \varphi^{(1)}(x_1)$, but $f^{(2)}(x_1) \neq \varphi^{(2)}(x_1)$. Then, taking $n=2$ in (3), we have

$$P'P = \tfrac{1}{2}(\Delta x)^2\{f^{(2)}(\bar{x}) - \varphi^{(2)}(\bar{x})\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

so that, assuming continuity of the second derivatives at x_1, $P'P$ is of the second order of smallness if $|\Delta x|$ is small. The curves now touch at P_1 and their separation is ultimately proportional to $(\Delta x)^2$.

It follows from (5) that in a certain neighbourhood of x_1, the curve $y=f(x)$ lies above (below) the curve $y=\varphi(x)$ on both sides of P_1 if $f^{(2)}(x_1)$ is greater (less) than $\varphi^{(2)}(x_1)$.

(iii) If, however, $f^{(2)}(x_1) = \varphi^{(2)}(x_1)$, but $f^{(3)}(x_1) \neq \varphi^{(3)}(x_1)$, we have, taking $n=3$ in (3),

$$P'P = \tfrac{1}{6}(\Delta x)^3\{f^{(3)}(\bar{x}) - \varphi^{(3)}(\bar{x})\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

and now $P'P$ is of the third order of smallness if $|\Delta x|$ is small. The separation of the curves is ultimately proportional to $(\Delta x)^3$ and the curves cross each other at P_1.

(iv) Generalizing, it appears that for a given small advance Δx from the point M_1 along Ox, the distance $P'P$ varies ultimately as the nth power of Δx when the first $n-1$ derivatives of the two functions are respectively equal at x_1 and the nth derivatives are unequal. In this case the curves are said to have *contact of the* $(n-1)$*th order at* (x_1, y_1). In particular, if $f^{(1)}(x_1) \neq \varphi^{(1)}(x_1)$, the order of contact is zero, while if $f^{(1)}(x_1) = \varphi^{(1)}(x_1)$ but $f^{(2)}(x_1) \neq \varphi^{(2)}(x_1)$, the order is the first, and so on. The curves cross or not according as n is odd or even, that is, according as the order of contact is even or odd.

The phraseology here introduced is applicable to the case considered in the first part of the present division, the tangent to the first curve, $y=f(x)$ say, replacing the second curve $y=\varphi(x)$. There is contact of the $(n-1)$th order at x_1 if the first of the derivatives after $f^{(1)}(x_1)$ which does not vanish is the nth. For example, at a point of contraflexure there is contact of the second order, at least.

Ex. 1. Find the constants of the parabola $y=a+bx+cx^2$ in order that it may have contact of the highest order with a given curve $y=f(x)$ at x_1.

The highest order will, in general, be the second since for the parabola we have $d^ny/dx^n=0$ when $n>2$. The equations to determine a, b, c are thus

$$f(x_1)=a+bx_1+cx_1^2, \quad f'(x_1)=b+2cx_1, \quad f''(x_1)=2c. \qquad \text{.............(7)}$$

These give

$$a=f(x_1)-x_1f'(x_1)+\tfrac{1}{2}x_1^2f''(x_1), \quad b=f'(x_1)-x_1f''(x_1), \quad c=\tfrac{1}{2}f''(x_1), \quad \text{...(8)}$$

and the required equation becomes

$$y=f(x_1)+(x-x_1)f'(x_1)+\tfrac{1}{2}(x-x_1)^2f''(x_1). \qquad \text{.................(9)}$$

Thus the approximation obtained by replacing the curve by this parabola is equivalent to replacing $f(x)$ by its Taylor polynomial of the second degree.

The question of finding the circle having contact of the highest order with a given curve is discussed in Chap. V, Art. 136.

116. Distance between Curve and Tangent Measured Parallel to the Axis Ox.

Let the curve AB (Fig. 84) pass through the origin O of coordinates and have slope ψ_0 there. If P, (x, y), is any other point on the curve, the distance $|PQ|$ between P and the tangent at O, measured parallel to Ox, is equal to $|x-y\cot\psi_0|$. Now, by the $I.D_2.T.$, we have

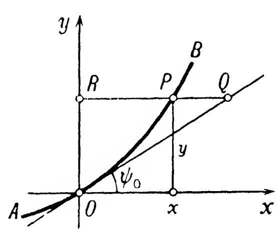


FIG. 84.

$$x=\left(\frac{dx}{dy}\right)_0 y+\frac{1}{2}\frac{d^2x}{dy^2}y^2, \qquad \text{..........(1)}$$

$$y=\left(\frac{dy}{dx}\right)_0 x+\frac{1}{2}\frac{d^2y}{dx^2}x^2, \qquad \text{..........(2)}$$

the second derivatives being taken at (different) intermediate points. From these we find, in turn,

$$|x-y\cot\psi_0|=\tfrac{1}{2}\left|\frac{d^2x}{dy^2}\right|y^2, \quad |x-y\cot\psi_0|=\tfrac{1}{2}\left|\frac{d^2y}{dx^2}\cot\psi_0\right|x^2. \quad \text{...(3)}$$

Now, by Art. 89. 3, we have

$$\frac{d^2x}{dy^2}=-\frac{d^2y/dx^2}{(dy/dx)^3}=-\frac{f''(x)}{\{f'(x)\}^3}, \qquad \text{......................(4)}$$

and we thus write

$$|x - y\cot\psi_0| = \tfrac{1}{2}\frac{|f''(\theta x)|}{|f'(\theta x)|^3}y^2, \quad\text{.....................(5)}$$

$$|x - y\cot\psi_0| = \tfrac{1}{2}\left|\frac{f''(\vartheta x)}{\tan\psi_0}\right|x^2. \quad\text{.......................(6)}$$

Hence, if $|f''(x)|_u$ is an upper barrier to $|f''(x)|$ in the range OP and $|f'(x)|_l$ a lower barrier to $|f'(x)|$ in the same range, we have the inequalities

$$|PQ| < \tfrac{1}{2}\frac{|f''(x)|_u}{\{|f'(x)|_l\}^3}y^2, \quad\text{.........................(7)}$$

$$|PQ| < \tfrac{1}{2}\frac{|f''(x)|_u}{|f'(x)|_l}x^2. \quad\text{.........................(8)}$$

These results are employed in the following division, Art. 119, in connection with the estimation of the error in Newton's method of approximating to a root of an equation.

IV. NEWTON'S METHOD FOR APPROXIMATING TO A ROOT OF AN EQUATION.

117. Description of the Method.

We here investigate a method, due to Newton, of approximating to a root of an algebraic or a transcendental equation. Let the equation be $f(x)=0$, and suppose that x_1 is an approximation to a root, found tentatively by substituting a succession of values for x in the function $f(x)$ or by sketching the graph of the function and observing the abscissae of the points of intersection with Ox, or in any other way. If the actual root in question is x_1+h, then $f(x_1+h)=0$, and, assuming the existence of a definite derivative $f'(x)$ in the neighbourhood of x_1, we have, by the $I.D_1.T.$,

$$f(x_1+h) = f(x_1) + hf'(\bar{x}) = 0, \quad\text{......................(1)}$$

which gives

$$h = -f(x_1)/f'(\bar{x}), \quad\text{.........................(2)}$$

where $\bar{x}$ lies in the range $[x_1, x_1+h]$.

On account of the presence of the undetermined number $\bar{x}$, this equation is not immediately applicable to the determination of h. An approximation to the true value is, however, obtained if, with Newton, we neglect the variation of $f'(x)$ in the range (x_1, x_1+h), as in Art. 103. 1, and take its value as equal to $f'(x_1)$. Employing the same symbol h to denote this approximate correction to the value x_1, we have

$$h = -f(x_1)/f'(x_1), \quad\text{..........................(3)}$$

and the resulting value of x_1+h, namely

$$x_1-\frac{f(x_1)}{f'(x_1)}, \qquad (4)$$

is Newton's formula for a second approximation to the root in question.

We now examine conditions under which this value is a *closer* approximation than x_1 to the root, that is, under which the success of Newton's method is *assured*. For this purpose we consider the geometrical equivalent of the process. Let the root in question be the value a of x, and let R (Fig. 85) be the point of intersection $x=a$ of the graph of the function $f(x)$ with Ox. The point P_1 on the graph corresponds to the value x_1, the first approximation to the root a. Draw the tangent P_1T_1 to the curve at P_1 to meet Ox in T_1. Then if ψ_1 is the angle xT_1P_1, we have

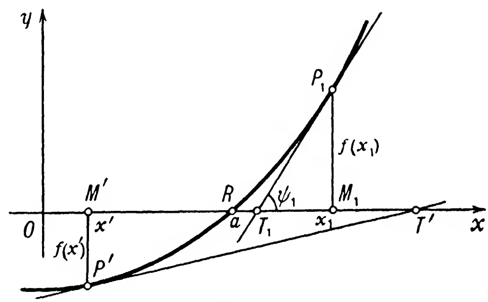


FIG. 85.

$$f'(x_1)=\tan\psi_1=\frac{M_1P_1}{T_1M_1}=\frac{f(x_1)}{T_1M_1}, \qquad (5)$$

so that

$$M_1T_1=-T_1M_1=-f(x_1)/f'(x_1)=h. \qquad (6)$$

Therefore T_1 is the point of abscissa x_1+h.

Thus Newton's approximation $x_1-f(x_1)/f'(x_1)$ to the root a is obtained, geometrically, by proceeding along the tangent at P_1 (to reach T_1) instead of along the curve itself (to reach R).

Now for the case illustrated, it is seen that starting from P_1, (or from any point between R and P_1), we necessarily obtain a closer approximation in this way, but if we start from a point to the left of R, such as P', the corresponding point T' reached may be farther from R than M' and the method may accordingly fail to give a closer approximation. The curve in Fig. 85 is concave upwards throughout, so that $f''(x)$ is positive, and it appears that if we start from a point on the curve for which $f(x)$ is also positive, we *must* get nearer to the root.

We may now state generally that a closer approximation is assured if at the starting point x_1 the values $f''(x_1)$, $f(x_1)$ of the second

derivative and the function have the same sign, provided that there is not a point of contraflexure between the point x_1 and the root a.

In Fig. 86 the case of $f''(x)$ negative throughout is illustrated, while in Fig. 87 there is a point of contraflexure C between P_1 and R, and the method may, in consequence, be unsuccessful. This happens when P_1 is taken as the starting point, even though $f(x)$,

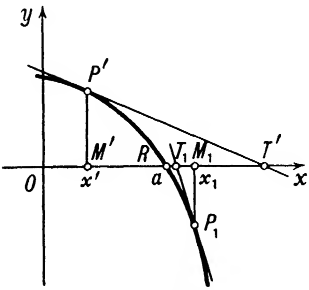


FIG. 86.

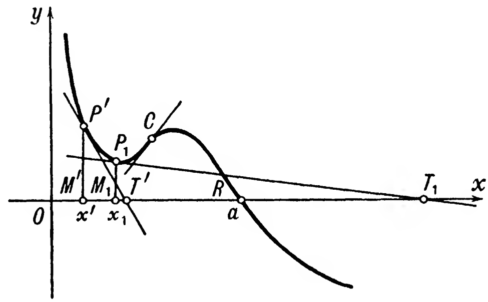


FIG. 87.

$f''(x)$ are both positive there. When P' is the starting point, however, the method does succeed.

118. Iteration of the Process.

Newton's method may be employed to obtain successive approximations to the root by iteration. Having determined the second approximation, x_2 say, by means of the formula

$$x_2 = x_1 - f(x_1)/f'(x_1), \quad \text{.........................(1)}$$

we take this as starting point and a third approximation x_3 is then given by

$$x_3 = x_2 - f(x_2)/f'(x_2), \quad \text{.........................(2)}$$

and so on. The geometrical equivalent of this iteration process is that at each stage we find a further approximation to the root by going along the tangent at the point corresponding to the last approximation found.

If we consider the case in which the second derivative has a constant sign throughout a range which includes the point x_1 and the root a, the second approximation x_2 is certainly closer to the root than x_1, provided that the signs of the second derivative and the function at x_1 are the same. The same conditions are now evidently satisfied for all points in the range (a, x_1), and hence each approximation will be closer to the root than the preceding one. This is illustrated in Fig. 88, where $f(x_1), f''(x_1)$ are both positive.

The objection may be raised to the above discussion that although it shews that each approximation is nearer to the root than the preceding one, it does not shew that the sequence $\{x_n\}$ of the approximations converges to the root rather than to another value on the same side of the root as x_1. But the convergence of $\{x_n\}$ requires that successive approximations x_n, x_{n+1} should close up to equality as n is increased indefinitely. Now the Newton formula $x_{n+1} - x_n = -f(x_n)/f'(x_n)$ shews that this closing up cannot occur inside a range in which $f'(x)$ is finite and $f(x)$ different from zero; that is, the convergence can only be to the root. This argument can be readily followed on a graph.

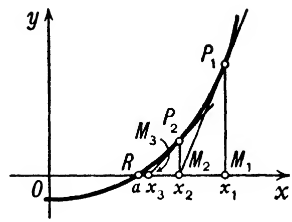


FIG. 88.

If the signs of $f(x_1)$, $f''(x_1)$ are opposite, the point x_2 reached may not be nearer to the root than x_1, but if the constancy of sign of $f''(x)$ extends from x_1 to x_2, each of the successive approximations after x_2 is closer to the root than the preceding one. This is illustrated in Fig. 89.

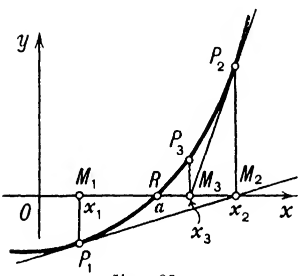


FIG. 89.

If the second derivative does not preserve a constant sign, the above simple criteria do not apply.

If $f(x_1)$, $f''(x_1)$ have the same sign and if the sign of $f''(x)$ is constant in the range (a, x_1), so that the iteration process is successful from the beginning, a simplification[1] in the numerical work involved in any practical case may be effected by taking the denominator in each correction term to be $f'(x_1)$. Geometrically, this is equivalent to replacing the tangents at the points P_2, P_3, ..., (Fig. 90) by lines parallel to the tangent at P_1. The number of successive approximations will, in general, be increased in this way, but the total amount of calculation involved in arriving at a given degree of approximation will probably be diminished.

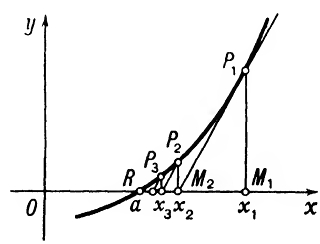


FIG. 90.

Ex. 1. *The equation* $x^4 - 2x^3 - 2x + 1 = 0$. We have proved, Art. 95, Ex. 7, p. 206, that there are two real roots, one lying between 0 and 1 and the other being its reciprocal. We approximate to the root in the range (0, 1). We have

$$f'(x) = 4x^3 - 6x^2 - 2, \quad f''(x) = 12x^2 - 12x. \quad \text{.....................(3)}$$

Let us try $x_1 = 1/2$ as the first approximation to the root. We have

$$f(\tfrac{1}{2}) = -3/16, \quad f'(\tfrac{1}{2}) = -3, \quad f''(\tfrac{1}{2}) = -3. \quad \text{.....................(4)}$$

[1] See Whittaker and Robinson, *Calculus of Observations*, (1929), Chap. VI, Art. 48.

Since $f''(x)$ does not vanish in the range $[0, 1]$, there is no point of contraflexure between x_1 and the root, and the process described gives

$$h_1 = -f(\tfrac{1}{2})/f'(\tfrac{1}{2}) = -1/16 = -0{\cdot}0625, \quad \text{.......................(5)}$$

so that the second approximation

$$x_2 = 0{\cdot}5 - 0{\cdot}0625 = 0{\cdot}4375 \approx 0{\cdot}438, \text{ say,} \quad \text{.....................(6)}$$

is also a closer approximation.

Repeating the process with $x_2 = 0{\cdot}438$ as starting point, we have

$$h_2 = -\frac{f(0{\cdot}438)}{f'(0{\cdot}438)} = -\frac{-0{\cdot}007251}{-2{\cdot}814953} \approx -0{\cdot}002576, \quad \text{.................(7)}$$

and hence the third approximation x_3 is given by

$$x_3 = 0{\cdot}438 - 0{\cdot}002576 = 0{\cdot}435424. \quad \text{........................(8)}$$

We shall shew below that this approximation is not in error by more than 4 in the sixth decimal place.

119. Error in the Newton Approximation.

If AB (Fig. 91) is the graph of the given function $f(x)$ in the neighbourhood of the point R, where the curve intersects Ox, upper barriers to the error committed in the determination of the root of the equation $f(x) = 0$ by taking it as the abscissa of T instead of that of R may be found from the formulae (7), (8) of Art. 116, which in the present case become

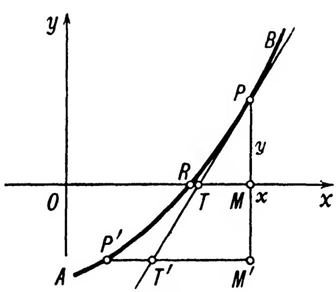


FIG. 91.

$$RT < \tfrac{1}{2}\frac{|f''(x)|_u}{\{|f'(x)|_l\}^3}(MP)^2, \quad \text{...(1)}$$

$$RT < \tfrac{1}{2}\frac{|f''(x)|_u}{|f'(x)|_l}(RM)^2, \quad \text{.....(2)}$$

respectively.

In using the result (1), the upper and lower barriers must be taken for an arc PP' extending from P to a point P' chosen beyond R, since R is unknown. Such a point P' will be known from the preliminary discussion in which a range of x containing R is determined.

In the result (2), RM is unknown so that the formula cannot be employed directly. Suppose, however, that we apply the result to the part PP' of the curve. Then we have

$$RT < P'T' < \tfrac{1}{2}\frac{|f''(x)|_u}{|f'(x)|_l}(P'M')^2. \quad \text{...................(3)}$$

This result can be used instead of (1).

Ex. 1. *The equation* $x^4 - 2x^3 - 2x + 1 = 0$. We test the accuracy of the approximation 0·435424 to a root, found in Ex. 1 above. The root lies between 0·43 and 0·438, for we find

$$f(0{\cdot}43) \approx 0{\cdot}0152, \quad f(0{\cdot}438) \approx -0{\cdot}0073. \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Now in the range (0·43, 0·438), $|f'(x)|$, $|f''(x)|$ are both up-way. We may therefore take

$$|f''(x)|_u = |f''(0{\cdot}438)| = 2{\cdot}954, \quad |f'(x)|_l = |f'(0{\cdot}43)| = 2{\cdot}791. \quad \ldots\ldots\ldots(5)$$

The above formula (1) now gives

$$RT < \tfrac{1}{2}\frac{2{\cdot}954}{(2{\cdot}791)^3}(0{\cdot}0073)^2 < 3{\cdot}7 \times 10^{-6}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

while, taking $P'M' = 0{\cdot}438 - 0{\cdot}43 = 0{\cdot}008$, the formula (3) gives

$$RT < \tfrac{1}{2}\frac{2{\cdot}954}{2{\cdot}791}(0{\cdot}008)^2 < 3{\cdot}4 \times 10^{-5}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

This, however, is not so good an estimate of the error as (6).

It thus appears that the approximation 0·435424 is not in error by more than 4 in the sixth place of decimals.

[An independent check is afforded in this Example in the following manner. Since $f(0) \neq 0$, we can divide throughout by x^2 to obtain the equation $x^2 + 1/x^2 - 2(x + 1/x) = 0$, and, on putting $y = x + 1/x$, $(y > 0)$, this becomes $y^2 - 2 - 2y = 0$, or $(y-1)^2 = 3$. Taking the positive root for y, we are led to the equation $x + 1/x = 1 + \sqrt{3}$, and the root in question, found from this equation, is given by

$$x = \tfrac{1}{2}(1 + \sqrt{3}) - \tfrac{1}{2} \, . \, 12^{1/4}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

From the tables we have $\sqrt{12} = 3{\cdot}464101615$, whence $12^{1/4} = 1{\cdot}8612097$. Also $1 + \sqrt{3} = 2{\cdot}73205081$, and therefore $x = 0{\cdot}4354205$.]

120. Condition for Success of the Newton Approximation when $f(x)$ and $f''(x)$ have Opposite Signs.

We have seen that if $f(x)$, $f''(x)$ have the same sign and if there is no point of contraflexure in the neighbourhood of the root, the success of Newton's method is assured. We now investigate a sufficient condition for the success of the method when the signs of the above numbers are different. It will sometimes be found that the numerical calculations are considerably simplified by approximating 'from the wrong end'.

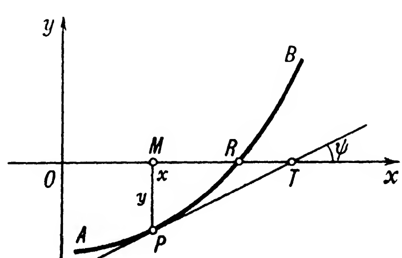


FIG. 92.

Let AB (Fig. 92) be the graph of the function $f(x)$ which, in the neighbourhood of the root R, has a positive second derivative $f''(x)$. Let OM be an approximation to the root, the ordinate MP being negative. The second approximation to the root is given by OT, and the condition that it should be a closer approximation is expressed by $MR > RT$.

Now in the Figure, PM is positive and d^2x/dy^2 negative. Therefore, having regard to the senses of the lines,

$$RT = MT - MR = PM\cot\psi - MR, \quad \text{.....................(1)}$$

and, according to (1), Art. 116,

$$MR = PM\cot\psi + \tfrac{1}{2}\frac{d^2x}{dy^2}(PM)^2, \quad \text{.........................(2)}$$

the second derivative being taken at an intermediate point. Therefore

$$MR - RT = PM\cot\psi + \overline{\frac{d^2x}{dy^2}}(PM)^2, \quad \text{.......................(3)}$$

and this expression is positive if $\cot\psi + \overline{\dfrac{d^2x}{dy^2}}PM > 0$, that is, if $PM < -\cot\psi \Big/ \overline{\dfrac{d^2x}{dy^2}}$, or if $|f(x)| < \dfrac{\cot\psi}{f''(\theta x)/\{f'(\theta x)\}^3}$, and hence if

$$|f(x)| < \frac{\cot\psi}{f''(x)_u/\{f'(x)_l\}^3}. \quad \text{..........................(4)}$$

All cases are covered by writing, instead of (4),

$$|f(x)| < \frac{|\cot\psi|}{|f''(x)|_u/\{|f'(x)|_l\}^3}. \quad \text{..........................(5)}$$

The required condition is thus expressed by

$$|f(x)|\,.\,|f''(x)|_u < \frac{\{|f'(x)|_l\}^3}{|f'(x)|}. \quad \text{..........................(6)}$$

In practice it is usually sufficient to replace $|f''(x)|_u$, $|f'(x)|_l$ by their values at P, the condition (6) then becoming

$$|f(x)|\,.\,|f''(x)| < \{f'(x)\}^2. \quad \text{..............................(7)}$$

Ex. 1. *The equation* $x^4 - 2x^3 - 2x + 1 = 0$. Suppose that we start with $x_1 = 0{\cdot}43$ as a first approximation to the smaller root. We have $f(0{\cdot}43) = 0{\cdot}0152$, and as above we may take $|f'(x)|_l = 2{\cdot}791$, $|f''(x)|_u = 2{\cdot}954$. These give

$$\left.\begin{aligned} &|f(x_1)|\,.\,|f''(x)|_u = 0{\cdot}0152 \times 2{\cdot}954 = 0{\cdot}0449, \\ &\{|f'(x)|_l\}^3/|f'(x_1)| = \{f'(0{\cdot}43)\}^2 = 2{\cdot}791^2 = 7{\cdot}790, \end{aligned}\right\} \quad \text{.............(8)}$$

and the condition (6) is satisfied.

Newton's method thus gives, as a closer approximation, the value

$$\begin{aligned} x_2 = 0{\cdot}43 - f(0{\cdot}43)/f'(0{\cdot}43) &= 0{\cdot}43 - (0{\cdot}0152)/(-2{\cdot}791) \\ &= 0{\cdot}43 + 0{\cdot}00544 = 0{\cdot}43544, \quad \text{...........(9)} \end{aligned}$$

which is actually correct to four places of decimals.

Ex. 2. Shew that if there are two nearly equal roots x_1, x_2 of $f(x) = 0$, so that there is a root x_3 of $f'(x) = 0$ between them, then x_1, x_2 are given approximately by $x_3 \pm \sqrt{\{-2f(x_3)/f''(x_3)\}}$.

[The $I.D_2.T.$ is here applied and the variation of $f''(x)$ from its value at x_3 is neglected. A limiting case is that in which the two roots x_1, x_2 close up to equality with x_3, so that $f(x_3) = 0$ as well as $f'(x_3) = 0$.]

V. *THEORY OF PROPORTIONAL PARTS. INTERPOLATION.*

121. The Method of Proportional Parts.

The method of proportional parts, here referred to by the abbreviation *M.P.P.*, is most naturally explained by considering a function which is *tabulated* only for a limited number of values of the argument in a certain range although it is *defined* for all values of the argument in that range. The tabular results are, in general, approximate, to a degree depending on the number of figures (places of decimals) used. The method in question is concerned with a rule for finding, from the tabulated values, intermediate values correct to the same number of places. The process of finding intermediate values is called *interpolation*, and the application of the rule here discussed is described as *interpolation by (the method of) proportional parts.*

The use of interpolation might conceivably be avoided in such a case by tabulating the function for so many arguments that its value at an intermediate value of the argument could be taken as equal to either of the adjacent tabulated results, but this would usually require quite unnecessarily extensive tables. For example, in an ordinary seven-figure table of common logarithms of integers from 10,000 to 100,000, the difference of consecutive logarithms at the beginning of the table is about 0·000,04, and the difference of 4 in the fifth place of decimals shews that the arguments are not close enough there for more than four-figure accuracy if interpolation is not employed.

The method of proportional parts supposes the possession by the function of the property that between two consecutive values of the argument the increment of the function is proportional to the increment of the argument to the desired degree of accuracy. Let $f(x)$ denote the function and let a, $a+h$ be consecutive values of the argument, $f(a)$, $f(a+h)$ being the corresponding tabular values of the function. The value $f(a+k)$ at an intermediate value $a+k$ of x, where $0<k<h$, is, according to the method, to be calculated from the proportion

$$\frac{f(a+k)-f(a)}{k}=\frac{f(a+h)-f(a)}{h}, \qquad\qquad (1)$$

that is, from the equation

$$f(a+k)=f(a)+\frac{k}{h}\{f(a+h)-f(a)\}. \qquad\qquad (2)$$

This is another example of the use of a linear law of variation in approximation, the gradient being now taken as the ratio of the increase of the tabular value to the increase of the argument.

In a graphical representation, the right-hand side of equation (2) is the ordinate at $a+k$ of the chord joining the points a, $a+h$ on the graph of $f(x)$. Thus in the application of the method, the ordinate of the graph at $a+k$ is taken to be equal to the ordinate of the chord.

A table of a function is regarded as *complete* when intermediate values can be calculated to the accuracy of the tabulated results by this process, that is, by *first differences*, as it is commonly called. The condition of completeness of such a table is obtained as the result of the investigation of the following Article.

Ex. 1. Find $\sqrt{(2{\cdot}1469)}$, given $\sqrt{21460}=146{\cdot}492321$, $\sqrt{21470}=146{\cdot}526448$.

We have, in this case,

$$\left.\begin{array}{l} a=2{\cdot}1460, \quad h=0{\cdot}001, \quad k=0{\cdot}0009, \\ f(a)=1{\cdot}46492321, \quad f(a+h)=1{\cdot}46526448. \end{array}\right\} \quad\ldots\ldots(3)$$

Hence, by the *M.P.P.*, we have

$$f(a+k)=f(a)+\tfrac{k}{h}\{f(a+h)-f(a)\}=1{\cdot}46492321+\tfrac{9}{10}\times 0{\cdot}00034127$$
$$=1{\cdot}46492321+0{\cdot}00030714=1{\cdot}46523035. \quad\ldots\ldots(4)$$

The accuracy of this result is investigated subsequently, Art. 123. 4.

122. Error made in the Application of the Method. Uncertainty of the Interpolated Value.

We proceed to find an expression, involving an intermediate value of $f''(x)$ in the range $(a, a+h)$, for the error made in interpolating by the *M.P.P.* for the value of the function when $x=a+k$. The expression so found leads to a formula for an upper barrier to this error.

Taking the excess of $f(a+k)$ over the value obtained for it by the *M.P.P.*, according to the formula (2) of the preceding Article, as the error E involved in applying the method, we have

$$E=f(a+k)-[f(a)+\tfrac{k}{h}\{f(a+h)-f(a)\}], \quad\ldots\ldots(1)$$

or

$$\frac{E}{k}=\frac{f(a+k)-f(a)}{k}-\frac{f(a+h)-f(a)}{h}, \quad (k\neq 0).\ldots\ldots(2)$$

Introducing the function $\varphi(x)$ defined by

$$\varphi(x)=\frac{f(a+x)-f(a)}{x}, \quad (x\neq 0), \quad\ldots\ldots(3)$$

we have

$$E/k = \varphi(k) - \varphi(h) = (k-h)\varphi'(\xi), \quad \text{.......................(4)}$$

by the $I.D_1.T.$, where ξ lies between k and h. Also

$$\varphi'(x) = \frac{f(a) - f(a+x) + xf'(a+x)}{x^2}, \quad \text{.....................(5)}$$

and employing the $I.D_2.T.$, we can write

$$f(a) = f(a+x-x) = f(a+x) - xf'(a+x) + \tfrac{1}{2}x^2 f''(a+\bar{x}), \quad \text{.........(6)}$$

where $a+\bar{x}$ lies between a and $a+x$. Thus (5) becomes

$$\varphi'(x) = \tfrac{1}{2}f''(a+\bar{x}), \quad \text{........................(7)}$$

and, in particular,

$$\varphi'(\xi) = \tfrac{1}{2}f''(a+\bar{\xi}), \quad \text{........................(8)}$$

where $a+\bar{\xi}$ lies between a and $a+\xi$, and therefore between a and $a+h$.

Hence, using (4) and (8), the error may be written

$$E = \tfrac{1}{2}k(k-h)f''(a+\theta h), \quad (0<\theta<1), \quad \text{............(9)}$$

and its magnitude is given by

$$|E| = \tfrac{1}{2}k(h-k)\,|f''(a+\theta h)|, \quad \text{.....................(10)}$$

which is the expression sought.

Interpreted geometrically, this is an expression for the magnitude of the difference between ordinate of curve and ordinate of chord at the intermediate point $a+k$ in the range $(a, a+h)$.

To investigate an upper barrier to the value of $|E|$ in the above range, we observe that since k lies in the range $[0, h]$, the greatest value of $k(h-k)$ is $\frac{1}{4}h^2$, corresponding to the value $\frac{1}{2}h$ of k. The magnitude of the error at any point of the range does not, therefore, exceed $\frac{1}{8}h^2|f''(x)|_{max}$, where $|f''(x)|_{max}$ is the greatest value of $|f''(x)|$ in the range.

The greatest numerical error which can arise in this way in the interpolated value of $f(x)$ in the range $(a, a+h)$ is called the *uncertainty*, (or *magnitude of uncertainty*), of $f(x)$ as determined by the *M.P.P.* Denoting this uncertainty by U, we have the formula

$$U = \tfrac{1}{8}h^2|f''(x)|_{max}. \quad \text{.............................(11)}$$

The validity of the method of proportional parts in any particular case is therefore demonstrated by shewing that the uncertainty U is less than the permissible error. If this condition is satisfied the table is complete. For example, if the tabulated values of $f(x)$ are correct to n decimal places in a range including $(a, a+h)$, the permissible

error in the intermediate value $f(a+k)$ is $5\times 10^{-(n+1)}$, and for completeness, U must be less than this number. In estimating U it is usually sufficient to replace $|f''(x)|_{max}$ by $|f''(a)|$.

The above discussion refers only to an upper barrier of the *absolute* error committed in the application of the rule to a table of a function. The question of the *fractional error* may be dealt with in a similar way. If the absolute error is E, the magnitude of the fractional error is $|E|/|f(a+k)|$, and this is not greater than $U/|f(x)|_{min}$, where $|f(x)|_{min}$ is the least value of $|f(x)|$ in the interval $(a, a+h)$. We define this expression to be the *fractional uncertainty* of $f(x)$, as determined by the *M.P.P.*, and denote it by U_f. Thus

$$U_f = \tfrac{1}{8}h^2\frac{|f''(x)|_{max}}{|f(x)|_{min}}, \quad \text{.........................(12)}$$

which, for purposes of estimation, can usually be replaced by the formula

$$U_f = \tfrac{1}{8}h^2\,|f''(a)|/|f(a)|. \quad \text{.........................(13)}$$

In the discussion of this Article the difference between the actual value and the tabulated value of $f(a)$ or of $f(a+h)$ has been treated as negligible. Any adequate discussion of the combination of errors arising from different sources is outside the scope of this Book. The student may be referred to Lüroth, *Numerisches Rechnen*, Kap. VI.

123. Application to Algebraic Functions.

We now apply the above considerations to certain tables of algebraic functions. Tables of other functions are dealt with as the functions are introduced.

123. 1. *Tables of Squares.* (*Barlow.*[1]) We have $f(x)=x^2$, $f''(x)=2$. The uncertainty U of an interpolated value of x^2, as determined by the *M.P.P.*, is therefore $\frac{1}{8}h^2\times 2$, that is, $\frac{1}{4}h^2$, while an upper barrier to the fractional error is $\frac{1}{4}h^2/a^2$, where a is the starting point for the application of the rule.

In Barlow's tables the squares are tabulated accurately for the integers from 1 to 10,000. The uncertainty of an interpolated value is 0·25 throughout the whole table, and interpolation by the *M.P.P.* is therefore justified if accuracy is merely required to the nearest integer, the permissible error then being 0·5. For example, if the value of $(475{\cdot}4)^2$ is found by the *M.P.P.* from the tabulated values

$$475^2=225{,}625, \quad 476^2=226{,}576, \quad \text{..................(1)}$$

[1] Barlow, *Tables of Squares, Cubes*, etc., London, 1930.

then, to the nearest integer,

$$(475{\cdot}4)^2 = 225{,}625 + \tfrac{4}{10} \times 951 = 225{,}625 + 380 = 226{,}005. \quad\ldots\ldots(2)$$

From the table we find $4{,}754^2 = 22{,}600{,}516$, so that the error introduced in (2) is only $0{\cdot}16$.

Practically, interpolation need only be carried out in the part of the table between 1,000 and 10,000. This fact is of importance as diminishing the fractional uncertainty, which is greatest when a has its least value and is $2{\cdot}5 \times 10^{-7}$. It follows that the fractional uncertainty cannot affect the sixth place of decimals even in the lower part of the range stated.

123. 2. *Table of Cubes.* (*Barlow.*) We have $f(x) = x^3, f''(x) = 6x$. The uncertainty of an interpolated value of x^3 is here $\frac{1}{8}h^2 \times 6(a+h)$, that is $0{\cdot}75h^2(a+h)$, while the fractional uncertainty is $0{\cdot}75h^2(a+h)/a^3$.

In Barlow's tables the cubes are tabulated accurately for the integers from 1 to 10,000. For interpolation by the *M.P.P.* in the portion of the range from 1,000 to 10,000, the fractional uncertainty is about $7{\cdot}5 \times 10^{-7}$.

123. 3. *Table of Reciprocals.* (*Barlow, Oakes,*[1] *Cotsworth.*[2]) We have $f(x) = 1/x$, $f''(x) = 2/x^3$. The uncertainty of an interpolated value of $1/x$ is $0{\cdot}25h^2/a^3$, and the fractional uncertainty is $0{\cdot}25h^2/a^2$.

In Barlow's tables the reciprocals are tabulated for all the integers from 1 to 10,000 to seven significant figures. In practice, interpolation is carried out in the part of the table between 1,000 and 10,000, the tabulated results being correct to ten decimal places, so that the permissible error in interpolation is 5×10^{-11}. If we put $0{\cdot}25/a^3 = 5 \times 10^{-11}$, we get $a \approx 1{,}710$. Therefore the table is complete only if $x > 1{,}710$. When $a = 1{,}000$, the uncertainty is $2{\cdot}5 \times 10^{-10}$, and accuracy to nine places only is assured when x lies between 1,000 and 1,710.

123. 4. *Table of Square Roots.* (*Barlow, Milne-Thomson.*[3]) Here $f(x) = \sqrt{x}$. $f''(x) = -1/(4x^{3/2})$. The uncertainty of an interpolated value of $\sqrt{x}$ is $\frac{1}{32}h^2/a^{3/2}$, and the fractional uncertainty is $\frac{1}{32}h^2/a^2$.

The range of the tables is the same as before, the values being given correct to seven places of decimals for values of x between 1 and 1,000, and to six places of decimals for values of x between 1,000 and 10,000. The permissible error in interpolation in the latter range is therefore 5×10^{-7}. If we interpolate by the *M.P.P.* in this

[1] Oakes, *Tables of Reciprocals.* [2] Cotsworth, *Direct Reciprocals.*

[3] Milne-Thomson, *Standard Table of Square Roots.*

range, the uncertainty of the interpolated value will vary from $\frac{1}{32} \times \frac{1}{1{,}000^{3/2}}$, or about 10^{-6}, when $x = 1{,}000$, to $\frac{1}{32} \times \frac{1}{10{,}000^{3/2}}$, or about 3×10^{-8}, when $x = 10{,}000$, being 5×10^{-7} when $x \approx 1{,}575$. The table is therefore only complete when x lies between 1,575 and 10,000.

A table of $\sqrt{(10x)}$, correct to six places of decimals, is also given for values of x ranging from 1,000 to 10,000. Writing $10x = x'$, the common difference of argument, h' say, being now 10, the uncertainty of an interpolated value of $\sqrt{(10x)}$, that is of $\sqrt{x'}$, is given by $\frac{1}{8}h'^2 \,|f''(x')|_{max}$, and it therefore never exceeds $\frac{1}{32} \times \frac{100}{10{,}000^{3/2}}$, or 3×10^{-6}, corresponding to the value 1,000 of x or 10,000 of x'. For the value 2,000 of x, the uncertainty is of the order of $\frac{1}{32} \times \frac{100}{20{,}000^{3/2}}$, or 10^{-6}. We conclude that the uncertainty of the result (4), Ex. 1, p. 256, is about 10^{-8}.

124. Interpolation with the Use of Second and Higher Differences.

124. 1. *Use of second differences.* In many cases where the use of proportional parts, or first differences, does not lead to the necessary accuracy, it is sufficient to employ a formula involving *second differences.* Suppose that the variation of $f(x)$ is represented in the double interval $(a, a + 2h)$ to the required degree of approximation by a quadratic expression

$$A + B(x - a) + C(x - a)^2, \quad \text{.......................(1)}$$

and that this formula reproduces the tabulated values of $f(x)$ for the values a, $a + h$, $a + 2h$ of x. Then we have

$$f(a) = A, \quad f(a+h) = A + Bh + Ch^2, \quad f(a + 2h) = A + 2Bh + 4Ch^2, \text{...(2)}$$

whence we find, on eliminating A,

$$f(a+h) - f(a) = Bh + Ch^2, \quad f(a+2h) - f(a+h) = Bh + 3Ch^2. \text{...(3)}$$

On eliminating B we now find

$$f(a+2h) - 2f(a+h) + f(a) = 2Ch^2, \quad \text{..................(4)}$$

and the values of A, B, C are thus determined. With the use of the difference symbol Δ, as applied to starting point a and increment h, we have the results

$$A = f(a), \quad C = \frac{1}{2h^2}\Delta^2 f(a), \quad B = \frac{1}{h}\Delta f(a) - \frac{1}{2h}\Delta^2 f(a), \text{......(5)}$$

and the approximation formula becomes

$$f(x) \approx f(a) + \frac{x-a}{h}\{\Delta f(a) - \tfrac{1}{2}\Delta^2 f(a)\} + \frac{(x-a)^2}{2h^2}\Delta^2 f(a)$$

$$= f(a) + \frac{x-a}{h}\Delta f(a) + \frac{(x-a)(x-a-h)}{2h^2}\Delta^2 f(a), \quad \text{.........(6)}$$

or $$f(a+k) \approx f(a) + \frac{k}{h}\Delta f(a) + \frac{k(k-h)}{2h^2}\Delta^2 f(a), \quad \text{......................(7)}$$

on writing $x = a + k$. This is the interpolation formula sought.

124. 2. *Use of higher differences. The interpolation formulae of Lagrange and Newton.* It is unnecessary to proceed to find a formula for the uncertainty of $f(a+k)$ as determined by the above approximation, as we now investigate a polynomial approximation of the nth degree in which the above quadratic approximation is included and for which an expression for the error will be found.

We begin with a formula given by Lagrange for the polynomial[1] of degree $n-1$ in x which is such that its values at the values $a_1, a_2, \ldots, a_n$ of x are equal to those of the function $f(x)$, that is to $f(a_1), f(a_2), \ldots, f(a_n)$, respectively. We shall suppose $a_1 < a_2 < \ldots < a_{n-1} < a_n$. The formula in question is

$$\begin{aligned}\varphi(x) \equiv {} & \frac{(x-a_2)(x-a_3)\ldots(x-a_n)}{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_n)}f(a_1) \\ & + \frac{(x-a_1)(x-a_3)\ldots(x-a_n)}{(a_2-a_1)(a_2-a_3)\ldots(a_2-a_n)}f(a_2) \\ & + \ldots + \frac{(x-a_1)(x-a_2)\ldots(x-a_{n-1})}{(a_n-a_1)(a_n-a_2)\ldots(a_n-a_{n-1})}f(a_n). \quad \ldots(8)\end{aligned}$$

The expression $\varphi(x)$ has a term corresponding to each of the values $f(a_1), f(a_2), \ldots, f(a_n)$. The denominator of the fraction multiplying $f(a_r)$ is obtained by subtracting from a_r each of the numbers of the set $a_1, a_2, \ldots, a_{r-1}, a_{r+1}, \ldots, a_n$ and forming the product. The numerator is formed by subtracting the same numbers from x instead of a_r.

The correctness of this formula is easily verified. If we put $x = a_1$, all the terms except the first vanish because they contain the factor $x - a_1$. The first term becomes $f(a_1)$, and thus $\varphi(a_1) = f(a_1)$. In the same way we shew generally that $\varphi(a_r) = f(a_r)$, $(r = 1, 2, \ldots, n)$. Each term of $\varphi(x)$ is a polynomial of degree $n-1$ in x so that $\varphi(x)$ is not of higher degree, and it is the unique polynomial satisfying the given

[1] The word *polynomial* is here used, for brevity, as equivalent to *rational integral function.*

conditions because two different polynomials of the $(n-1)$th (or lower) degree cannot be equal to one another for n values of the variable, otherwise their difference would have n roots.

The interpolation formula of Lagrange is arrived at by assuming that $f(x)$ is represented with sufficient accuracy by the polynomial $\varphi(x)$ in the range (a_1, a_n), that is we put

$$f(x) \approx \sum_{r=1}^{n} \frac{(x-a_1)\dots(x-a_{r-1})(x-a_{r+1})\dots(x-a_n)}{(a_r-a_1)\dots(a_r-a_{r-1})(a_r-a_{r+1})\dots(a_r-a_n)} f(a_r). \quad \dots\dots(9)$$

We now proceed to find an expression for the magnitude of the error committed in this approximation. Let $R(x)$ denote the difference $f(x)-\varphi(x)$. Observing that this difference vanishes for the values $a_1, a_2, \dots, a_n$ of x, we may write

$$R(x) = \frac{1}{n!}(x-a_1)(x-a_2)\dots(x-a_n)\psi(x), \quad \dots\dots\dots\dots\dots(10)$$

where $\psi(x)$ is a function of x which we determine as follows. Consider the function

$$F(\xi) \equiv f(\xi) - \varphi(\xi) - \frac{1}{n!}(\xi-a_1)(\xi-a_2)\dots(\xi-a_n)\psi(x), \quad \dots\dots\dots(11)$$

where ξ is another variable in the same range (a_1, a_n). This function vanishes for the values $a_1, a_2, \dots, a_n$ of ξ and also for the value x of ξ since $F(x) = f(x) - \varphi(x) - R(x) = 0$ by definition of $R(x)$. Therefore by Rolle's Theorem, $F^{(1)}(\xi)$ must vanish for at least n values of ξ lying between the successive zeros[1] of $F(\xi)$. In the same way we find that $F^{(2)}(\xi)$ vanishes for at least $n-1$ values of ξ lying between the successive zeros of $F^{(1)}(\xi)$, and so on. Finally we find that $F^{(n)}(\xi)$ must vanish for at least one value of ξ in the range $[a_1, a_n]$. Let $\bar{x}$ be such a value. Now the nth derivative of $\frac{1}{n!}(\xi-a_1)(\xi-a_2)\dots(\xi-a_n)$ with respect to ξ is plainly 1 and the nth derivative of $\varphi(\xi)$ vanishes since it is a polynomial of degree $n-1$. Therefore

$$f^{(n)}(\bar{x}) - \psi(x) = 0, \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots(12)$$

and hence, from (10),

$$R(x) = \frac{1}{n!}(x-a_1)(x-a_2)\dots(x-a_n)f^{(n)}(\bar{x}), \quad \dots\dots\dots\dots(13)$$

where it is to be noted that $\bar{x}$ is a function of x.

The magnitude of this expression for $R(x)$ gives the magnitude of the error in $f(x)$ as determined by the interpolation formula (9).

We now write

$$f(x) = \varphi(x) + \frac{1}{n!}(x-a_1)(x-a_2)\dots(x-a_n)f^{(n)}(\bar{x}), \quad \dots\dots\dots(14)$$

[1] A 'zero' of a function $f(x)$ is a value of x for which $f(x)$ vanishes, that is, it is a root of the equation $f(x)=0$.

which is the expression of *Lagrange's Interpolation Theorem*, where $\bar{x}$ lies in the range $[a_1, a_n]$.

Let us now suppose that $a_2 - a_1 = a_3 - a_2 = \dots = a_n - a_{n-1} = h$, say. We get, on writing a for a_1,

$$f(x) = \varphi(x) + \frac{1}{n!}(x-a)(x-a-h)\dots\{x-a-(n-1)h\}f^{(n)}(\bar{x}). \quad \dots(15)$$

In this case the polynomial $\varphi(x)$ can be written in the form

$$\varphi(x) = f(a) + \frac{x-a}{h}\Delta f(a) + \frac{1}{2!}\frac{(x-a)(x-a-h)}{h^2}\Delta^2 f(a) + \dots$$
$$+ \frac{1}{(n-1)!}\frac{(x-a)(x-a-h)\dots\{x-a-(n-2)h\}}{h^{n-1}}\Delta^{n-1}f(a). \quad \dots(16)$$

To shew this, observe that the polynomial on the right is of degree $n-1$ in x, and when x has the successive values $a, a+h, a+2h, \dots, a+(n-1)h$ it takes the successive values $f(a)$, $f(a)+\Delta f(a)$, $f(a)+2\Delta f(a)+\Delta^2 f(a)$, $\dots, f(a)+(n-1)\Delta f(a) + \frac{(n-1)(n-2)}{2!}\Delta^2 f(a) + \dots + \Delta^{n-1}f(a)$. Now according to Art. 60. 2, these latter values are $f(a)$, $f(a+h)$, $f(a+2h)$, $\dots, f\{a+(n-1)h\}$, so that the polynomial takes the required values at the respective values of x. Since the polynomial in question is unique, the identification with the Lagrange polynomial for this case is complete.

The interpolation theorem of Lagrange now becomes

$$f(x) = f(a) + \frac{x-a}{h}\Delta f(a) + \frac{1}{2!}\frac{(x-a)(x-a-h)}{h^2}\Delta^2 f(a) + \dots$$
$$+ \frac{1}{(n-1)!}\frac{(x-a)(x-a-h)\dots\{x-a-(n-2)h\}}{h^{n-1}}\Delta^{n-1}f(a)$$
$$+ \frac{1}{n!}(x-a)(x-a-h)\dots\{x-a-(n-1)h\}f^{(n)}(\bar{x}), \quad \dots\dots\dots\dots(17)$$

where x, $\bar{x}$ lie in the range $[a, a+(n-1)h]$. The theorem in this form is also known as the *Gregory-Newton Interpolation Theorem*.[1]

In the case of interpolation with second differences, the theorem takes the form

$$f(x) = f(a) + \frac{x-a}{h}\Delta f(a) + \frac{1}{2!}\frac{(x-a)(x-a-h)}{h^2}\Delta^2 f(a)$$
$$+ \frac{1}{3!}(x-a)(x-a-h)(x-a-2h)f^{(3)}(\bar{x}), \quad \dots\dots(18)$$

[1] The following book may be referred to in the present connection, F. Klein, *Elementary Mathematics*, etc., (1932), p. 229.

or, if we put $x = a + k$,

$$f(a+k) = f(a) + \frac{k}{h}\Delta f(a) + \frac{1}{2!}\frac{k(k-h)}{h^2}\Delta^2 f(a)$$
$$+ \frac{1}{3!}k(k-h)(k-2h)f^{(3)}(\bar{x}). \qquad \text{.....................(19)}$$

Since the greatest value of $|k(k-h)(k-2h)|$ in the range $(0, 2h)$ of k is $2h^3/(3\sqrt{3})$, this result shews that the uncertainty in $f(a+k)$ as determined by the interpolation formula (7) is $\frac{1}{9\sqrt{3}}h^3|f^{(3)}(x)|_{max}$, where $|f^{(3)}(x)|_{max}$ is the greatest value of $|f^{(3)}(x)|$ in the range $(a, a+2h)$ of x. For purposes of estimation the uncertainty can usually be taken as $\frac{1}{15}h^3|f^{(3)}(a)|$.

125. Tabulation of Inverse Functions. Tables Used in Reverse.

A table of a function $f(x)$, or y, for a range of the independent variable x is usually constructed for equidistant values of the argument. A separate table of the inverse function $\arg f(y)$, or x, if constructed, will thus naturally proceed by equidistant values of the variable y, which is now the argument. Usually, however, a separate table is not constructed but the table for $f(x)$ is employed for the inverse function also, being read backwards, or in reverse, to find x when y is given. The student is familiar with this process in the case of a table of common logarithms. If the given value of y occurs in the table, the value of the argument x against which it is tabulated is taken as the value of the inverse function.[1] When the given value of y, say y^*, falls between the two consecutive tabulated values y_1, y_2, the value x^* taken for the inverse function is found by interpolation between the values x_1, x_2 of the argument against which y_1, y_2 are tabulated. If we interpolate by the *M.P.P.*, we have

$$x^* = x_1 + \frac{y^* - y_1}{y_2 - y_1}(x_2 - x_1). \qquad \text{.......................(1)}$$

Two questions which now arise are considered in turn.

125. 1. *The uncertainty of x_1 taken as the value of the inverse function corresponding to y_1.* Suppose that the tabulated values of y are correct to n decimal places, so that the uncertainty of y_1 is $5 \times 10^{-(n+1)}$. The variation of y_1 by this amount corresponds to a variation of x_1 by the amount $\left(\frac{dx}{dy}\right)_{y_1} \times 5 \times 10^{-(n+1)}$, if we neglect

[1] If the given value of y occurs more than once we may take any one of the values of the argument x against which it is tabulated or, better, their mean.

the variation of dx/dy in the small range of y. Thus the uncertainty of x_1 is of amount $\dfrac{1}{(dy/dx)_{x_1}} \times 5 \times 10^{-(n+1)}$; it is large compared with the uncertainty of y_1 only if $(dy/dx)_{x_1}$ is small.

The meaning of this result is easily seen from a graph of the relation $y=f(x)$, in which equidistant points of division of the axis Ox correspond to equidistant arguments of the table, and the tabulated values of y are represented by the ordinates of the graph through these points. The use of the table in reverse corresponds to the determination of the value of the abscissa x when the ordinate y is given. Now it is evident that it is in the part of the range where y varies slowly, that is, where dy/dx is small, that uncertainty in the determination of x is large. The extreme case is that in which y remains constant in a part of the range, so that $dy/dx=0$ and x becomes uncertain to the complete extent of that part.

125. 2. *The uncertainty of an intermediate value x^* determined by the M.P.P.* According to the general formula of Art. 122, the error involved in finding x^*, corresponding to y^*, by this method, is not greater than $\frac{1}{8}k^2\left|\dfrac{d^2x}{dy^2}\right|_{max}$, where k is the difference $|y_2 - y_1|$ of the tabulated values between which y^* lies. Now

$$\frac{d^2x}{dy^2} = -\frac{d^2y}{dx^2}\Big/\left(\frac{dy}{dx}\right)^3, \quad \text{.........................(2)}$$

and if h is the common difference of the argument x, we have

$$k = |y_2 - y_1| = \left|\left(\frac{dy}{dx}\right)_{x_1}\right| h, \quad \text{.........................(3)}$$

ignoring the variation of dy/dx in the range (x_1, x_2). Thus, if we ignore also the variation of d^2y/dx^2 in the same range, we get the formula

$$\tfrac{1}{8}h^2\left|\left(\frac{d^2y}{dx^2}\right)_{x_1}\right| \Big/ \left|\left(\frac{dy}{dx}\right)_{x_1}\right| \quad \text{.........................(4)}$$

for the uncertainty of the value x^* which is involved in the interpolation.

The ratio of the uncertainties for direct and inverse reading has therefore in both of the above cases the value $|(dy/dx)_{x_1}|$, or k/h, that is, the ratio of corresponding small increments of the function and its argument.

If we denote the uncertainty of x_1 (or x_2) taken as the value of the inverse function corresponding to y_1 (or y_2) by U, and that

of an intermediate value x^* determined by the *M.P.P.* by U^*, the condition of completeness for inverse reading of the table is expressed by $U^* \leqslant U$. The value of x^* is then given to the same accuracy as x_1 (or x_2). Since the factor $1 \Big/ \left(\frac{dy}{dx}\right)_{x_1}$ appears in both U^* and U, the condition of completeness for inverse reading is equivalent to that obtained in Art. 122 for direct reading.

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126. Taylor's Series and Taylor's Theorem.

We have seen how Taylor polynomials of low degree may be used for the approximate evaluation of functions. It remains now to consider the case, (which is of the greatest interest from the analytical point of view), where the approximate evaluation for a given value of x can be carried to any desired degree of refinement by sufficiently increasing the degree of the polynomial whose value is taken for that of the function.

The question here discussed is not whether, for a given value of x, a value of n can be found such that the difference between the function and the polynomial of degree $n-1$ is less than an assigned ε (however small), but whether a value of n can be found such that the difference between the function and *each of the polynomials of degree not less than* $n-1$ is less than ε. Stated concisely, what is required is that the Taylor remainder $R_n(x)$ should converge to zero as n increases indefinitely or that the sequence of values of the successive polynomials should converge to a focus (or limit) which is equal to the function. Since the remainder in question lies between $\left|\frac{1}{n!}mx^n\right|$ and $\left|\frac{1}{n!}Mx^n\right|$ in the case of the polynomial of degree $n-1$, it is *sufficient* for the purpose that both these numbers should be less than ε for values of n greater than an appropriate integer N. Here m, M are the algebraically least and greatest values of $f^{(n)}(x)$ in the range $(0, x)$.

It should be noticed that if this condition is satisfied for a certain value x_1 of x, it is satisfied for all values of x in the range $(0, x_1)$, for $|x^n|$ decreases as $|x|$ decreases and m cannot decrease nor M increase because they are extreme values for the whole range.

When the sequence of values of the successive polynomials converges in the manner described, we may say that the function is defined by the polynomial of indefinitely high degree, since the sequence of polynomials is obtained from it by stopping at the successive terms. In other words, we can obtain the value of

the function to any desired accuracy by taking enough terms of the indefinitely continued polynomial.

The expression arrived at in this way, namely

$$f(0)+xf^{(1)}(0)+\frac{x^2}{2!}f^{(2)}(0)+\ldots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+\ldots, \quad \ldots\ldots\ldots(1)$$

is called the *Taylor Infinite Series*, or the *Taylor expansion*, of the function for the range in which the convergence holds.[1]

In the same case the function is said to be equal to the Taylor Series, the equality being expressed symbolically by the equation

$$f(x)=f(0)+xf^{(1)}(0)+\ldots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+\ldots. \quad \ldots\ldots\ldots(2)$$

This equality, or the symbolic expression of it, is called *Taylor's Theorem*.

When the remainder $R_n(x)$ after n terms is inserted on the right-hand side, the equation

$$f(x)=f(0)+xf^{(1)}(0)+\ldots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+R_n(x) \quad \ldots\ldots\ldots(3)$$

is called *Taylor's Theorem with Remainder*. The value $\frac{x^n}{n!}f^{(n)}(\theta x)$ of $R_n(x)$ is called *Lagrange's Form of Remainder* for Taylor's Series, and with this form of $R_n(x)$, Taylor's Theorem is said to have the Lagrangian Form.

It is to be observed that it is a *necessary*, but not a *sufficient*, condition for the convergence of $R_n(x)$ to zero when $n\to\infty$ that the term of the nth degree in the Taylor polynomials, that is, $\frac{x^n}{n!}f^{(n)}(0)$, should converge to zero when $n\to\infty$. For

$$\frac{x^n}{n!}f^{(n)}(0)=R_n(x)-R_{n+1}(x), \quad \ldots\ldots\ldots(4)$$

and if $R_n(x)$, $R_{n+1}(x)$ converge to zero, so does the difference.

127. The Binomial Theorem.

Many illustrations of Taylor's Theorem will be given later. We here consider the function $f(x)=(1+x)^m$, the starting point being $x=0$. When m is a positive integer, the result obtained constitutes

[1] The student should note that it is not proved (and, in fact, it is not true) that the value of the function considered is given by Taylor's Series merely in virtue of its convergence. It is necessary and sufficient that the remainder term should converge to zero as n is indefinitely increased. See, for example, Pierpont, *Functions of a Real Variable*, Vol. II, p. 214.

the Elementary Binomial Theorem. We may suppose this known and proceed to the discussion when m is positive and fractional, or negative.

As in Ex. 1, Art. 86, p. 181,

$$f^{(r)}(x)=m(m-1)\dots(m-r+1)(1+x)^{m-r}, \quad \text{.............(1)}$$

and we therefore have the following results

$$\left.\begin{aligned} f(0)=1,\quad f^{(1)}(0)=m,\quad \dots,\quad f^{(n-1)}(0)=m(m-1)\dots(m-n+2),\\ f^{(n)}(\theta x)=m(m-1)\dots(m-n+1)(1+\theta x)^{m-n}. \end{aligned}\right\} \quad \text{.....(2)}$$

The expression of Taylor's Theorem with Remainder in this case is thus

$$(1+x)^m=1+mx+\frac{m(m-1)}{2!}x^2+\dots$$
$$+\frac{m(m-1)\dots(m-n+2)}{(n-1)!}x^{n-1}+R_n(x), \quad \text{......(3)}$$

where

$$R_n(x)=\frac{m(m-1)\dots(m-n+1)}{n!}x^n(1+\theta x)^{m-n}, \quad (0<\theta<1). \quad \text{...(4)}$$

(i) Suppose, at first, that $0<x<1$. We can write $R_n(x)$ as the product, P say, of n terms of the form

$$\left(\frac{m+1}{r}-1\right)\frac{x}{1+\theta x}, \quad (r=1, 2, \dots, n),$$

multiplied by $(1+\theta x)^m$. Here m may be positive or negative. Writing $|m+1|=\mu$, we have

$$\left|\left(\frac{m+1}{r}-1\right)\frac{x}{1+\theta x}\right|<\left(1+\frac{\mu}{r}\right)x, \quad \text{....................(5)}$$

and if we take a number ξ between x and 1, we have

$$\left(1+\frac{\mu}{r}\right)x<\xi \text{ provided } r>\mu\frac{x}{\xi-x}. \quad \text{....................(6)}$$

Let r_1 be the smallest value of r for which (6) is satisfied. Then supposing, as we may, that $n>r_1-1$, we have

$$|P|<|Q|\xi^{n-r_1+1}, \quad \text{.............................(7)}$$

where Q is the product of the first r_1-1 factors of P and is independent of n. We conclude that

$$|R_n(x)|<|Q|(1+x)^m\xi^{n-r_1+1}. \quad \text{....................(8)}$$

Since $\xi<1$, ξ^{n-r_1+1} diminishes indefinitely as n increases in-

definitely, and $R_n(x)$ can be made less than an arbitrarily assigned number ε for all values of n greater than N by taking

$$|Q|(1+x)^m\xi^{N-r_1+1}<\varepsilon, \quad\text{..........................(9)}$$

an inequality which can always be satisfied by (an appropriate) choice of N. The condition for Taylor's Series is therefore satisfied in this case, and we can write

$$(1+x)^m=1+mx+\frac{m(m-1)}{2!}x^2+\ldots$$
$$+\frac{m(m-1)\ldots(m-n+1)}{n!}x^n+\ldots \quad\text{.......(10)}$$

This equation expresses the Binomial Theorem for a positive value of x less than 1.

(ii) If x is negative, the above method of examination of $R_n(x)$ succeeds only if $|x|<1/2$. Writing $x=-x'$, we have, with this restriction,

$$\left|\frac{x}{1+\theta x}\right|=\left|\frac{x'}{1-\theta x'}\right|<\frac{x'}{1-x'}<1. \quad\text{..................(11)}$$

Putting $x'/(1-x')=\eta$, so that $\eta<1$, the above argument applies if we introduce η for x in the inequalities ; thus we have

$$\left|\left(\frac{m+1}{r}-1\right)\frac{x}{1+\theta x}\right|<\left(1+\frac{\mu}{r}\right)\eta, \quad\text{..................(12)}$$

and $$\left(1+\frac{\mu}{r}\right)\eta<\xi \text{ provided } r>\mu\frac{\eta}{\xi-\eta}, \quad\text{................(13)}$$

where $\eta<\xi<1$. The equation (10) thus still holds if $-\frac{1}{2}<x<0$. It is proved below, Art. 128. 1, that it is also valid if $-1<x<0$.

(iii) We consider finally the case where $|x|>1$. The Taylor polynomials do not now satisfy the necessary condition for the validity of Taylor's Theorem that the general term $\frac{1}{n!}m(m-1)\ldots(m-n+1)x^n$ should converge to zero when $n\to\infty$. For if we write

$$u_n=\frac{1}{n!}|m(m-1)\ldots(m-n+1)|\,.\,|x|^n, \quad\text{..............(14)}$$

we have $$\frac{u_{n+1}}{u_n}=\frac{|n-m|}{n+1}|x|, \quad\text{..........................(15)}$$

which is greater than 1 if n is at once greater than m and $(m|x|+1)/(|x|-1)$.

Thus, after a certain value of n, $|u_n|$ increases with n and therefore cannot converge to zero. The Binomial Theorem, as expressed by equation (10), therefore does not hold if $|x|>1$.

The Binomial Theorem also holds when $x=1$ provided that $m>-1$, and also when $x=-1$ provided that $m>0$. The proofs of these latter statements cannot, however, be given at this stage; see Chap. XVIII, Art. 347. 1.

If the function considered is $(a+x)^m$, the corresponding Binomial Theorem is obtained by writing the function in the form $a^m(1+x/a)^m$, or $a^m(1+x')^m$ say, where $x'=x/a$. We then have

$$(a+x)^m = a^m\{1+mx'+\frac{m(m-1)}{2!}x'^2+\ldots\}$$

$$=a^m+ma^{m-1}x+\frac{m(m-1)}{2!}a^{m-2}x^2+\ldots, \quad \ldots\ldots(16)$$

the remainder after n terms being given by

$$R_n(x)=\frac{1}{n!}m(m-1)\ldots(m-n+1)x^n(a+\theta x)^{m-n}, \quad (0<\theta<1). \quad \ldots\ldots(17)$$

Ex. 1. *Expansion of* $\sqrt{(1+x)}$. Here $m=\frac{1}{2}>0$, and the infinite series is valid if $-1\leqslant x\leqslant 1$. We have

$$(1+x)^{1/2}=1+\tfrac{1}{2}x+\frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2+\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3+\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}x^4+\ldots$$

$$+\frac{\frac{1}{2}(-\frac{1}{2})\ldots(\frac{1}{2}-n+1)}{n!}x^n+\ldots$$

$$=1+\tfrac{1}{2}x-\frac{1}{2^2 2!}x^2+\frac{1.3}{2^3 3!}x^3-\frac{1.3.5}{2^4 4!}x^4+\ldots$$

$$+(-1)^{n-1}\frac{1.3.5\ldots(2n-3)}{2^n n!}x^n+\ldots. \quad \ldots\ldots(18)$$

The first four Taylor polynomials are

$$\left.\begin{array}{l}\text{(i) } 1+\tfrac{1}{2}x, \quad \text{(ii) } 1+\tfrac{1}{2}x-\tfrac{1}{8}x^2, \quad \text{(iii) } 1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3, \\ \text{(iv) } 1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3-\tfrac{5}{128}x^4.\end{array}\right\} \quad \ldots\ldots(19)$$

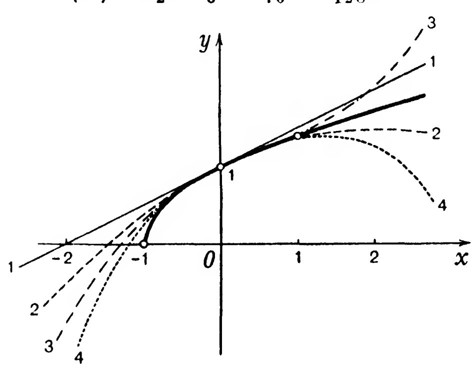

FIG. 93.

The closeness of approximation of these polynomials to the function is illustrated in Fig. 93, where the heavy line is the graph of the given function

and the lightly-drawn numbered lines are those of the successive polynomials. The range of convergence, $(-1, 1)$, is clearly indicated.

Ex. 2. *Expansion* of $1/\sqrt{(1+x)}$. Here $m = -\frac{1}{2} > -1$ and the infinite series is valid only for the range $(-1, 1)$ of x. We have

$$(1+x)^{-1/2} = 1 - \tfrac{1}{2}x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}x^2 - \dots + \frac{(-\frac{1}{2})(-\frac{3}{2})\dots(-\frac{1}{2}-n+1)}{n!}x^n + \dots$$

$$= 1 - \tfrac{1}{2}x + \frac{1.3}{2^2 2!}x^2 - \frac{1.3.5}{2^3 3!}x^3 + \dots + (-1)^n \frac{1.3\dots(2n-1)}{2^n n!}x^n + \dots . \quad (20)$$

Ex. 3. *Expansion of* $1/(1+x)^2$. Here $m = -2$ and the infinite series is only valid in the range $[-1, 1]$ of x. We have

$$(1+x)^{-2} = 1 - 2x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3 + \dots$$

$$+ \frac{(-2)(-3)\dots(-n-1)}{n!}x^n + \dots$$

$$= 1 - 2x + 3x^2 - 4x^3 + \dots + (-1)^n(n+1)x^n + \dots . \quad \dots\dots(21)$$

Ex. 4. A circle of radius r touches the axis Ox at the origin O (Fig. 94) and P is the point (x, y) on it. Shew that the ordinate y is approximately equal to $\dfrac{x^2}{2r} + \dfrac{x^4}{8r^3}$ when $\dfrac{x}{r}$ is small.

From the Figure we have

$$y = OA - NA = r - \sqrt{(r^2 - x^2)} = r - r(1 - x^2/r^2)^{1/2}$$

$$= r - r\left(1 - \frac{1}{2}\frac{x^2}{r^2} - \frac{1}{8}\frac{x^4}{r^4} + R\right)$$

$$= \frac{x^2}{2r} + \frac{x^4}{8r^3} - rR, \quad \dots\dots(22)$$

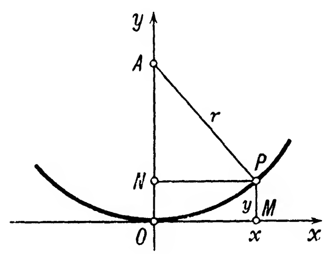

FIG. 94.

where, using the Lagrange form of remainder,

$$|R| = \frac{1}{16}\frac{x^6}{r^6}\frac{1}{(1-\theta^2 x^2/r^2)^{5/2}} < \frac{1}{16}\frac{x^6}{r^6}\frac{1}{(1-x^2/r^2)^{5/2}} . \quad \dots\dots(23)$$

For small values of x/r, the remainder is small compared with the other terms and we may write

$$y \approx \frac{x^2}{2r} + \frac{x^4}{8r^3} . \quad \dots\dots(24)$$

To take a numerical case, let $r = 40$ feet, $x = 10$ feet. From (24) we have

$$y \approx 40\left(\frac{1}{2}\cdot\frac{1}{16} + \frac{1}{8}\cdot\frac{1}{16^2}\right) = 1{\cdot}2695 \text{ feet}, \quad \dots\dots(25)$$

and the error involved is less than $\dfrac{10}{4} \times \left(\dfrac{1}{4}\right)^6 \times \dfrac{1}{(1-\frac{1}{16})^{5/2}} \approx 0{\cdot}0007$ feet.

128. Alternative Forms for the Remainder in Taylor's Theorem.

Many formulae of the same kind as Lagrange's have been given for the remainder after n terms in Taylor's Theorem. We here give

a demonstration of Taylor's Theorem with Remainder which includes the forms of the remainder most commonly employed. The proof depends on a rule, due to Cauchy, which follows directly from Rolle's Theorem.

Let $F(x)$, $G(x)$ be two continuous functions of x in a range (x_1, x_2), each possessing a definite derivative at all points in the range $[x_1, x_2]$. We construct a function

$$F(x) + AG(x) + B, \quad \text{.........................(1)}$$

linear in $F(x)$, $G(x)$ and which vanishes at the ends x_1, x_2 of the range. This requires that the disposable constants A, B should satisfy the equations

$$F(x_1) + AG(x_1) + B = 0, \quad F(x_2) + AG(x_2) + B = 0, \quad \text{.........(2)}$$

whence, supposing $G(x_2) \neq G(x_1)$, we find

$$\left.\begin{aligned} A &= -\frac{F(x_2) - F(x_1)}{G(x_2) - G(x_1)}, \\ B &= -F(x_1) + \frac{F(x_2) - F(x_1)}{G(x_2) - G(x_1)} G(x_1). \end{aligned}\right\} \quad \text{.................(3)}$$

The required function is thus

$$F(x) - F(x_1) - \frac{F(x_2) - F(x_1)}{G(x_2) - G(x_1)} \{G(x) - G(x_1)\}. \quad \text{...........(4)}$$

By Rolle's Theorem the derivative of this function vanishes for some value $\bar{x}$ in the range $[x_1, x_2]$, and we therefore have

$$F'(\bar{x}) - \frac{F(x_2) - F(x_1)}{G(x_2) - G(x_1)} G'(\bar{x}) = 0. \quad \text{....................(5)}$$

Suppose, now, that $G'(x)$ does not vanish in $[x_1, x_2]$, so that $G'(\bar{x}) \neq 0$. Then replacing x_2 by a variable end point x in the range $[x_1, x_2)$, we deduce, from (5), the relation

$$\frac{F(x) - F(x_1)}{G(x) - G(x_1)} = \frac{F'(\bar{x})}{G'(\bar{x})}, \quad \{G(x) \neq G(x_1),\ G'(\bar{x}) \neq 0\}, \quad \text{........(6)}$$

where now $\bar{x}$ lies in $[x_1, x]$. This is the symbolic expression of the required rule. To secure the satisfaction of the inequalities in (6), it is sufficient that the derivative $G'(x)$ should be one-signed in $[x_1, x_2]$.

For the present purpose it is sufficient to suppose the lower terminal x_1 of the range to be zero, and we choose the functions $F(x)$, $G(x)$ in such a way that $F(0) = 0$, $G(0) = 0$. We then have, from (6),

$$F(x) = G(x) \frac{F'(\theta x)}{G'(\theta x)}, \quad (0 < \theta < 1). \quad \text{...................(7)}$$

Now let $f(x)$, the function to which it is proposed to apply Taylor's Theorem, possess definite derivatives of all orders up to the $(n-1)$th inclusive at all points in the given range $(0, x_2)$ and a definite nth derivative throughout $[0, x_2]$. From $f(x)$ define a function $\varphi(x)$ by means of the equation

$$\varphi(x) = f(x_2 - x), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

so that as x ranges from 0 to x_2, $\varphi(x)$ ranges from $f(x_2)$ to $f(0)$, that is, $\varphi(x)$ takes on the same values as the function $f(x)$ in the range but in the reverse order. The introduction of the function $\varphi(x)$ effects a simplification in the succeeding investigation.

In order to lead to the standard forms of Taylor's Theorem with Remainder, we choose special forms for the functions $F(x)$, $G(x)$. The function $F(x)$ is expressed in terms of $\varphi(x)$ and its derivatives by means of the equation

$$F(x) = -\varphi(0) + \varphi(x) - x\varphi^{(1)}(x) + \frac{x^2}{2!}\varphi^{(2)}(x) - \ldots$$
$$+ (-1)^{n-1}\frac{x^{n-1}}{(n-1)!}\varphi^{(n-1)}(x), \quad \ldots\ldots(9)$$

and we take

$$G(x) = x^p, \quad (p \text{ a positive integer}), \quad \ldots\ldots\ldots\ldots\ldots(10)$$

which is a choice of $G(x)$ consistent with the conditions imposed on it. From these we have $F(0)=0$, $G(0)=0$. The function $F(x)$ possesses a definite derivative at all points in $[0, x_2]$ and it has been so formed that its first derivative involves the nth derivative only of $\varphi(x)$. We have, in fact,

$$F^{(1)}(x) = (-1)^{n-1}\frac{x^{n-1}}{(n-1)!}\varphi^{(n)}(x). \quad \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

On applying the rule (7), we now obtain the result

$$-\varphi(0) + \varphi(x) - x\varphi^{(1)}(x) + \ldots + (-1)^{n-1}\frac{x^{n-1}}{(n-1)!}\varphi^{(n-1)}(x)$$
$$= x^p \frac{1}{p(\theta x)^{p-1}}(-1)^{n-1}\frac{(\theta x)^{n-1}}{(n-1)!}\varphi^{(n)}(\theta x)$$
$$= (-1)^{n-1}\frac{\theta^{n-p}x^n}{p(n-1)!}\varphi^{(n)}(\theta x), \quad (0<\theta<1). \ldots\ldots\ldots\ldots(12)$$

This gives

$$\varphi(0) = \varphi(x) - x\varphi^{(1)}(x) + \ldots + (-1)^{n-1}\frac{x^{n-1}}{(n-1)!}\varphi^{(n-1)}(x)$$
$$+ (-1)^n\frac{\theta^{n-p}x^n}{p(n-1)!}\varphi^{(n)}(\theta x), \quad \ldots\ldots\ldots\ldots\ldots(13)$$

and, in particular,

$$\varphi(0)=\varphi(x_2)-x_2\varphi^{(1)}(x_2)+\dots+(-1)^{n-1}\frac{x_2^{\,n-1}}{(n-1)!}\varphi^{(n-1)}(x_2)$$
$$+(-1)^n\frac{\theta^{n-p}x_2^{\,n}}{p(n-1)!}\varphi^{(n)}(\theta x_2). \quad \text{......(14)}$$

This result is now to be expressed in terms of the original function $f(x)$ and its derivatives. Since $\varphi(x)=f(x_2-x)$, we have the results

$$\left.\begin{aligned}\varphi^{(r)}(x)=(-1)^r f^{(r)}(x_2-x),\quad \varphi^{(r)}(x_2)=(-1)^r f^{(r)}(0),\\ \varphi^{(n)}(\theta x_2)=(-1)^n f^{(n)}(x_2-\theta x_2)=(-1)^n f^{(n)}(\vartheta x_2),\ \text{say,}\end{aligned}\right\} \quad \text{......(15)}$$

where $\vartheta=1-\theta$, so that $0<\vartheta<1$. We therefore obtain

$$f(x_2)=f(0)+x_2 f^{(1)}(0)+\frac{x_2^{\,2}}{2!}f^{(2)}(0)+\dots+\frac{x_2^{\,n-1}}{(n-1)!}f^{(n-1)}(0)$$
$$+\frac{(1-\vartheta)^{n-p}x_2^{\,n}}{p(n-1)!}f^{(n)}(\vartheta x_2), \quad \text{......(16)}$$

which is Taylor's Theorem for a starting point 0 and end point x_2 with the special form of remainder sought.

On introducing a variable end point x in the range $(0, x_2)$, we have

$$f(x)=f(0)+xf^{(1)}(0)+\frac{x^2}{2!}f^{(2)}(0)+\dots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+R_n(x), \quad \text{...(17)}$$

where $R_n(x)$, the remainder after n terms, is given by

$$R_n(x)=\frac{(1-\vartheta)^{n-p}x^n}{p(n-1)!}f^{(n)}(\vartheta x). \quad \text{......(18)}$$

This expression for $R_n(x)$ is commonly referred to as the *Schlömilch-Roche formula* for the remainder. The Lagrange formula is immediately recovered from it by putting $p=n$. A special form of frequent occurrence arises when p is put equal to 1, namely,

$$R_n(x)=\frac{(1-\vartheta)^{n-1}x^n}{(n-1)!}f^{(n)}(\vartheta x), \quad \text{......(19)}$$

which is known as the *Cauchy formula* for the remainder after n terms.

128. 1. *The Binomial Theorem for the range* $[-1, 0)$ *of* x. We can now supply the demonstration that the Binomial Theorem for the expansion of $(1+x)^m$ holds when x lies in the range $[-1, 0)$. Using Cauchy's formula for the remainder for the range stated, we have

$$R_n(x)=\frac{m(m-1)\dots(m-n+1)}{(n-1)!}x^n(1-\vartheta)^{n-1}(1+\vartheta x)^{m-n}. \quad \text{...(20)}$$

As in Art. 127, we can write this as the product of $n-1$ factors of the form $\left(\frac{m}{r}-1\right)\frac{x(1-\vartheta)}{1+\vartheta x}$, $(r=1, 2, \ldots, n-1)$, multiplied by $mx(1+\vartheta x)^{m-1}$. Writing $|m|=\mu$, we have

$$\left|\left(\frac{m}{r}-1\right)\frac{x(1-\vartheta)}{1+\vartheta x}\right| = \left|\frac{m}{r}-1\right|\frac{|x|(1-\vartheta)}{1-\vartheta|x|} < \left(1+\frac{\mu}{r}\right)|x|, \quad \ldots .(21)$$

since $\vartheta|x|<\vartheta<1$. The discussion then proceeds exactly as before.

129. Taylor Polynomials for Combinations of Functions.

It is often convenient to obtain the earlier Taylor polynomials of a combination of functions by algebraic operations from the Taylor polynomials of the separate functions instead of by direct determination of derivatives. For this purpose the following elementary theorems are employed, the starting point being always $x=0$.

THEOREM I. *If u, v are two functions of x, the Taylor polynomial of degree n for the function $u+v$ is the sum of the Taylor polynomials of the same degree for u and v.*

This is clear from the definition of the polynomials.

THEOREM II. *The Taylor polynomial of degree n for the function uv is found by forming the product of the Taylor polynomials of degree n for u and v, and rejecting the terms in it of degree higher than the nth.*

This follows since, by Leibniz's Theorem, Art. 90, the mth derivative of uv is given by

$$\frac{1}{m!}(uv)^{(m)} = \frac{1}{m!}u^{(m)}v + \frac{1}{(m-1)!}u^{(m-1)}v^{(1)} + \frac{1}{2!}\frac{1}{(m-2)!}u^{(m-2)}v^{(2)} + \ldots$$
$$+\frac{1}{m!}uv^{(m)}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and if $m \leqslant n$ the right-hand side is the coefficient of x^m in the product

$$\left(u + u^{(1)}x + \frac{1}{2!}u^{(2)}x^2 + \ldots + \frac{1}{n!}u^{(n)}x^n\right)$$
$$\times\left(v + v^{(1)}x + \frac{1}{2!}v^{(2)}x^2 + \ldots + \frac{1}{n!}v^{(n)}x^n\right) \ldots(2)$$

of the Taylor polynomials for u and v.

THEOREM III. *The Taylor polynomial of degree n for the function u/v is found by dividing that for u by that for v and rejecting terms in the quotient which are of degree higher than the nth. It is here assumed that v does not vanish at $x=0$.*

Let $a_0+a_1x+\ldots+a_nx^n$ be the polynomial obtained by dividing the Taylor polynomial for u by that for v, as described. The coefficients $a_0, a_1, \ldots, a_n$ can be obtained by equating the coefficient of each power of x, up to the nth, in the product

$$\left(v+v^{(1)}x+\ldots+\frac{1}{n!}v^{(n)}x^n\right)\left(a_0+a_1x+\ldots+a_nx^n\right), \quad \ldots\ldots\ldots\ldots(3)$$

to the coefficient of the same power of x in the polynomial $u+u^{(1)}x+\ldots+\frac{1}{n!}u^{(n)}x^n$. This gives $n+1$ linear equations to determine $a_0, a_1, \ldots, a_n$. Now according to Theorem II, if we write $u/v=w$, so that $vw=u$, the coefficients of the Taylor polynomial of the nth degree for u are equal to the respective coefficients of the powers of x in the product

$$\left(v+v^{(1)}x+\ldots+\frac{1}{n!}v^{(n)}x^n\right)\left(w+w^{(1)}x+\ldots+\frac{1}{n!}w^{(n)}x^n\right). \quad \ldots\ldots(4)$$

Thus the same set of linear equations is found for the coefficients $w, w^{(1)}, \ldots, \frac{1}{n!}w^{(n)}$ as for $a_0, a_1, \ldots, a_n$, to which, therefore, they are respectively equal.

Generalizing these results, we deduce that a Taylor polynomial for a function compounded algebraically (by addition, multiplication and division) of a set of functions may be obtained by algebraic operations applied to the Taylor polynomials of the functions of the set.

In a similar manner, Taylor polynomials for algebraic expressions which contain powers of polynomials can be found algebraically, the Taylor polynomials for the rational powers being found by the use of the Binomial Theorem.

Ex. 1. Find the Taylor polynomial of the third degree for $1/\sqrt{(1+x)}$, given that the corresponding polynomial for $\sqrt{(1+x)}$ is

$$1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3. \quad \ldots\ldots\ldots\ldots(5)$$

According to Theorem III, the required polynomial is given correctly by dividing 1 by the expression (3) and rejecting terms in the quotient which involve powers of x higher than the third. The result obtained in this way is

$$1-\tfrac{1}{2}x+\tfrac{3}{8}x^2-\tfrac{5}{16}x^3. \quad \ldots\ldots\ldots\ldots(6)$$

See also Ex. 2, p. 271.

Instead of dividing out, we may employ the method of disposable coefficients and use Theorem II. Thus, assuming the form $\alpha+\beta x+\gamma x^2+\delta x^3$ for the polynomial sought, we are to have

$$(\alpha+\beta x+\gamma x^2+\delta x^3)(1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3)=1, \quad \ldots\ldots\ldots\ldots(7)$$

where in the product we reject terms involving powers of x above the third.

The process of equating coefficients of powers of x on each side of (7) then leads to the equations

$$\alpha=1, \quad \beta+\tfrac{1}{2}\alpha=0, \quad \gamma+\tfrac{1}{2}\beta-\tfrac{1}{8}\alpha=0, \quad \delta+\tfrac{1}{2}\beta-\tfrac{1}{8}\gamma+\tfrac{1}{16}\alpha=0, \quad \text{........(8)}$$

whence we find $\alpha=1$, $\beta=-1/2$, $\gamma=3/8$, $\delta=-5/16$.

THEOREM IV. *Taylor polynomial for a function of a function.* Let the function $f(y)$ and its first n derivatives with respect to y be finite at $y=0$. Also let $y=\varphi(x)$, where $\varphi(0)=0$ and $\varphi(x)$ has its first n derivatives with respect to x finite at $x=0$. Then there is a Taylor polynomial of the nth degree,

$$P_n(y)\equiv f(0)+f^{(1)}(0)\,y+\frac{1}{2!}f^{(2)}(0)\,y^2+\ldots+\frac{1}{n!}f^{(n)}(0)\,y^n, \quad \text{......(9)}$$

for $f(y)$, and a Taylor polynomial of the nth degree,

$$Q_n(x)\equiv\varphi^{(1)}(0)\,x+\frac{1}{2!}\varphi^{(2)}(0)\,x^2+\ldots+\frac{1}{n!}\varphi^{(n)}(0)\,x^n, \quad \text{......(10)}$$

for $\varphi(x)$ or y. We shew that *the Taylor polynomial of degree n for the function $f\{\varphi(x)\}$ is also the Taylor polynomial for the function $P_n\{Q_n(x)\}$*, and it can therefore be found by substituting $Q_n(x)$ for y in $P_n(y)$ and neglecting powers of x higher than the nth.

To prove this statement we have to shew that the first n derivatives of $f\{\varphi(x)\}$ at $x=0$ are respectively equal to the first n derivatives of $P_n\{Q_n(x)\}$ there. Consider the mth derivative of $f(y)$, or $f\{\varphi(x)\}$, with respect to x, where $m\leqslant n$. It is found, by repeated differentiation, as a polynomial in the first m derivatives of $f(y)$ with respect to y and the first m derivatives of y, or $\varphi(x)$, with respect to x. Also, the first m derivatives of $\varphi(x)$ and $Q_n(x)$ are respectively equal at $x=0$, and the first m derivatives of $f(y)$ and $P_n(y)$ with respect to y are equal at $y=0$ and therefore when $x=0$, since $\varphi(0)=0$.

Therefore the mth derivative of $f\{\varphi(x)\}$ at $x=0$ is correctly found by replacing $f(y)$ by $P_n(y)$ and $\varphi(x)$ by $Q_n(x)$. Since this is true provided $m\leqslant n$, the identity of the nth polynomials for $f\{\varphi(x)\}$ and $P_n\{Q_n(x)\}$ is demonstrated.

We proceed to illustrate two methods of applying this theorem to the determination of Taylor polynomials of a function defined implicitly.

Ex. 2. Find the Taylor polynomial of the sixth degree for y when y is defined implicitly in terms of x by means of the equation

$$y+\sqrt{(1+y)}=1+\tfrac{1}{2}x^2, \quad \text{..............................(11)}$$

with the condition $y=0$ when $x=0$.

This equation can be solved for y and the function for which the Taylor polynomial is to be found is $\frac{1}{2}(3+x^2)-\frac{1}{2}\surd(9+2x^2)$. Thus the results obtained below can be derived by means of the Binomial Theorem. The Example is used here to illustrate methods which can be employed also where the function cannot be explicitly expressed.

(i) *The method of disposable coefficients.* Assuming a form

$$y=a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6, \quad\text{.....(12)}$$

equation (11) gives

$$1+y=(1+\tfrac{1}{2}x^2-y)^2, \quad\text{.....(13)}$$

that is

$$\begin{aligned}1+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6\\ =[1-\{a_1x+(a_2-\tfrac{1}{2})x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6\}]^2,\end{aligned} \quad\text{.....(14)}$$

where on the right-hand side terms involving powers of x higher than the sixth are to be rejected. Comparing coefficients we get the set of equations

$$\left.\begin{array}{l}a_1=-2a_1, \quad a_2=a_1^2-2(a_2-\tfrac{1}{2}), \quad a_3=2a_1(a_2-\tfrac{1}{2})-2a_3,\\ a_4=2a_1a_3+(a_2-\tfrac{1}{2})^2-2a_4, \quad a_5=2a_1a_4+2(a_2-\tfrac{1}{2})a_3-2a_5,\\ a_6=2a_1a_5+2(a_2-\tfrac{1}{2})a_4+a_3^2-2a_6,\end{array}\right\} \quad\text{.....(15)}$$

whence we find

$$a_1=0=a_3=a_5, \quad a_2=1/3, \quad a_4=1/108, \quad a_6=-1/972. \quad\text{.....(16)}$$

The required Taylor polynomial is therefore

$$y=\frac{1}{3}x^2+\frac{1}{108}x^4-\frac{1}{972}x^6. \quad\text{.....(17)}$$

It may be remarked that since even powers only of x occur in the development of (13), the powers of x of odd index in (12) could have been omitted.

(ii) *The method of successive substitutions.* An alternative method is to proceed in steps, employing the Taylor polynomials for the function $\surd(1+y)$ in terms of y of increasing degree at each step. The earlier polynomials are obtained from Ex. 1, Art. 127, p. 270. Employing the polynomial of the first degree, $1+\frac{1}{2}y$, the Taylor polynomial of the second degree for y in terms of x is given by

$$y=1+\tfrac{1}{2}x^2-(1+\tfrac{1}{2}y), \quad\text{.....(18)}$$

and is therefore $\frac{1}{3}x^2$. If we employ the polynomial $1+\frac{1}{2}y-\frac{1}{8}y^2$ for $\surd(1+y)$, writing $y=1+\frac{1}{2}x^2-(1+\frac{1}{2}y-\frac{1}{8}y^2)$, that is,

$$y=\tfrac{1}{3}x^2+\tfrac{1}{12}y^2, \quad\text{.....(19)}$$

the Taylor polynomial of the fourth degree for y in terms of x is given by

$$y=\tfrac{1}{3}x^2+\tfrac{1}{12}(\tfrac{1}{3}x^2)^2=\tfrac{1}{3}x^2+\tfrac{1}{108}x^4. \quad\text{.....(20)}$$

Similarly, writing $y=1+\frac{1}{2}x^2-(1+\frac{1}{2}y-\frac{1}{8}y^2+\frac{1}{16}y^3)$, that is,

$$y=\tfrac{1}{3}x^2+\tfrac{1}{12}y^2-\tfrac{1}{24}y^3, \quad\text{.....(21)}$$

the Taylor polynomial of the sixth degree for y in terms of x is obtained from the equation

$$y=\tfrac{1}{3}x^2+\tfrac{1}{12}(\tfrac{1}{3}x^2+\tfrac{1}{108}x^4)^2-\tfrac{1}{24}(\tfrac{1}{3}x^2+\tfrac{1}{108}x^4)^3 \quad\text{.....(22)}$$

when we reject terms involving powers of x higher than the sixth. This gives

$$y = \tfrac{1}{3}x^2 + \tfrac{1}{108}x^4 - \tfrac{1}{972}x^6, \quad \text{.........}(23)$$

as before.

130. Extreme Values of a Polymetric Function.

An adequate treatment of the extreme values of a polymetric function on the same lines as that given for a monometric function belongs to a later stage as it requires the employment of the Theorem of Intermediate Derivatives (Taylor's Theorem with Remainder) for polymetric functions. It seems desirable, however, to consider here the *stationary conditions* which are constantly employed to determine the positions of extreme values. As in the case of monometric functions, the special features of a particular geometrical or physical problem often enable us to decide the nature of the extreme values without the application of general criteria. These are examined here only so far as to find the second-derivative criteria for a dimetric function.

A polymetric function $f(x_1, x_2, \dots, x_n)$ is said to have a *proper local upper extreme* or *maximum* at a point A, $(a_1, a_2, \dots, a_n)$, if there exists a neighbourhood [1] of the point such that the value of the function at any point of it other than A is algebraically less than the value at A; the inequality being reversed in the case of a *proper local lower extreme* or *minimum*.

For simplicity we consider, at first, the case of a dimetric function $f(x, y)$ and suppose that it has a maximum value at (x_1, y_1). Then, treating x, y as rectangular coordinates in a plane, there must exist a neighbourhood, which we may take to be a circle with its centre at (x_1, y_1), such that the value of the function at any other point of it is less than that at the centre. Consider, now, the points of the neighbourhood which are on the line $y = y_1$ through the centre. The monometric function $f(x, y_1)$ is a maximum at (x_1, y_1), since its value at any other point of the line, inside the circle, is less than the value at the centre. It follows, from Art. 94, that the derivative $\partial f/\partial x$, which is assumed to exist, must vanish at the centre. The same argument applied to the line $x = x_1$ shews that the derivative $\partial f/\partial y$ must vanish at the centre. If the argument is applied to any other line through the centre, no additional conditions are found. To shew this, take a line at an angle α with the axis Ox. Along it $y = y_1 + (x - x_1)\tan\alpha$ and on substituting this value of y in $f(x, y)$

[1] A neighbourhood is here *some* domain within which $(a_1, a_2, \dots, a_n)$ lies. It may be defined, for example, by the inequation $(x_1 - a_1)^2 + (x_2 - a_2)^2 + \dots + (x_n - a_n)^2 \leqslant r^2$.

we obtain a function of x, $\varphi(x)$ say, which, as before, is a maximum [1] at the centre. Thus $\varphi'(x)$ must vanish at that point. Now according to Art. 82,

$$\varphi'(x)=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx} \quad \text{.........................(1)}$$

$$=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\tan\alpha, \quad \text{.....................(2)}$$

and the condition $\varphi'(x_1)=0$ is consequently satisfied in virtue of the conditions $\partial f/\partial x=0$, $\partial f/\partial y=0$, already laid down. The above argument clearly applies equally if the function $f(x, y)$ has a minimum value at (x_1, y_1).

The conditions

$$\left(\frac{\partial f}{\partial x}\right)_{(x_1, y_1)}=0, \quad \left(\frac{\partial f}{\partial y}\right)_{(x_1, y_1)}=0 \quad \text{....................(3)}$$

thus obtained are called the *stationary conditions* and the value of the function at a point where they are satisfied is called a *stationary value*. In simple cases these conditions usually determine only a finite number of points at which the function *may* have an extreme value.

In the case of a polymetric function $f(x_1, x_2, \dots, x_n)$, exactly the same argument can be applied. If there is an extreme value at A, $(a_1, a_2, \dots, a_n)$, the monometric function $f(x_1, a_2, \dots, a_n)$ has an extreme value when $x_1=a_1$ and the partial derivative $\partial f/\partial x_1$, assumed to exist, must vanish there; similarly for the other variables $x_2, \dots, x_n$. We thus obtain the stationary conditions

$$\frac{\partial f}{\partial x_1}=\frac{\partial f}{\partial x_2}=\dots=\frac{\partial f}{\partial x_n}=0, \quad \text{.......................(4)}$$

where the values of the derivatives are those at $(a_1, a_2, \dots, a_n)$. As in the case of a dimetric function, no other conditions are found in this manner. In simple cases these stationary conditions usually determine only a finite number of points at which the function *may* have an extreme value.

Ex. 1. Find the values of the n positive numbers $x_1, x_2, \dots, x_n$ for which the product $x_1x_2 \dots x_n$ is greatest, the sum s of the numbers being given.

Here we have $x_1+x_2+\dots+x_n=s$, a given number, and we can therefore

[1] The distance of any point of the line from the centre is $(x-x_1)\sec\alpha$, not x, but the use of this variable instead of the distance does not affect the argument.

It may be recalled that the conditions under which the equation (1) is valid have not been examined in Art. 82 and, in fact, it has only been proved for a special form of $f(x, y)$.

write $x_1x_2 \dots x_n = x_1x_2 \dots x_{n-1}(s - x_1 - x_2 - \dots - x_{n-1})$. Denoting the last function by $f(x_1, x_2, \dots, x_{n-1})$, the stationary conditions are

$$\frac{\partial f}{\partial x_1} = x_2x_3 \dots x_{n-1}(s - x_1 - x_2 - \dots - x_{n-1}) - x_1x_2 \dots x_{n-1}$$

$$= x_2x_3 \dots x_{n-1}x_n - x_1x_2 \dots x_{n-1} = 0, \qquad (5)$$

$$\frac{\partial f}{\partial x_2} = x_1x_3 \dots x_{n-1}x_n - x_1x_2 \dots x_{n-1} = 0, \qquad (6)$$

. . . .

$$\frac{\partial f}{\partial x_{n-1}} = x_1x_2 \dots x_{n-2}x_n - x_1x_2 \dots x_{n-1} = 0. \qquad (7)$$

Now for the required solution the numbers are all positive, and on removing common factors in these equations, we get

$$x_1 = x_2 = x_3 = \dots = x_n. \qquad (8)$$

Each of the numbers sought is therefore s/n and their product is s^n/n^n. This must correspond to a maximum value of the product, which evidently exists.

130. 1. *Discrimination of extreme values of a dimetric function.* The argument applied above shews that if the function $f(x, y)$ is a maximum at (x_1, y_1), the monometric function $f(x, y_1)$ is a maximum at x_1 and, if the conditions of Art. 106 are satisfied, it follows that $\partial^2 f/\partial x^2$ must be negative or zero at (x_1, y_1). We will suppose that the latter (critical) value does not occur.[1] In the same way, excluding the critical case, we find that $\partial^2 f/\partial y^2$ must be negative at (x_1, y_1).

If, now, we apply the same argument to the corresponding monometric functions along all lines through (x_1, y_1), we find an additional condition. Considering, as above, the line for which $y = y_1 + (x - x_1)\tan\alpha$, we have, as in Art. 88,

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \tan\alpha\,\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) + \tan\alpha\left\{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) + \tan\alpha\,\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\right\}$$

$$= \frac{\partial^2 f}{\partial x^2} + 2\tan\alpha\,\frac{\partial^2 f}{\partial x\,\partial y} + \tan^2\alpha\,\frac{\partial^2 f}{\partial y^2}, \qquad (9)$$

and again excluding the critical case, this function must be negative at (x_1, y_1) for all values of $\tan\alpha$.

According to the theory of quadratic expressions, this gives the additional condition that $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2$ must be positive. Hence, with the exclusion of the critical case, we arrive at the second derivative condition for a maximum,

$$\frac{\partial^2 f}{\partial x^2} < 0, \quad \frac{\partial^2 f}{\partial y^2} < 0, \quad \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 > 0. \qquad (10)$$

Similarly, for a minimum we arrive at the conditions

$$\frac{\partial^2 f}{\partial x^2} > 0, \quad \frac{\partial^2 f}{\partial y^2} > 0, \quad \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 > 0. \qquad (11)$$

In either case the conditions stated are *sufficient* for a maximum (minimum) at (x_1, y_1), provided that the second derivatives are continuous in a neighbourhood of that point. The proof is not completed here.

[1] When it does occur, further discussion, for which we are not prepared, is required.

Ex. 2. *The function* $x^2y^2-4x^2-6xy-4y^2$. We have

$$\left.\begin{aligned}&\frac{\partial f}{\partial x}=2xy^2-8x-6y,\quad \frac{\partial f}{\partial y}=2x^2y-6x-8y,\\&\frac{\partial^2 f}{\partial x^2}=2y^2-8,\quad \frac{\partial^2 f}{\partial x\,\partial y}=4xy-6,\quad \frac{\partial^2 f}{\partial y^2}=2x^2-8.\end{aligned}\right\}\quad\text{.............(12)}$$

The stationary conditions $xy^2-4x-3y=0$, $x^2y-3x-4y=0$ give, on subtracting,

$$xy(y-x)+(y-x)=0,\quad\text{..............................(13)}$$

from which we find $y=x$ or $y=-1/x$. These lead to the values

$$x=0,\ y=0;\quad x=\pm\sqrt{7},\ y=\pm\sqrt{7};\quad x=\pm 1,\ y=\mp 1.$$

(i) When $x=0=y$ we have $\partial^2 f/\partial x^2=-8$, $\partial^2 f/\partial y^2=-8$, $\partial^2 f/(\partial x\,\partial y)=-6$ and $\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2}-\left(\dfrac{\partial^2 f}{\partial x\,\partial y}\right)^2>0$. The corresponding value of the function is therefore a maximum.

(ii) When $x=\pm\sqrt{7}=y$, $\partial^2 f/\partial x^2=6$, $\partial^2 f/\partial y^2=6$ and $\partial^2 f/(\partial x\,\partial y)=22$. Therefore $\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2}-\left(\dfrac{\partial^2 f}{\partial x\,\partial y}\right)^2<0$. The corresponding value of the function is not an extreme.

(iii) When $x=\pm 1$, $y=\mp 1$, we have $\partial^2 f/\partial x^2=-6$, $\partial^2 f/\partial y^2=-6$, $\partial^2 f/(\partial x\,\partial y)=-10$. The corresponding value of the function is not an extreme.

EXAMPLES XVII

DISTRIBUTION OF ROOTS OF EQUATIONS[1]

1. Investigate the distribution of the roots of the following equations:
(i) $x^3+7x-1=0$; (ii) $4x^3-15x^2+12x+1=0$;
(iii) $x^4+2x^3+7x^2+4x-5=0$. [(i) One; (ii) three; (iii) two roots.]

2. Shew that the equation $x^4+4px^3+q=0$, $(p, q>0)$, has two real roots or none according as $27p^4\gtrless q$.

3. Prove that the equation $(x+1)^5=m(x^5+1)$ has three real roots only if $0<m<16$ and one only (equal to -1) if $m<0$ or $m>16$.

4. Prove that if $(n-1)a_1^2-2na_2<0$, the number of real roots of the equation $x^n+a_1x^{n-1}+\ldots+a_n=0$ is less than n.
[The condition makes the roots of the equation $f^{(n-2)}(x)=0$ imaginary.]

5. Shew that the reduction of the equation $f(x)\equiv x^3+3ax^2+3bx+c=0$, $(a, b, c$ real), to the form $\xi^3+\alpha\xi^2+\gamma=0$, ($\alpha, \gamma$ real), by writing $x=\xi+h$, is possible only when the roots of the equation $f'(x)=0$ are real. Verify that in this case $h+a=\pm\sqrt{(a^2-b)}$.

[1] These Examples are intended to be treated as in Art. 95. 1. General theorems for the determination of the number and the distribution of the real roots of an equation (Sturm's Theorem, etc.) lie outside the scope of this Book; see Perron, *Algebra*, Vol. II.

EXAMPLES XVIII

INTERMEDIATE DERIVATIVE THEOREMS

1. Assuming that the fourth derivative $f^{(4)}(x)$ of a function $f(x)$ has the same sign throughout the range $(x-h, x+h)$, shew that

$$|f(x)-\tfrac{2}{3}f(x+\tfrac{1}{2}h)-\tfrac{2}{3}f(x-\tfrac{1}{2}h)+\tfrac{1}{6}f(x+h)+\tfrac{1}{6}f(x-h)|<\tfrac{1}{72}h^4|f^{(4)}(x)|_{max}.$$

2. In the expression $f(a+h)=f(a)+hf'(a)+\frac{1}{2}h^2 f''(a+\theta h)$ of the $I.D_2.T.$, shew that $\theta\to\frac{1}{3}$ as $h\to 0$, assuming that $f'''(x)$ is continuous in the range $(a, a+h)$.

If $f(a+h)=(a+h)^n$, prove that when $|h/a|$ is small, a second approximation to θ is $\dfrac{1}{3}+\dfrac{n-3}{36}\dfrac{h}{a}$.

3. Establish the result $\dfrac{f(h)-f(0)}{h}=\frac{1}{2}\{f'(h)+f'(0)\}-\frac{1}{12}h^2f'''(\theta h)$, $(0<\theta<1)$.

4. Prove that if the first three derivatives of $f(x)$ are continuous in the range $(0, h)$,

$$f(x)-f(0)=\frac{x}{h}\{f(h)-f(0)\}+\tfrac{1}{2}x(x-h)f''(0)+\tfrac{1}{6}x(x^2-h^2)f'''(\theta h),$$

where $0<\theta<1$ and $0<x<h$.

5. A chord of the curve $y=f(x)$ parallel and near to the tangent at P, (ξ, η), meets the curve at the points Q, R, near to P. Prove that the gradient of the line joining P to the middle point of QR is approximately

$$f'(\xi)-3\{f''(\xi)\}^2/f'''(\xi).$$

6. If $f(0)=0$ and $f''(x)>0$ in the positive range $(0, a)$, prove that $f(x)/x$ is up-way in the range. Explain the geometrical significance of this result.

7. Prove that the polynomial $x^3-6x^2+15x+3$ is up-way in any range of x.

8. Investigate the ranges in which the polynomial $2x^3-9x^2+12x-6$ is one-way. [Up-way in the ranges $[-\infty, 1)$, $(2, \infty]$; down-way in $(1, 2)$.]

9. At two points P_1, P_2 of a curve, straight lines are drawn making equal angles α with the corresponding tangents and on the same sides of them. Find the coordinates of their ultimate point of intersection when P_2 moves up to P_1.

[Take axes along the tangent and normal at P_1. The required coordinates are then given by $\xi=\dfrac{\sin\alpha\cos\alpha}{(d^2y/dx^2)_1}$, $\eta=\dfrac{\sin^2\alpha}{(d^2y/dx^2)_1}$.]

10. Through the points of a curve $y=f(x)$ straight lines are drawn, the gradient of the line through P, (x, y), being $\gamma(x)$, a given function of x. If P_1, P_2 are any two points on the curve, find the coordinates of the point of intersection of the lines through P_1, P_2, and the coordinates of the ultimate point of intersection when P_2 moves up to P_1.

[Taking axes as in the preceding Example, $\xi=\left(\dfrac{\gamma}{d\gamma/dx}\right)_1$, $\eta=\left(\dfrac{\gamma^2}{d\gamma/dx}\right)_1$.]

11. A plane curve C is slightly changed by shifting each point of it a small distance which varies in magnitude and direction in a continuous manner along the curve. Shew that a point of C which is shifted in the direction of the tangent to C is a point of ultimate intersection of C with the deformed

curve. What form does the statement take if each point P of C is shifted along the tangent at P?

12. At each point P of the line AB, whose equation is $x=h$, a line $Q'PQ$ is drawn of gradient γ connected with the gradient g of the line OP, drawn from the origin O, by the relation $\gamma=f(g)$. Shew that the ultimate intersection of the line $Q'PQ$ with the corresponding lines from adjacent points on AB has coordinates (x, y) given by $x=h\{1-1/f'(g)\}$, $y=h\{g-f(g)/f'(g)\}$.

Hence shew that if the relation between g and γ is

$$1/\gamma^2-\mu^2/g^2=\mu^2-1,\ (\mu>1,\ \gamma>0),\text{ we have } y=-h(1-1/\mu^2)g^3.$$

[The special relation between g and γ is that for the refraction of light at the boundary AB of a medium whose coefficient of refraction is μ. The formulae determine the focus after refraction of a narrow pencil of light in the plane xOy and emanating from O.]

EXAMPLES XIX

APPROXIMATIONS. SMALL CORRECTIONS

1. Use the approximation (27), Art. 103, to find approximations to the values of $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{11}$. Find an upper barrier to the error in each case.

[If $N=3$ take $a=2$, so that $\sqrt{3}\approx\frac{7}{4}\left(1-\frac{1}{2\times 7^2}\right)$; etc.]

2. Obtain the approximations $\sqrt{\frac{5}{7}}\approx 0{\cdot}8452$, $\sqrt{\frac{3}{8}}\approx 0{\cdot}6125$, $\sqrt{2}\approx 1{\cdot}4142$.

[Write $\sqrt{\frac{5}{7}}=\frac{6}{7}\sqrt{(1-\frac{1}{36})}$, $\sqrt{\frac{3}{8}}=\frac{5}{8}\sqrt{(1-\frac{1}{25})}$, $\sqrt{2}=\sqrt{\frac{288}{144}}=\frac{17}{12}\sqrt{(1-\frac{1}{289})}$, and employ Taylor polynomials of the first degree in each case.]

3. Shew that $(26{\cdot}834)^{1/3}\approx 2{\cdot}99385$ with an error less than $1{\cdot}5\times 10^{-5}$ and that $(27{\cdot}166)^{1/3}\approx 3{\cdot}00615$ with an error less than 2×10^{-5}.

4. Shew that the Taylor polynomial of the second degree for $\sqrt{(x_1+h)}$ is

$$\sqrt{x_1}+\frac{h}{2x_1^{1/2}}-\frac{h^2}{8x_1^{3/2}}.$$

Employ this polynomial to find an approximation to $\sqrt{50}$. Find the error in this approximation, given that $\sqrt{2}=1{\cdot}4142136$.

5. Shew that the Taylor polynomial of the second degree for $(x_1+h)^{1/3}$ is $x_1^{1/3}+\frac{h}{3x_1^{2/3}}-\frac{h^2}{9x_1^{5/3}}$. Use this polynomial to find an approximation to $\sqrt[3]{70}$.

Investigate also the corresponding polynomial for $(x_1+h)^{1/m}$.

6. Calculate 1000/988 correct to six places of decimals by writing it in the form $1\Big/\left(1-\frac{12}{1000}\right)$ and employing the Taylor polynomial of the third degree for $1/(1-x)$. [1·012146.]

7. Shew that the Taylor polynomial of the second degree for the function $\frac{\sqrt{(9+5x)}\,.\,(1-x^2)^{2/3}}{(8-x)^{4/3}}$ is $\frac{3}{16}\left(1+\frac{4}{9}x-\frac{1645}{2592}x^2\right)$.

[Employ the theorems of Art. 129 with Taylor polynomials of the second degree for each of the factors.]

8. The measured value of the radius of a sphere is subject to an error of $p\%$ at most. Find the greatest percentage errors in the calculated values of the volume and surface. [$3p$, $2p$.]

9. Justify the approximation $\dfrac{(1+x)^m(1+y)^n}{(1+z)^p} \approx 1+mx+ny-pz.$

The work that must be done to propel a ship of displacement D for a distance s in time t is proportional to $s^3D^{2/3}/t^2$. Find approximately the percentage increase of work necessary when the distance is increased by 1%, the time diminished by 1% and the displacement diminished by 3%. Compare with a direct calculation employing four-figure logarithms.

10. A formula for the number Q of cubic feet of water passing per minute over a weir of breadth b feet and depth h feet is $Q=3{\cdot}33\,(b-0{\cdot}2h)h^{3/2}$. If b is measured as 8 and h as 1·25, determine the maximum error made in estimating Q due to errors of 1 inch in the measurements of both b and h. [3·9 cu. ft.]

11. The side a of the triangle ABC is calculated from the formula $a^2=b^2+c^2-2bc\cos A$. If there is an error δc in the measured value of c, find the corresponding error δa in the calculated value of a.

If the triangle is nearly equilateral, shew that $\delta a \approx \frac{1}{2}\delta c$.

EXAMPLES XX

EXTREME VALUES OF MONOMETRIC FUNCTIONS

1. Investigate the local extreme values of the following functions:

(i) $x+a^3/x^2$; (ii) $\sum_{r=1}^{n} \lambda_r(x_r-x)^2$, ($\lambda_r$ constants); (iii) $x/(a^2+x^2)$;

(iv) x^n+a^{2n}/x^n; (v) $2x^3-3x^2-36x+10$; (vi) $x\sqrt{(ax-x^2)}$, $(a>0)$;

(vii) $x/(a+x)^n$, $(n>1)$; (viii) x^4+4x^3+5; (ix) $(x+1)^4(x-2)^3$;

(x) x^5-15x^3+3; (xi) $\dfrac{x^2-8}{(x-3)(x-4)}$; (xii) $\dfrac{(x+1)^5}{x^5+1}$;

(xiii) $\dfrac{(x+a)(x+b)}{(x-a)(x-b)}$, $(a, b>0)$; (xiv) $\sqrt{\{x(x-1)(2-x)\}}$.

2. Find the greatest and least values of the polynomial x^3-18x^2+96x in the range (0, 9) of x. [160, 0.]

3. Shew that the difference between the extreme values of the function $(\alpha-x-1/\alpha)(4-3x^2)$, ($\alpha$ constant), is $\frac{4}{9}(\alpha+1/\alpha)^3$. For what values of α is this difference least? [$\alpha=\pm 1$.]

4. The value a is an approximation to a root of the equation $f'(x)=0$. Shew that, in general, $f(a)-\{f'(a)\}^2/f''(a)$ is a closer approximation than $f(a)$ to the corresponding stationary value of $f(x)$.

Hence determine, correct to two places of decimals, the minimum value of $2x^4-58x^2+133x$ which occurs in the neighbourhood of $x=3$. [38·99.]

5. Investigate the extreme values of the function $\dfrac{ax^2+2hx+b}{Ax^2+2Hx+B}$, where a, A are positive. Shew that the only case in which there is no extreme value

occurs when the numerator and denominator have real roots, those of the numerator separating those of the denominator.

If the (real) roots of the numerator are α, β and those of the denominator are α', β', sketch the graphs of the function for the cases (i) $\alpha<\beta<\alpha'<\beta'$, (ii) $\alpha'<\alpha<\beta<\beta'$, (iii) $\alpha<\alpha'<\beta<\beta'$.

If the function is denoted by y and the stationary values y_1, y_2 exist, prove that $(Ax^2+2Hx+B)\,dy/dx=k\sqrt{\{(y_2-y)(y-y_1)\}}$, and find the value of the constant k.

6. Shew that if $0<\beta<\alpha<\frac{1}{2}\pi$, the maximum and minimum values of the function $\dfrac{1+2x\cos\alpha+x^2}{1+2x\cos\beta+x^2}$ are $\dfrac{1-\cos\alpha}{1-\cos\beta}$ and $\dfrac{1+\cos\alpha}{1+\cos\beta}$, respectively.

7. Find the shortest distance of the point (h, k) from the line $ax+by+c=0$.

[Minimize the square of the distance of (h, k) from any point (x, y) of the line.]

8. Find the lines of stationary length drawn from the point $(5a, 2a)$ to the parabola $y^2=4ax$. Shew that there is one extreme length and discuss the results obtained.

[Factorize the equation in y corresponding to the stationary values.]

9. If $ax^2+2hxy+by^2=c$, investigate the extreme values of y.

10. Shew that the lengths r_1, r_2 of the semi-axes of the ellipse

$$ax^2+2hxy+by^2=1 \text{ satisfy the equation } r^4(ab-h^2)-r^2(a+b)+1=0.$$

Hence shew that the area of the ellipse is $\pi/\sqrt{(ab-h^2)}$.

[Find the extremes of $r^2=x^2+y^2$ where $ax^2+2hxy+by^2=1$. Here y is a function of x and the stationary condition gives $x+y\,dy/dx=0$ where $ax+hy+(hx+by)\,dy/dx=0$. Now eliminate dy/dx. The area of an ellipse is π times the product of the semi-axes.]

11. Shew that the areas of ellipses which pass through the points $(p, 0)$, (q, r) and have their centres at the origin, have the minimum value πpr.

12. If a, b are positive and $a+b=k$, shew that the maximum value of $a^m b^n$, $(m, n>1)$, is $m^m n^n k^{m+n}/(m+n)^{m+n}$.

13. The (x, y) equation of a curve being given, shew how to find the points on it which are at an extreme distance from a given line in the same plane.

Shew that the extreme distances of points on the ellipse $x^2/a^2+y^2/b^2=1$ from the chord joining the points $(-a, 0)$, $(0, b)$ are given by

$$\frac{ab}{\sqrt{(a^2+b^2)}}(\sqrt{2}\pm 1).$$

14. A parabola of latus-rectum $4p$ moves so as always to touch two fixed lines at inclination $\operatorname{arccot}\mu$. Shew that the least distance r of a point of contact of the parabola with either line from the intersection of the lines is given by $\lambda^2-(2+\frac{9}{2}\mu^2)\lambda+1=0$, where $(4+\frac{9}{2}\mu^2)\lambda=1+\frac{4}{9}r^2/p^2$.

[If α, β are the distances of the points of contact in any position, the length of the latus-rectum is $4\alpha^2\beta^2(1+\mu^2)^{-1/4}\{(\alpha^2+\beta^2)\sqrt{(1+\mu^2)}+2\alpha\beta\mu\}^{-3/2}$.]

15. Find the proportions of a closed cylindrical can having the greatest volume for a given total area of surface. [Height equal to diameter.]

16. A paddock of given area is to be fenced off along the bank of a straight river, no fence being required along the bank. Find the proportions of the rectangle requiring least fence. [Length twice the breadth.]

17. Two points A, B and a line CD are coplanar, the points being at distances a, b from the line and on the same side of it. If P is a variable point on CD, find the position of P when $AP+BP$ is a minimum.

[When AP, BP make equal angles with the normal to CD at P.]

18. A fixed point A is taken on Ox and OP, PQ are equal variable distances along Oy. Shew that the maximum angle subtended at A by PQ is about $19^\circ\ 28'$. [Find the maximum value of the tangent of the angle.]

19. Find the greatest difference of the angles x, x', each lying in the range $(0, \frac{1}{2}\pi)$, which are connected by the relation $\tan x' = \tan^2 x$.

[The greatest difference corresponds to a maximum value of $\tan(x-x')$. Express this in terms of $\tan x$, or m, and find the extremes with respect to m. The equation for the determination of m is $m^4 - 2m^3 - 2m + 1 = 0$, the roots of which are $0{\cdot}4354$, $2{\cdot}2967$, Art. 118, Ex. 1. For the former value the second derivative of $\tan(x-x')$ with respect to m is negative, which indicates a maximum. This value is $0{\cdot}2232$ radians and it occurs when x has the values $0{\cdot}4106$, $1{\cdot}1601$.]

20. It is required to find the semi-vertical angle of the right circular cone which has a maximum volume for a given area of surface (base included). Shew that if the ratio of the total area of surface to the area of base of a cone is denoted by x, the ratio of the square of the volume to the cube of the total area is $(x-2)/(9\pi x^2)$. Deduce that the required angle is arsin $\frac{1}{3}$.

21. A tree trunk of length l is a frustum of a cone, the radii of the ends being a, b, $(a>b)$. It is required to cut from it a beam of uniform square section, the centre line of the beam being along the axis of the cone. Prove that the beam of greatest volume has the length $\frac{1}{3}al/(a-b)$.

22. A closed container with square ends is of a fixed capacity whose measure is made 1 by choice of the unit of length. The cost of the container is estimated as $2c$ per unit area of surface, plus c per unit length of the seams (one along the length and one around each end). Shew that the cost is a minimum when the side x of the ends satisfies the equation $x^3(x+1) = x + \frac{1}{4}$.

23. The area A and the semi-perimeter s of a triangle are fixed. Prove that for one of the sides x to be an extreme it must be a root of the equation $s(x-s)x^2 + 4A^2 = 0$. Hence show that there is one maximum and one minimum.

24. On a given area S of land a portion $S-x$ is used to produce a material A while the rest is used to obtain a material B, all of which is applied to the production of A. The amount of each material produced is proportional to the area employed, but the amount of A per unit area is proportional to the ratio of two expressions each of which differs from the amount of B applied to it by a constant. Find the proportion of the areas for which the production of A is a maximum.

25. Find the cuboid of greatest volume the total length of whose edges and the total area of whose faces are given.

[Express the volume in terms of the length of one edge.]

26. A container is in the shape of a prism of square section terminated by two equal pyramids on the square ends of the prism. Shew that for a given area a^2 of cross-section of the prism and a given volume V of the container, the area of its surface is least when the height of each pyramid is $a/\sqrt{5}$.

Assuming this proportion and varying a, shew that for a given volume of the container the area of its surface is least when $a^3 = 3V/\sqrt{5}$, and find the height of the prism.

27. The gate in front of a man's house is 30 yards from a tram line. If the man walks at 4 miles per hour and the tram travels at 15 miles per hour, where should he jump off in order to reach the gate as soon as possible?

[Maximize the time saved by jumping off. Result 8·3 yards.]

28. The sides of a wooden trough are each 1 ft. long and are equally inclined to the bottom of the trough, which is 9 in. wide, the ends being vertical. What must be the width across the top for the capacity to be a maximum? [1·84 ft.]

29. Through the point A, (h, k), a straight line PAQ is drawn, intersecting Ox at P and Oy at Q. Find (i) the minimum area of the triangle OPQ, (ii) the minimum value of the sum $OP + OQ$, (iii) the minimum length of PQ.

[(i) $2hk$; (ii) $h + k + 2\sqrt{(hk)}$; (iii) $(h^{2/3} + k^{2/3})^{3/2}$.]

30. A rod PQ of length l passes through a fixed point A, the end P being on a fixed line Ox. Find the position of the rod when the distance of Q from A, measured parallel to Ox, is a maximum.

[The distance of P from A, measured parallel to Ox, is $h\{(l/h)^{2/3} - 1\}^{1/2}$, where h is the distance of A from Ox.]

31. Two variable radii OA, OB are drawn to a curve. The tangent at A intersects the line OB (produced) at C and M is the foot of the perpendicular from C on OA. If BN is the foot of the perpendicular from B on OA, shew that $NB/(OA)^m$ is stationary if the tangent at B is parallel to OA and $OM = mMA$.

32. Two points A, B are taken in the plane of a curve $\mathcal{C}$. Shew that if P is a point on $\mathcal{C}$ for which the sum of the lengths AP, PB is stationary, the lines AP, PB are equally inclined to the tangent at P.

33. If ABC is a triangle, no angle of which exceeds 120°, find the position of a point P for which the sum of the distances PA, PB, PC is a minimum.

[Suppose, at first, that P lies on a circle of centre A, and minimize the sum $PB + PC$. Then, according to the preceding Example, PB, PC must be equally inclined to AP. It follows that for the unrestricted case each of the angles APB, BPC, CPA is $\frac{2}{3}\pi$. The minimum value of the sum is $\sqrt{\{\frac{1}{2}(a^2 + b^2 + c^2) - 2\sqrt{3}S\}}$, where a, b, c are the sides of the triangle and S is its area.]

34. A straight line Ox and two points A, B lie in one plane. If P is a variable point in the plane and PM is always perpendicular to Ox, find the minimum value of the sum $PA + PB + PM$.

35. A tug leaves port to intercept a liner travelling at k knots on a straight course which, at the nearest point, is p miles from the port. The tug starts when the liner is a miles from the port and has not yet reached the nearest point. Shew that the tug must have a uniform speed of at least kp/a knots if it is to reach the liner and that its course at that speed should be perpendicular to the bearing of the liner at the start.

EXAMPLES XXI

CONCAVITY AND CONVEXITY. CONTACT. ROOTS OF EQUATIONS. INTERPOLATION

1. Find the ranges of x for which the curve $y=2x^4-9x^3+11x^2$ is concave upwards. $\left[x<\frac{27-\sqrt{201}}{24},\quad x>\frac{27+\sqrt{201}}{24}.\right]$

2. Investigate the nature of the following graphs in the neighbourhood of the point for which $x=1$:

(i) $3x^4-18x^2+23x-7$, (ii) $x(x-1)^2$, (iii) $\frac{1}{2}x^5-x^3-2x^2+x+1$.

3. Determine the points of contraflexure of the graphs of the functions given in Ex. 1, Set XX, and sketch characteristic portions of the graphs.

4. Find the condition that the curve $x=f(t)$, $y=\varphi(t)$ may have upward concavity at the point t.

5. Determine the point of contraflexure and sketch the graph of the function given by the equation $\frac{x}{x-y}=\frac{1}{\sqrt{(x+y)}}$. $[x=-9]$.

6. Shew that the graph $y=x/(x^2+bx+c)$ has three points of contraflexure if $b^2<4c$. Shew that those of the graph $y=2x/(x^2+2x+2)$ lie on a straight line.

7. A curve is defined by an equation of the form

$$y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5.$$

It passes through $(1, 0)$, touches Ox at $(2, 0)$ and has a point of contraflexure with unit gradient at O. Shew that its equation is

$$y=-\tfrac{1}{4}(x-2)^2(x-1)x(2x+1).$$

Find the other points of contraflexure and sketch the curve.

8. Find the polynomial curve of the form $y=a_0+a_1x+\ldots+a_nx^n$ having the highest possible order of contact at the origin with the curve $y=f(x)$ which touches Ox there. (The function $f(x)$ is assumed to have derivatives of any desired order.) $[a_r=f^{(r)}(0)/r!.]$

9. Shew that the parabola $bx^2-2a^2(b-y)=0$ has third order contact with the ellipse $x^2/a^2+y^2/b^2=1$ at the point $(0, b)$.

10. The equation of a curve is of the nth degree in x, y, and of the form

$$a_0+a_1x+b_1y+a_2x^2+b_2xy+c_2y^2+\ldots=0.$$

Find the conditions that must be satisfied by the coefficients in order that the curve may have second order contact at the origin with a circle whose centre is at $(-1, 1)$. $[a_0=0,\ a_1+b_1=0,\ a_1-a_2-b_2-c_2=0.]$

11. Two curves $y=\varphi(x)$, $y=f(x)$ meet at the point P_1. A line $P'P''$, of slope θ to Ox, cuts the curves at P', P'', respectively, this slope being different from the slope ψ_1 of the curve $y=f(x)$ at P_1. If P', P'' have coordinates (ξ, η), (x, y), respectively, use the method of infinitesimals to shew that when P', P'' are close to P_1,

$$P'P''\approx\frac{f(\xi)-\varphi(\xi)}{\sin\theta-f^{(1)}(x_1)\cos\theta}.$$

Find the least distance of P' from the curve $y=f(x)$ and shew that the

direction of the shortest distance is approximately normal to the curve $y=f(x)$ at P_1.

12. Shew that the equation $x^3-2x^2+3x-4=0$ has only one real root, near 1·7. Shew that two successive applications of Newton's method leads to the approximation 1·650631 to the root, and that the error of this approximation is less than $1{\cdot}23\times10^{-6}$.

13. Shew that 0·85 is a rough approximation to the positive root of the equation $x^3(x+1)=x+\frac{1}{4}$ and employ Newton's method to find the root correct to four decimal places. [0·8398.]

14. Interpolation by the method of proportional parts is made in the interval $(a, a+h)$. Shew that if $|f''(x)|$ takes its greatest value at the middle of the interval the greatest error allowed by the formula (11), Art. 122, may be made, but that for the function $f(x)=x^4$ with interpolation between $x=0$ and $x=h$ the greatest error is, in fact,

$$\frac{h^2\,|f''(x)|_{max}}{16\times4^{1/3}}.$$

15. Given

$$\sqrt{1246}=35{\cdot}29872\,51909,\ \sqrt{1247}=35{\cdot}31288\,71660,\ \sqrt{1248}=35{\cdot}32704\,34655,$$

find, by interpolation with the aid of second differences, the value of $\sqrt{1246{\cdot}54}$ to ten decimal places, and estimate the uncertainty of the result obtained.

[35·30637 33624 ; $4{\cdot}5\times10^{-10}$. The result correct to ten decimal places is 35·30637 33623.]

EXAMPLES XXII

BINOMIAL EXPANSIONS

1. Investigate the Binomial Series for the following functions, stating the ranges of convergence :

(i) $1/\sqrt{(1+x^2)}$; (ii) $1/(1+x)$; (iii) $1/(1-ax)$, $(a>0)$; (iv) $1/(1+x^2)$;
(v) $(1+x)^{1/n}$; (vi) $(1+x)^{-1/n}$; (vii) $(1-x)^{2/3}$, (viii) $(1+x)^{1/3}$;
(ix) $1/\sqrt{(1-x)}$; (x) $\sqrt{(1-2x)}$.

2. Calculate $\sqrt{600}$ and $1/\sqrt{600}$ correct to six decimal places by means of the Binomial Theorem.

3. Find the Taylor polynomial of the second degree for $\sqrt{(1+x)}$ by the method of disposable coefficients.

[Assume $\sqrt{(1+x)}=1+ax+bx^2$, square both sides and ignore terms involving powers of x above the second. Then equate coefficients.]

4. If $N=x^n(1+\varepsilon)$, (ε small compared with 1), shew that

$$x_1=x\frac{(n-1)x^n+(n+1)N}{(n+1)x^n+(n-1)N}$$

is an approximation to $N^{1/n}$ correct to the second order in ε.

Taking $x=1{\cdot}4$, find an approximation to $\sqrt{2}$ in this way.

[The expression for x_1 reduces to $x\dfrac{1+\frac{1}{2}\varepsilon(n+1)/n}{1+\frac{1}{2}\varepsilon(n-1)/n}$. Using the Taylor polynomial of the second degree for the denominator, shew that this expression

takes the same form, to the second order, as that obtained by expanding $x(1+\varepsilon)^{1/n}$.]

5. If y is expressed as a power series in x of the form

$$y = x - b_1x^2 - b_2x^3 - b_3x^4 - \dots,$$

shew how to express x as a power series in y of the form

$$x = y + c_1y^2 + c_2y^3 + c_3y^4 + \dots,$$

where
$$c_1 = b_1,\quad c_2 = b_2 + 2b_1^2,\quad c_3 = b_3 + 5b_1b_2 + 5b_1^3,$$
$$c_4 = b_4 + 6b_1b_3 + 3b_2^2 + 21b_1^2b_2 + 14b_1^4.$$

6. If the length l of a metal bar at temperature θ is given by $l = a + b\theta + c\theta^2$, find the coefficient of expansion at this temperature. Shew that to the second degree in θ, this is equal to $\frac{b}{a}\left\{1 + \left(\frac{2c}{b} - \frac{b}{a}\right)\theta + \left(\frac{b^2}{a^2} - \frac{3c}{a}\right)\theta^2\right\}$.

EXAMPLES XXIII

EXTREMES OF DIMETRIC FUNCTIONS

1. Shew that the least value of the sum of the squares of the perpendiculars from a point inside a triangle to the three sides is $\frac{1}{4}(a^2+b^2+c^2) - \dfrac{a^4+b^4+c^4}{2(a^2+b^2+c^2)}$, where a, b, c are the lengths of the sides.

2. Find the least distance of the point (x_1, y_1, z_1) from the plane $ax + by + cz + d = 0$, assuming that the coordinates are such that the square of the distance between the points (x_1, y_1, z_1), (x_2, y_2, z_2) is

$$\Sigma k_{11}(x_2 - x_1)^2 + 2\Sigma k_{23}(y_2 - y_1)(z_2 - z_1).$$

Shew that the result takes the form $(ax_1 + by_1 + cz_1 + d)/\sqrt{(a^2+b^2+c^2)}$, when $k_{rr} = 1$, $k_{rs} = 0$.

3. A water channel having a flat bottom and flat sloping sides is to be made so that for a given capacity the surface wetted shall be as small as possible. Shew that if α is the area of the surface wetted per unit length the required capacity per unit length is $\alpha^2/(4\sqrt{3})$.

4. A plane makes positive intercepts on the axes Ox, Oy, Oz. A point P, whose coordinates x, y, z are all positive, lies on the plane and through P planes parallel to the coordinate planes are drawn. Find the position of P so that the cuboid formed has the greatest possible area of surface.

[If the plane is $lx + my + nz = 1$, the required coordinates are in the ratios $m + n - l : n + l - m : l + m - n$, provided these numbers are positive.]

5. Find the position of a straight line in the plane xOy such that the sum of the squares of the distances of the points $(0, 0)$, $(a, 0)$, $(0, a)$ from it is a minimum.

[It makes equal angles with the axes and is at the distance $\frac{1}{3}a\sqrt{2}$ from the origin.]

6. Prove that the point P within a triangle, such that the sum of the squares of the distances from P to the vertices is a minimum, is the centroid. Shew also that if P is restricted to lie on a given circle, it will lie on the line joining the centre of the circle to the centroid.

7. If $z=(p/x)^k+(x/y)^k+(y/q)^k$, prove that z has a stationary value when p, x, y, q are in a geometric progression.

8. Straight lines are drawn from the origin to points on the surface $x^2+7y^2+4z^2+8xy=c^2$. Shew that there are two stationary lengths, namely $\frac{1}{2}c$, $\frac{1}{3}c$.

9. A tent has a rectangular floor of length y and breadth $2x$, and the shape of each end is a rectangle of width $2x$ surmounted by an isosceles triangle of vertical angle 2θ, the mean height of the tent being u. Find the volume V and the area S of canvas, and shew that

$$S=\frac{V}{x}+\frac{2-\cos\theta}{2u\sin\theta}V+4xu.$$

Find the value of θ and the ratios $x:y:u$ for the tent using least material for a given volume.

10. Shew that the distance of a point P of the surface $2z=x^2/a+y^2/b$, $(a, b>0)$, from the point $(0, 0, c)$, $(c>0)$, has a minimum value when P is at the origin, provided that a, b are each greater than c.

11. Shew that the distance between two points, one on each of two given curves, is stationary if the join of the points is normal to both curves at the ends.

12. Find the stationary point for the function

$$x^2-4xy+6y^2+3z^2-zx-18y+3. \qquad [(12,\ 5{\cdot}5,\ 2).]$$

13. Find the stationary values of z, where

$$2x^2+4y^2+z^2+8yz+6zx+6xy+12=0. \qquad [\pm 2.]$$

14. Find the least value of the sum $\sum_{r=1}^{n}(x_r-x)^2+\sum_{r=1}^{n}(y_r-y)^2+\sum_{r=1}^{n}(z_r-z)^2$, where $(x_1, y_1, z_1), (x_2, y_2, z_2), \dots, (x_n, y_n, z_n)$ are the coordinates of n points each of which lies on the plane $ax+by+cz+d=0$.

CHAPTER V

CURVATURE OF A PLANE CURVE. CARTESIAN COORDINATES

131. Introduction and Definition of Curvature.

In ordinary language the word *curvature*, as associated with a line, is merely an equivalent of *non-straightness*. In Mathematics we are concerned with curvature as a measurable quantity, that is, with the introduction of *measures of curvature*.[1] We consider in the present Chapter the *measure of curvature at a point* of a plane curve. The measure of curvature of any finite portion of a curve is considered later, Chap. XII, Art. 284. 2.

The notion of greater or less curvature at a point of a curve is commonly associated either with greater or less rapidity of recession (or deflection) from the tangent at the point, or with greater or less rapidity of change of direction of the curve (rotation of the tangent) as the curve is continuously described. In the present Chapter the curvature at a point is defined by a consideration of the recession from the tangent of arcs having one end at the point. The treatment is based on the formula for the deviation of a curve from its tangent which has been investigated in Art. 103. 2 and which we now proceed to put into a form suitable for the present purpose.

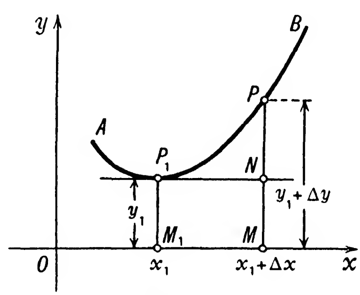

FIG. 95.

Considering a curve AP_1B (Fig. 95), suppose that rectangular axes Ox, Oy are chosen so that Ox is parallel to (or coincides with) the tangent at P_1, the point at which the curvature is to be considered. Let the equation of the curve referred to these axes be $y=f(x)$. The coordinates of P_1 are (x_1, y_1) and those of P, a neighbouring point on the curve, $(x_1+\Delta x, y_1+\Delta y)$. The

[1] As in all similar cases, the word *curvature* is constantly used alone for *measure of curvature* where no misunderstanding can arise.

tangent P_1N at P_1 meets the ordinate MP to P in N. We have $f'(x_1)=0$, and, from the $I.D_2.T.$, supposed applicable to the range $(x_1, x_1+\Delta x)$, we deduce the result

$$NP=f(x_1+\Delta x)-f(x_1)=\tfrac{1}{2}(\Delta x)^2 f''(x_1+\theta\Delta x), \quad (0<\theta<1). \quad \text{....(1)}$$

If, now, we suppose that $f''(x)$ is continuous at x_1 and take for Δx an increment δx so small that the variation of $f''(x)$ in the range $(x_1, x_1+\theta\delta x)$ can be neglected, the formula (1) gives [1]

$$NP\approx\tfrac{1}{2}(\delta x)^2 f''(x_1), \quad \text{.........................(2)}$$

the corresponding exact formula being

$$\lim_{\Delta x\to 0}\frac{2NP}{(\Delta x)^2}=f''(x_1). \quad \text{.........................(3)}$$

Thus for the case considered, the deflection of the curve from the tangent at P_1 for a given small advance $|\delta x|$ in either direction along the tangent, is proportional to the value of the second derivative of $f(x)$ at P_1.

Now the *measure of the curvature* is chosen so that it is greater or less according as the deflection from the tangent is greater or less for a given small advance along the tangent. Hence we may, in the present case, take the measure of the curvature as *proportional* to the magnitude of the second derivative at P_1. In practice, the factor of proportionality is taken to be unity, so that the measure of the curvature is actually equal to the magnitude of $f''(x_1)$.

The definition of curvature as an algebraic quantity is completed by taking into account the *sign* of the deflection. For the case considered, the curvature is taken positive or negative according as the deflection NP is positive or negative with respect to the direction Oy. The algebraic curvature is therefore measured by the algebraic value of $f''(x)$ at P_1. Denoting this measure of curvature by the letter κ, we have at P_1,

$$\kappa_1=f''(x_1), \quad \text{..............................(4)}$$

and κ is therefore positive or negative according as the curve is concave or convex upwards at P_1.

Since $f''(x)=dg/dx$, where g is the gradient dy/dx, we can write (4) in the equivalent form

$$\kappa_1=(dg/dx)_1, \quad \text{..............................(5)}$$

which expresses the curvature at P_1 as the value there of the rate

[1] If P_1 is taken as the origin of coordinates, the axes being along the tangent and normal, the approximation becomes $NP\approx\frac{1}{2}(\delta x)^2 f''(0)$.

of increase of the gradient with respect to x. This applies to any point of a curve when the axes of coordinates are chosen in the particular manner described above.

We are now prepared to give a formal definition of the measure of the curvature at any point of a curve. *The curvature of a curve at any point is measured by the rate of increase there of the gradient of the curve with respect to distance along the tangent at the point.*

It is sometimes necessary to consider one-sided curvatures at a point P of a curve. A one-sided curvature is defined by means of the change of gradient or the deflection from the tangent in proceeding along the part of the curve on one side only of P. If P is the end point of the curve or a branch of the curve, the one-sided curvature is called an *end-curvature*. In the formulae for one-sided curvature, one-sided derivatives take the place of ordinary derivatives.

Ex. 1. *Curvature at the vertex of a parabola.* Taking the axes along and perpendicular to the tangent at the vertex, the equation of the parabola may be put into the form $y=ax^2$. We then have $d^2y/dx^2=2a$, and hence at the vertex, where $x=0$, we have

$$\kappa_0=2a. \quad\text{..........}(6)$$

Ex. 2. *Curvature at any point of a circle.* Taking the axes Ox, Oy along the tangent and normal at P_1 (Fig. 96), the equation of the circle is $x^2+(y-a)^2=a^2$, or

$$x^2+y^2-2ay=0, \quad\text{..........}(7)$$

where a is the radius. On differentiating this equation we find

$$x+(y-a)\frac{dy}{dx}=0, \quad 1+\left(\frac{dy}{dx}\right)^2+(y-a)\frac{d^2y}{dx^2}=0, \quad\text{..........}(8)$$

and when $x=0$, $y=0$, these give $(dy/dx)_0=0$, $(d^2y/dx^2)_0=1/a$. Hence

$$\kappa_0=1/a. \quad\text{..........}(9)$$

This formula is evidently true for all points on the circle, since equation (7) is the same wherever P_1 is chosen on it. The curvature of a circle is thus equal to the reciprocal of its radius when the axes are chosen so that Oy is drawn into the circle, as here. If Oy is drawn outwards from the circle, $\kappa=-1/a$. In either case the magnitude of the curvature is equal to $1/a$. This property of a circle is utilized subsequently in defining the *radius of curvature* of a curve, (Art. 134).

132. Newton's Formula for the Curvature of a Curve.

The equation

$$\lim_{\Delta x\to 0}\frac{2NP}{(\Delta x)^2}=f''(x_1) \quad\text{..........}(1)$$

of the preceding Article can now be written

$$\kappa_1=\lim_{P_1N\to 0}\frac{2NP}{(P_1N)^2}, \quad\text{..........}(2)$$

and this expresses the curvature in a form substantially due to Newton. This form is plainly applicable to the curvature at any point of a curve since it involves only distances along and perpendicular to the tangent at the point considered.

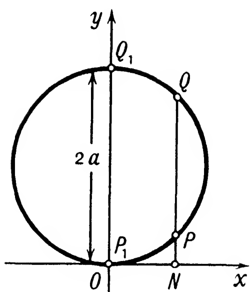

FIG. 96.

In the principles of Dynamics of Huyghens and Newton the curvature of a curve plays an essential part. The force required to hold a particle of mass m moving with velocity v to a curved path is $m\kappa v^2$ at right angles to the path, κ being the curvature at the point considered.

Ex. 1. *Curvature at any point of a circle.* Referring to Fig. 96, we have, from elementary Geometry,

$$P_1N^2 = NP \, . \, NQ, \quad \text{.................(3)}$$

Q being the point where NP, produced, meets the circle again. Newton's formula now gives

$$\kappa_1 = \lim_{P_1N \to 0} \frac{2NP}{(P_1N)^2} = \lim_{P_1N \to 0} \frac{2}{NQ} = \frac{2}{P_1Q_1} = \frac{1}{a}, \quad \text{.................(4)}$$

where P_1Q_1, the diameter through P, is of length $2a$.

133. Formula for the Curvature of a Curve Referred to any Rectangular Axes.

In the preceding discussion the formula for the curvature has been given with respect to special axes. We now proceed to express the curvature in terms of coordinates referred to general rectangular axes.

Let the equation of the curve AB (Fig. 97) referred to any rectangular axes Ox, Oy be $y=f(x)$, and suppose that at P_1, (x_1, y_1), the point at which the curvature is to be considered, the gradient g_1, or $\tan\psi_1$, is positive and finite, so that the corresponding slope ψ_1 lies in the range $[0, \frac{1}{2}\pi]$. Through P_1 draw axes P_1x', P_1y', the former along the tangent to the curve at P_1, as shewn, and the latter along the normal, being obtained from the former by rotation in a positive (counter-clockwise) sense through a right-angle.

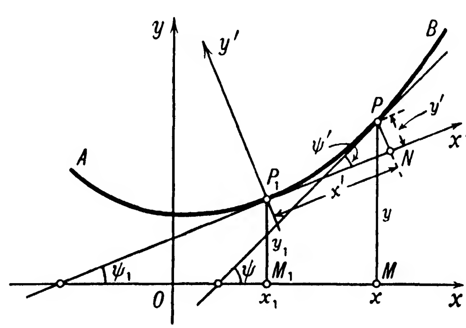

FIG. 97.

Let P, a neighbouring point on the curve, have coordinates (x, y), (x', y') in the respective coordinate systems, the slope of the tangent

there being ψ relative to Ox and ψ' relative to P_1x'. If g, g' are the corresponding gradients, we have

$$\tan\psi' = \tan(\psi - \psi_1) = (\tan\psi - \tan\psi_1)/(1+\tan\psi\tan\psi_1), \quad \text{.....(1)}$$

and thus

$$g' = \frac{g-g_1}{1+gg_1}. \quad \text{..............................(2)}$$

Now according to the general definition of curvature given in Art. 131, the curvature at P_1 is measured by the value of dg'/dx' there, that is,

$$\kappa_1 = (dg'/dx')_{x'=0}. \quad \text{..............................(3)}$$

We can write this

$$\kappa_1 = \left(\frac{dg'}{dx}\frac{dx}{dx'}\right)_{x=x_1} = \left(\frac{dg'}{dx}\right)_{x_1}\left(\frac{dx}{dx'}\right)_{x_1}. \quad \text{..................(4)}$$

From (2) we have

$$\left(\frac{dg'}{dx}\right)_{x_1} = \left\{\frac{1}{1+gg_1}\frac{d}{dx}(g-g_1) + (g-g_1)\frac{d}{dx}\frac{1}{1+gg_1}\right\}_{x_1}$$

$$= \frac{(dg/dx)_{x_1}}{1+g_1^2}, \quad \text{..(5)}$$

since $g = g_1$ when $x = x_1$. Now from Ex. 8, Art. 71, p. 133, we have, for the particular case here considered,

$$\left.\begin{array}{l} x' = (x-x_1)\cos\psi_1 + (y-y_1)\sin\psi_1, \\ (dx'/dx)_{x_1} = \sec\psi_1 = \sqrt{(1+g_1^2)}, \end{array}\right\} \quad \text{................(6)}$$

so that

$$\left(\frac{dx}{dx'}\right)_{x_1} = \frac{1}{\sqrt{(1+g_1^2)}}. \quad \text{..........................(7)}$$

Hence we may write (4) in the form

$$\kappa_1 = \frac{(dg/dx)_{x_1}}{(1+g_1^2)^{3/2}}, \quad \text{..............................(8)}$$

which gives the algebraic curvature at P_1 under the convention of sign introduced in Art. 131 for the special axes P_1x', P_1y'.

On writing dy/dx for g and dropping the indication of the special point P_1, this formula becomes

$$\kappa = \frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}. \quad \text{..........................(9)}$$

The magnitude of the curvature at *any*[1] point P on the curve is obtained from the same formula. The question of the sign to be

[1] Provided the gradient there is finite. The case of an infinite gradient, that is, where the tangent at P is parallel to Oy, is considered in Sub-article 3 below.

attributed to the algebraic curvature at any point is considered in the following Sub-article.

It is desirable to apply a dimensional check to an equation such as (9). Suppose that the unit of length is changed to $1/k$ of itself. Then the measures x, y, x', y' are changed to kx, ky, kx', ky' and are all of the same dimensions. By definition, κ is equal to d^2y'/dx'^2 and is of dimensions -1 and dy/dx is unaltered, being of zero dimensions. Thus $1+(dy/dx)^2$ is of zero dimensions and the whole formula is of dimensions -1, that is, of the dimensions of κ. The equation (9) is therefore dimensionally correct.[1]

It may be observed here that since the curvature vanishes with the second derivative, a point of contraflexure at which there is a definite zero value of the second derivative is also a point of zero curvature.

A formula corresponding to (9) for the curvature of a curve referred to oblique coordinates is proved in Art. 137 below.

If, in the preceding discussion, we interchange (the rôles of) the axes Ox, Oy, we get the formula

$$\frac{\dfrac{d^2x}{dy^2}}{\left\{1+\left(\dfrac{dx}{dy}\right)^2\right\}^{3/2}} \quad \text{.................................(10)}$$

for the curvature, apart from sign.

133. 1. *Different conventions for the sign of the curvature at a point.* There are two main conventions for the determination of the sign of the curvature at a point of a curve. These, as we shall see, have their origin in two different modes of approach to the subject. In the first, regard is paid to the concavity or convexity relative to the fixed direction Oy, as explained in Art. 112. According to that Article, a curve is concave (convex) with respect to the direction Oy, that is, is concave (convex) 'upwards', if d^2y/dx^2 is positive (negative) at the point considered. Under the first convention, the sign of the curvature is taken positive (negative) if the curve is concave (convex) upwards, so that κ has the same sign as d^2y/dx^2, and is therefore given by

$$\kappa=\frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}} \quad \text{........................(11)}$$

$$=\frac{dg/dx}{(1+g^2)^{3/2}}, \quad \text{...............................(12)}$$

[1] It may be well to remark that (9) is given as a general formula which is true whatever unit of length is chosen. If, then, κ changes in any ratio on a change of unit, the right-hand side must change in the same ratio if the formula is correct. The dimensional correctness, of course, forms only a partial check.

the denominator being positive in all cases. The student should carefully observe that according to this convention, the curvature of a circle changes sign at the points where the tangent is parallel to Oy, and the same is true generally of any curve at such a point if it is not a point of contraflexure. The discontinuity in the algebraic curvature arising in this manner is considered below.

In introducing the second convention, we suppose a curve to be described by the continuous motion of a point on it in a definite sense. The sign of the curvature at any point P is then determined relative to rectangular axes Px', Py', the axes Px' being taken along the tangent and in the sense of description of the curve at P, while the sense of Py' is obtained from that of Px' by positive rotation through a right-angle. The convention for the sign of the curvature at P relative to these axes is now the same as that introduced in Art. 131. Thus the curvature is to be positive when the deflection of the curve in the neighbourhood of P from the tangent there is in the sense of Py'. In the case of a circle, the curvature now has a constant sign throughout, being positive if the axis Py' is drawn towards the centre.

Figure 98 illustrates the case where the two conventions lead to opposite signs for the curvature at P, but if, as in Fig. 99, the sense of description of the arc is reversed, the two conventions give the same sign for the curvature.

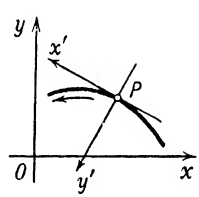

FIG. 98.

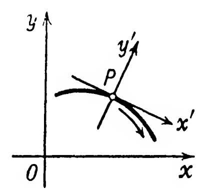

FIG. 99.

In the following general discussion the first only of these conventions is adopted. The second is considered subsequently; see Chap. VIII, Art. 204, Chap. XII, Art. 284. Some of the transformations employed in the present Chapter will take simpler forms when the differentiation of the trigonometrical functions is discussed; see Chap. VI, Art. 144.

Ex. 1. Prove that the curvature may be expressed as an x-rate in the form

$$\kappa = \frac{d}{dx}\left\{\frac{g}{(1+g^2)^{1/2}}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

an equivalent form being

$$\kappa = \frac{d}{dx}\left(\frac{y}{t}\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

where t is the length of the tangent from the point of contact to Ox.

133. 2. *Case of small gradient throughout.* **When the gradient of a curve in a range is so small that its square is negligible compared with 1, we may employ the approximation $\kappa \approx d^2y/dx^2$ throughout.**

An important example of this occurs in the theory of the bending of beams, which, in constructional work, must remain very nearly straight when carrying their loads. According to the usual theory, the curvature of the axis of the beam at any point is proportional to the moment, about the point, of the forces on the part of the beam on either side of it. We suppose that the direction of the axis of x is chosen to make so small an angle with the tangent to the axis of the beam at any (every) point of its length that we can neglect $(dy/dx)^2$ compared with 1 in the formula for the curvature. If, then, $M(x)$ denotes the moment of the forces about the point x in the sense of rotation from Ox towards Oy, we have

$$K\frac{d^2y}{dx^2}=M(x), \quad \text{.......(15)}$$

where K is the factor of proportionality of moment to curvature and is independent of x if, as we assume, the beam is of uniform section.

We now suppose that the forces acting are (or can be treated as) perpendicular to Ox, and consider two special cases.

(i) For a part of the beam along which no external forces (loads or reactions at supports) are applied, the moment $M(x)$ is linear in x, so that $M(x)=Bx+C$, where B, C are constants. Thus

$$K\frac{d^2y}{dx^2}=Bx+C, \quad \text{.......(16)}$$

and therefore, (Ex. 3, Art. 98. 3),

$$K\frac{dy}{dx}=\tfrac{1}{2}Bx^2+Cx+D, \quad \text{.......(17)}$$

$$Ky=\tfrac{1}{6}Bx^3+\tfrac{1}{2}Cx^2+Dx+E, \quad \text{.......(18)}$$

where D, E are constants. Thus the axis of this part of the beam is (approximately) bent to a cubical parabola.

(ii) For a part of the beam on which the force is a uniform load per unit length, the moment $M(x)$ is quadratic in x, so that $M(x)=Ax^2+Bx+C$, where A, B, C are constants. Thus

$$K\frac{d^2y}{dx^2}=Ax^2+Bx+C, \quad \text{.......(19)}$$

and therefore

$$K\frac{dy}{dx}=\tfrac{1}{3}Ax^3+\tfrac{1}{2}Bx^2+Cx+D, \quad \text{.......(20)}$$

$$Ky=\tfrac{1}{12}Ax^4+\tfrac{1}{6}Bx^3+\tfrac{1}{2}Cx^2+Dx+E. \quad \text{.......(21)}$$

Thus the axis of this part of the beam is (approximately) bent to a quartic parabola.

When the whole beam consists of parts for each of which either the condition (i) or the condition (ii) is satisfied, the values of the constants A, B, ... for the successive parts must be such that the deflection y and the gradient dy/dx are continuous.

Ex. 2. A uniform beam, of length l, is supported (without constraint of direction) at its ends and carries a single load W at a point at distance a from one end, which is taken as origin. Shew that the deflection from the line of supports due to the load W is given by

$$12lKy = Wal(l^2 - a^2) + Wxa(2a - l)(a - l) \\ + W\{l\,|x - a|^3 - (l - a)x^3 - a(l - x)^3\}, \quad \text{.............(22)}$$

y being measured downwards.

[Here the forces of support at the ends are $W(l - a)/l$, Wa/l. The moment of the forces about the point x is $-Wx(l - a)/l$ if $x < a$ and $-W(l - x)a/l$ if $x > a$].

133. 3. *Case of infinite gradient.* If the tangent at P_1 (Fig. 100) is parallel to Oy, so that g becomes infinite there, the general formula (11) or (12) for κ is not available for the determination of the curvature at P_1 except by the application of a limit-process. Considering one side of P_1 we have

$$\frac{dg/dx}{(1+g^2)^{3/2}} = \pm\frac{dg'}{dx'}. \quad \text{..........(23)}$$

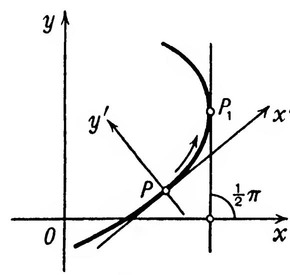

FIG. 100.

Suppose that the positive sign applies, as at P in the Figure. Then if dg'/dx' varies continuously and has the finite value κ_1 at P_1, the expression $\dfrac{dg/dx}{(1+g^2)^{3/2}}$, being always equal to dg'/dx' on the side of P_1 considered, must have the finite limit κ_1 as P moves up to P_1. Similarly, if the negative sign in (23) applies on the side of P_1 considered, the expression on the left-hand side converges to $-\kappa_1$ as P moves up to P_1.

If in this case we employ y as independent variable, the curvature at P_1, as defined with respect to the fixed direction Ox, is given by

$$\kappa_1 = (d^2x/dy^2)_1. \quad \text{..........................(24)}$$

The curvature is now positive (negative) if the curve is concave (convex) with respect to the direction Ox.

Ex. 3. *The circle* $x^2 + y^2 = a^2$. Here we have $dy/dx = -x/y$, $d^2y/dx^2 = -a^2/y^3$, and hence, provided that $|x| \neq a$,

$$\kappa = \frac{d^2y/dx^2}{\{1+(dy/dx)^2\}^{3/2}} = -\frac{a^2/y^3}{(1+x^2/y^2)^{3/2}} = -\frac{a^2/y^3}{(a^2/y^2)^{3/2}} = \mp\frac{1}{a}, \quad \text{........(25)}$$

the upper (lower) sign being taken if y is positive (negative).

Since for $y > 0$ we have

$$\frac{dg/dx}{(1+g^2)^{3/2}} = -\frac{1}{a}, \quad \text{..............................(26)}$$

the value to which the left-hand side converges when $y \to 0$ is $-1/a$, and this is therefore the curvature at the points $x = \pm a$, $y = 0$, regarded as approached from above. In the same way we prove that the curvature at these points, regarded as approached from below, is $1/a$.

Ex. 4. *The parabola.* (i) *Axis parallel to Oy.* If the equation is $y = ax^2$, we find $dy/dx = 2ax$, $d^2y/dx^2 = 2a$, and hence we have

$$\kappa = \frac{2a}{(1+4a^2x^2)^{3/2}} = \frac{2a}{(1+4ay)^{3/2}}. \qquad \text{.......................(27)}$$

If we consider a range $(-h, h)$ of x such that $4a^2h^2$ is negligibly small compared with 1, the curvature is equal to $2a$ throughout that range.

(ii) *Axis parallel to Ox.* Taking $y^2 = 4ax$ as the equation, we find $dy/dx = 2a/y$, $d^2y/dx^2 = -4a^2/y^3$, and therefore, provided $x \neq 0$,

$$\kappa = \frac{-4a^2/y^3}{(1+4a^2/y^2)^{3/2}} = \frac{-4a^2/y^3}{\pm(y^2+4a^2)^{3/2}/y^3} = \mp\frac{4a^2}{(y^2+4a^2)^{3/2}}, \qquad \text{...........(28)}$$

the upper (lower) sign being taken if y is positive (negative).

When $y \to +0$, the expression $-4a^2/(y^2+4a^2)^{3/2}$ has the limiting value $-1/(2a)$. This, therefore, is the curvature at the vertex regarded as approached from above. If the vertex is approached from below, the corresponding limiting value is $1/(2a)$. We observe that at O the value of d^2x/dy^2 is $1/(2a)$.

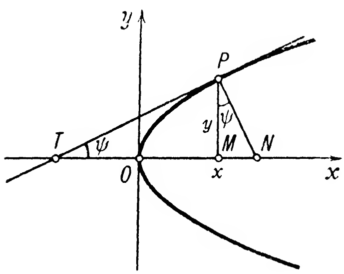

FIG. 101.

The curvature at any point P, (x, y), can also be expressed simply in terms of the length of the normal PN (Fig. 101). Taking NP as of the same sign as y, we have

$$NP = y\,|\sec\psi| = y\sqrt{(1+g^2)}, \qquad \text{.....(29)}$$

and hence $(1+g^2)^{3/2} = (NP/y)^3$. Therefore

$$\kappa = -\frac{4a^2/y^3}{(1+g^2)^{3/2}} = -\frac{4a^2}{NP^3}. \qquad \text{...........................(30)}$$

Ex. 5. *The ellipse and hyperbola.* For the ellipse $x^2/a^2 + y^2/b^2 = 1$, we have, on differentiating the equation twice,

$$\frac{x}{a^2} + \frac{y}{b^2}\frac{dy}{dx} = 0, \quad \frac{1}{a^2} + \frac{1}{b^2}\left(\frac{dy}{dx}\right)^2 + \frac{y}{b^2}\frac{d^2y}{dx^2} = 0, \qquad \text{...................(31)}$$

whence we find

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}, \quad \frac{d^2y}{dx^2} = -\frac{b^2}{y}\left(\frac{1}{a^2} + \frac{1}{b^2}\frac{b^4x^2}{a^4y^2}\right) = -\frac{b^4}{a^2y^3}. \qquad \text{................(32)}$$

Therefore

$$\kappa = \frac{-b^4/(a^2y^3)}{\left(1+\dfrac{b^4x^2}{a^4y^2}\right)^{3/2}} = \frac{-b^4/(a^2y^3)}{\pm(a^4y^2+b^4x^2)^{3/2}/(a^6y^3)} = \mp\frac{a^4b^4}{(b^4x^2+a^4y^2)^{3/2}}, \qquad \text{.......(33)}$$

the upper (lower) sign being taken if y is positive (negative).

This expresses the curvature in terms of the coordinates (x, y). We can

also express the curvature in terms of the length, p, of the perpendicular OR (Fig. 102) from the centre O to the tangent RPT at P. We have

$$p = OT\sin(\pi-\psi) = OT\sin\psi,\quad MT = y\cot(\pi-\psi) = -y\cot\psi, \qquad (34)$$

where $\tan\psi = -b^2x/(a^2y)$. Therefore

$$\begin{aligned} p^2 &= (OM+MT)^2\sin^2\psi \\ &= (x-y\cot\psi)^2\sin^2\psi \\ &= \left(x-\frac{y}{\tan\psi}\right)^2\frac{\tan^2\psi}{1+\tan^2\psi} \\ &= \frac{a^4b^4}{b^4x^2+a^4y^2}, \end{aligned} \qquad (35)$$

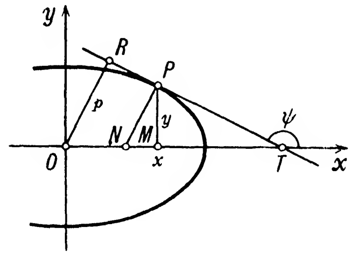

FIG. 102.

so that, p being always regarded as positive,

$$p^3 = a^6b^6/(b^4x^2+a^4y^2)^{3/2}. \qquad (36)$$

Therefore

$$\kappa = \mp p^3/(a^2b^2), \qquad (37)$$

the negative (positive) sign being taken if y is positive (negative).

Again, if PN is the normal at P, we have, as in Ex. 4 above,

$$NP^3 = y^3(1+g^2)^{3/2}, \qquad (38)$$

and hence

$$\kappa = \frac{-b^4/(a^2y^3)}{(1+g^2)^{3/2}} = -\frac{b^4/(a^2y^3)}{NP^3/y^3} = -\frac{b^4}{a^2NP^3}. \qquad (39)$$

The same formula is obtained for all points on the curve, the positive sense for the measurement of NP being that of Oy.

The results (33), (37), (39) also hold for the hyperbola $x^2/a^2 - y^2/b^2 = 1$, and, recalling the result (30) above, we can state generally that the curvature of a conic section varies inversely as the cube of the normal.

Ex. 6. A curve of the form $y = a_1x + a_3x^3 + a_5x^5$ is to touch the parallel lines shown in Fig. 103 at P, P' and to have zero curvature at these points. Given $OA = OA' = c$, $OQ = OQ' = h$, $\angle QAP = \theta$, find the appropriate values of the disposable constants a_1, a_3, a_5.

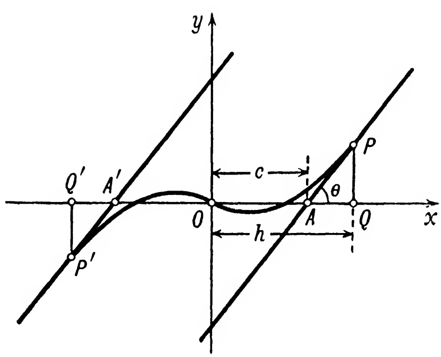

FIG. 103.

On account of the symmetry we need only consider the portion OP. The coordinates of P being $\{h, (h-c)\tan\theta\}$, we have, since P is on the curve,

$$(h-c)\tan\theta = a_1h + a_3h^3 + a_5h^5. \qquad (40)$$

Again, since the gradient of the curve is $\tan\theta$ when $x = h$, we have

$$\tan\theta = a_1 + 3a_3h^2 + 5a_5h^4. \qquad (41)$$

Also the second derivative vanishes when $x = h$, since $\kappa = 0$ there, and so

$$0 = 6a_3h + 20a_5h^3. \qquad (42)$$

From equations (40), (41), (42) we find

$$a_1=\left(1-\frac{15c}{8h}\right)\tan\theta,\quad a_3=\frac{5c}{4h^3}\tan\theta,\quad a_5=-\frac{3c}{8h^5}\tan\theta. \quad\ldots\ldots\ldots\ldots(43)$$

The above conditions would naturally be satisfied by a line joining two parallel tracks of railway, although in practice the equation of the transition curve is not of the simple algebraic form assumed.

Ex. 7. *Formula for the curvature when the coordinates are given as functions of a parameter.* If $x=f(t)$, $y=\varphi(t)$, we have

$$\frac{dy}{dx}=\frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}},\quad \frac{d^2y}{dx^2}=\frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{\left(\dfrac{dx}{dt}\right)^3}. \quad\ldots\ldots\ldots\ldots(44)$$

Hence
$$\kappa=\frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{\left\{1+\left(\dfrac{dy/dt}{dx/dt}\right)^2\right\}^{3/2}\left(\dfrac{dx}{dt}\right)^3}=\pm\frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2}-\dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{\left\{\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2\right\}^{3/2}}, \quad\ldots\ldots\ldots(45)$$

where, according to the first convention as to the sign of the curvature, the upper or lower sign is to be taken according as dx/dt is positive or negative.

If we put $x=at$, $y=\pm b\sqrt{(1-t^2)}$, the curve is an ellipse, and the student will easily recover the formula (33) of Ex. 5 by means of (45).

Ex. 8. *Formula for the curvature when the curve is defined implicitly.* If a curve is defined by the relation $f(x, y)=0$, its curvature at any point can be conveniently expressed in terms of the partial derivatives of $f(x, y)$ by the use of the equations (4), (6) of Art. 88. We have

$$\left.\begin{aligned}&\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0,\\ &\frac{\partial^2 f}{\partial x^2}+\left(\frac{\partial^2 f}{\partial y\,\partial x}+\frac{\partial^2 f}{\partial x\,\partial y}\right)\frac{dy}{dx}+\frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2+\frac{\partial f}{\partial y}\frac{d^2y}{dx^2}=0.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(46)$$

Hence
$$\kappa=\frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}=\mp\frac{\left(\dfrac{\partial f}{\partial y}\right)^2\dfrac{\partial^2 f}{\partial x^2}-\dfrac{\partial f}{\partial x}\dfrac{\partial f}{\partial y}\left(\dfrac{\partial^2 f}{\partial y\,\partial x}+\dfrac{\partial^2 f}{\partial x\,\partial y}\right)+\left(\dfrac{\partial f}{\partial x}\right)^2\dfrac{\partial^2 f}{\partial y^2}}{\left\{\left(\dfrac{\partial f}{\partial x}\right)^2+\left(\dfrac{\partial f}{\partial y}\right)^2\right\}^{3/2}}, \quad(47)$$

where the upper or lower sign is to be taken according as $\partial f/\partial y$ is positive or negative.

We have already mentioned that, under normal conditions, $\dfrac{\partial^2 f}{\partial y\,\partial x}=\dfrac{\partial^2 f}{\partial x\,\partial y}$.

134. Circle of Curvature. Its Radius and Centre.

We have already seen that the radius of a circle is equal to the reciprocal of the magnitude of its curvature. We can therefore at once draw at any point of a curve, at which the curvature is known, a circle touching the curve and having the same curvature as the curve in the same sense. This circle is naturally called the *circle of curvature* of the curve at the point considered. If the curvature of the curve there is κ, the radius of this circle is $1/|\kappa|$.

It is clear that this circle will approximate more closely to the curve in the neighbourhood of the point than a circle of different radius. For the deflections of any two curves from a common tangent at the same distance from the point of contact are ultimately in the ratio of the curvatures, and are thus in the ultimate ratio of equality if the curvatures are equal, and not otherwise. The circle of curvature is therefore also called the *circle of closest fit* at the point.

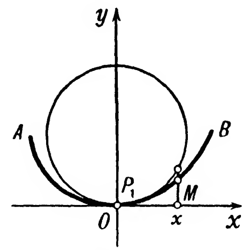

FIG. 104.

The proximity of this circle to the curve in the neighbourhood of the point may be investigated as follows. Taking axes P_1x, P_1y along the tangent and normal to the curve AB (Fig. 104) at P_1, then for the point (x, y) either on AB or on the circle of curvature at P_1, we have, by the $I.D_3.T.$,

$$y = \tfrac{1}{2}\kappa_1 x^2 + \tfrac{1}{6}\frac{\overline{d^3y}}{dx^3}x^3, \qquad (1)$$

where the derivative is taken as some point in $[0, x]$. If, now, x is small and we neglect the variation of d^3y/dx^3 in the range $(0, x)$, we obtain from (1) the approximation

$$y \approx \tfrac{1}{2}\kappa_1 x^2 + \tfrac{1}{6}\left(\frac{d^3y}{dx^3}\right)_1 x^3. \qquad (2)$$

Also, from the formula $\kappa = \dfrac{d^2y/dx^2}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}$, we find

$$\left(\frac{d\kappa}{dx}\right)_1 = \left(\frac{d^3y}{dx^3}\right)_1, \qquad (3)$$

so that (2) becomes

$$y \approx \tfrac{1}{2}\kappa_1 x^2 + \tfrac{1}{6}\left(\frac{d\kappa}{dx}\right)_1 x^3. \qquad (4)$$

For the circle, $d\kappa/dx$ vanishes. Hence the difference of the ordinates of curve and circle is ultimately in a ratio of equality to $\tfrac{1}{6}(d\kappa/dx)_1 x^3$, and is thus of the third order of smallness unless the rate of variation of the curvature of the curve AB vanishes at the point. This shews that the contact of a curve with its circle of curvature is, in general, of the second order only. We investigate below, Art. 136, conditions that the contact may be of higher order.

The radius of the circle of curvature at any point is called the *radius of curvature* of the curve at the point. It is usually denoted by ϱ. The radius of curvature of the curve is therefore $1/|\kappa|$, and we have

$$\varrho = \frac{1}{|\kappa|} = \frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}}{\left|\dfrac{d^2y}{dx^2}\right|}. \qquad (5)$$

The centre of the circle of curvature is called the *centre of curvature* of the curve at the point considered. Its coordinates are readily found. We consider, for simplicity, the case in which $\kappa > 0$ and ψ lies in the range $(0, \frac{1}{2}\pi)$, so that $dy/dx > 0$. If P (Fig. 105) is the point (x, y), and C, (ξ, η), is the centre of curvature, then $CP = \varrho = 1/\kappa$, and we have

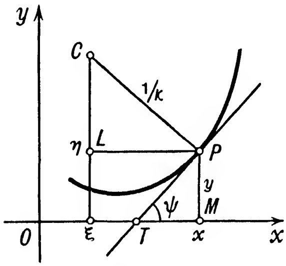

FIG. 105.

$$\xi = x - LP = x - \frac{1}{\kappa}\sin\psi$$
$$= x - \frac{1}{\kappa}\frac{dy/dx}{\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}}$$
$$= x - \frac{\frac{dy}{dx}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{dx^2}}. \quad \text{..........(6)}$$

In a similar way we find

$$\eta = y + LC = y + \frac{1}{\kappa}\cos\psi = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}. \quad \text{..................(7)}$$

The formulae (6), (7) for the coordinates (ξ, η) hold generally, whatever the signs of dy/dx or κ.

To prove this it is convenient to define ψ in the trigonometrical manner as the angle which the directed axis Px' (Fig. 106) makes with the directed axis Ox. Then if κ' is the curvature at P *measured with respect to the axes* Px', Py', we have always for the coordinates (ξ, η) of the centre of curvature C, the relations

$$\xi = x - \frac{1}{\kappa'}\sin\psi, \quad \eta = y + \frac{1}{\kappa'}\cos\psi, \quad \text{.......(8)}$$

since PC is along Py' if κ' is positive.

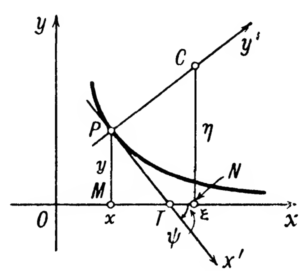

FIG. 106.

Now it is seen at once that $\kappa' = (\operatorname{sig}\cos\psi)\,\kappa$, and therefore we have generally

$$\xi = x - (\operatorname{sig}\cos\psi)\frac{1}{\kappa}\sin\psi, \quad \eta = y + (\operatorname{sig}\cos\psi)\frac{1}{\kappa}\cos\psi. \quad \text{............(9)}$$

Since $\sin\psi = \tan\psi\cos\psi$, these become

$$\xi = x - \frac{1}{\kappa}|\cos\psi|\tan\psi, \quad \eta = y + \frac{1}{\kappa}|\cos\psi|, \quad \text{.................(10)}$$

where $\tan\psi = dy/dx$, $|\cos\psi| = 1\Big/\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{1/2}$. We are thus led back to the formulae (6), (7) above.

It should be observed, however, that the formulae $\xi = x - \sin\psi/\kappa$, $\eta = y + \cos\psi/\kappa$ do not hold generally under the convention previously introduced for the measurement of the slope ψ of the (undirected) tangent, namely that ψ lies in the range $(0, \pi)$.

When $\psi = \frac{1}{2}\pi$ we have evidently $\eta = y$ and $\xi = x \pm 1/|\kappa|$ according as the concavity is to the right or left.

The chord of the circle of curvature drawn in a definite direction through the point of the curve at which the curvature is being investigated is called the *chord of curvature* in the given direction.

Ex. 1. *The ellipse* $x^2/a^2 + y^2/b^2 = 1$. As in Ex. 5, p. 302, we have

$$\kappa = \mp \frac{a^4b^4}{(b^4x^2 + a^4y^2)^{3/2}}, \qquad (11)$$

the upper (lower) sign being taken if y is positive (negative). The radius of curvature is therefore given by

$$\varrho = \frac{(b^4x^2 + a^4y^2)^{3/2}}{a^4b^4}. \qquad (12)$$

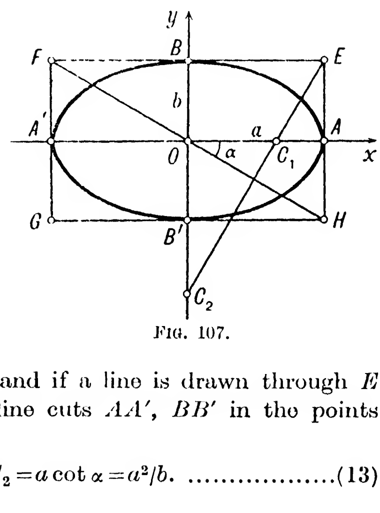

FIG. 107.

At the ends of the minor axis, where $x = 0$, $y^2 = b^2$, the radii of curvature are equal to a^2/b. Also the expression on the right-hand side of (12) has the definite limit b^2/a as $y \to 0$, that is, as $x^2 \to a^2$, and so the radii of curvature at the ends of the major axis are equal to b^2/a. Referring to Fig. 107, the centres of curvature C_1, C_2 for the points A, B lie along AA', BB', respectively. If the enveloping rectangle is drawn as shown, and if a line is drawn through E perpendicular to the diagonal FH, this line cuts AA', BB' in the points C_1, C_2. For we have $\tan\alpha = b/a$, so that

$$AC_1 = b\tan\alpha = b^2/a. \quad BC_2 = a\cot\alpha = a^2/b. \qquad (13)$$

Ex. 2. A chain is stretched between two posts 100 feet apart and the sag in the middle is 1 foot. Assuming the curve to have a parabolic form, find the radius of the circle of curvature at the middle point. Shew that the difference between the ordinates of the two curves at the positions of the posts is approximately 1/200 inch.

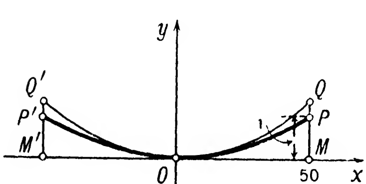

FIG. 108.

Taking as axes the tangent and normal at the lowest point, as in Fig. 108, where $P'P$ represents the chain (not to scale), the equation of the parabola is of the form $y = ax^2$, the constant a being determined by the fact that P, whose coordinates are (50, 1), is on the curve;

thus $a = 1/2500$. Since $d^2y/dx^2 = 2a$ we have $\kappa_0 = 2a$, and the radius of curvature at O is thus given by

$$\varrho_0 = 1/(2a) = 1250 \text{ feet.} \quad\quad (14)$$

The ordinate to any point of the parabola is thus equal to $x^2/(2\varrho_0)$, that is, $x^2/2500$.

Using (x, y) now to denote any point on the circle of curvature, part of which is represented by $Q'Q$, its equation is, as in Ex. 2, p. 295, $x^2 + y^2 - 2\varrho_0 y = 0$, and for it we thus have

$$y = \frac{x^2}{2\varrho_0} + \frac{y^2}{2\varrho_0}. \quad\quad (15)$$

When $x = 50$ feet, we have $y = MQ$, $x^2/(2\varrho_0) = MP = 1$ foot, and the difference PQ between the ordinates is thus equal to $y^2/(2\varrho_0)$, where y refers to the circle. In the calculation of this small term we may, as a sufficiently close approximation, take for y the corresponding value of the ordinate of the parabola, namely 1 foot, and we thus obtain

$$PQ = \frac{1}{2\varrho_0} = \frac{1}{2500} \text{ feet} = \frac{1}{200} \text{ inch.} \quad\quad (16)$$

This Example illustrates the principle that when the gradient is small throughout, the original curve may be replaced by its circle of curvature at any point with considerable accuracy.

135. Centre of Curvature as the Intersection of Coalescing Normals.

We have obtained in Ex. 2, Art. 95, p. 203, expressions for the coordinates of the ultimate point of intersection of adjacent normals to a curve. We now shew that this limiting point is the centre of curvature at the point of the curve from which the coalesced normals are drawn.

Let P_1 be the point in question and let the axes of x and y be taken along the tangent and normal at P_1. The coordinates (ξ^*, η^*) of the ultimate point of intersection R^* of adjacent normals, that is, the limiting position of the point of intersection R of the normals at P_1 and a neighbouring point P as P moves up to P_1, are, according to equation (7) of the Example referred to, given by

$$\xi^* = 0, \quad \eta^* = \frac{1}{(dg/dx)_1}. \quad\quad (1)$$

Now $(dg/dx)_1 = \kappa_1$ and hence

$$\xi^* = 0, \quad \eta^* = \frac{1}{\kappa_1} = \varrho_1, \quad\quad (2)$$

and so R^* coincides with the centre of curvature at P_1.

It is therefore natural to speak of the centre of curvature as the *point of intersection of coalescing*[1] *normals*, the working notion being

[1] The traditional adjective employed here is *consecutive*, but it is now usually regarded as inappropriate.

that the point of intersection of the normals from two adjacent points on the curve can usually be taken, with sufficient precision, as the centre of curvature at either point, as in the case of a circle.

Ex. 1. If PR, $P'R$ are the normals at adjacent points P, P' of a curve and if the circle of centre R and radius RP intersects RP' in Q, shew that the chords PP', PQ are ultimately in a ratio of equality when P' moves up to P, and that the projections of these chords on the tangent at P are also ultimately in a ratio of equality.

Let $\angle PRP'=\alpha$, $\angle P'PR=\beta$, $\angle QPR=\gamma$. Then

$$PP'=\frac{\sin\alpha}{\sin\beta}RP', \quad PQ=\frac{\sin\alpha}{\sin\gamma}RQ, \quad \text{......................(3)}$$

so that the ratio of the chords is $\frac{\sin\gamma}{\sin\beta}\frac{RP'}{RQ}$. Now when P' (and therefore also Q) moves up to P, R converges to the centre of curvature at P and RP'/RQ converges to 1. Also β, γ each converge to $\frac{1}{2}\pi$ and therefore $\sin\gamma/\sin\beta$ converges to 1. The ultimate ratio of the chords is therefore one of equality.

Bearing in mind the cosine-law of projection, the second part of the Example at once follows.

136. Order of Contact of a Curve with its Circle of Curvature.

We have shewn (Art. 134) that, in general, a curve and its circle of curvature have contact of the second order. We may utilize this fact to determine analytically the coordinates (ξ, η) of the centre of the circle of curvature and the radius of curvature ϱ. Writing the equation of the circle as $(x-\xi)^2+(y-\eta)^2=\varrho^2$, we have

$$(x-\xi)+(y-\eta)\frac{dy}{dx}=0, \quad 1+\left(\frac{dy}{dx}\right)^2+(y-\eta)\frac{d^2y}{dx^2}=0, \quad \text{.........(1)}$$

giving, for the circle,

$$\frac{dy}{dx}=-\frac{x-\xi}{y-\eta}, \quad \frac{d^2y}{dx^2}=-\frac{\varrho^2}{(y-\eta)^3}. \quad \text{..................(2)}$$

If the equation of the given curve is $y=f(x)$, the equations to be satisfied for contact of the second order at the point of abscissa x are

$$(x-\xi)^2+\{f(x)-\eta\}^2=\varrho^2, \quad f'(x)=-\frac{x-\xi}{f(x)-\eta}, \quad f''(x)=-\frac{\varrho^2}{\{f(x)-\eta\}^3}, \text{...(3)}$$

and from these we can determine ξ, η, ϱ in terms of x, $f(x)$, $f'(x)$, $f''(x)$. We find without difficulty

$$\xi=x-\frac{f'(x)[1+\{f'(x)\}^2]}{f''(x)}, \quad \eta=f(x)+\frac{1+\{f'(x)\}^2}{f''(x)}, \quad \varrho=\frac{[1+\{f'(x)\}^2]^{3/2}}{|f''(x)|}, \text{..(4)}$$

as in Art. 134.

We now investigate conditions that the contact may be of higher

order than the second. We first express the successive derivatives of the gradient of the circle of curvature at the point of contact in terms of the gradient and its first derivative there, which are the same for the curve and the circle. To do this we observe that for *any* curve we have, writing g', g'', g''' for dg/dx, d^2g/dx^2, d^3g/dx^3,

$$\kappa = \frac{g'}{(1+g^2)^{3/2}}, \qquad \text{.......(5)}$$

$$\frac{d\kappa}{dx} = \frac{g''}{(1+g^2)^{3/2}} - \frac{3gg'^2}{(1+g^2)^{5/2}}, \qquad \text{.......(6)}$$

$$\frac{d^2\kappa}{dx^2} = \frac{g'''}{(1+g^2)^{3/2}} - \frac{9gg'g''+3g'^3}{(1+g^2)^{5/2}} + \frac{15g^2g'^3}{(1+g^2)^{7/2}}, \qquad \text{.......(7)}$$

and so on. If, now, these formulae are applied to the circle of curvature, the derivatives of the curvature vanish, and from (6), (7) we have

$$\frac{d^3y}{dx^3} = g'' = \frac{3gg'^2}{1+g^2}, \qquad \text{.......(8)}$$

$$\frac{d^4y}{dx^4} = g''' = \frac{3(1+5g^2)g'^3}{(1+g^2)^2}, \qquad \text{.......(9)}$$

and so on.

Since for third order of contact of the given curve and its circle of curvature the third derivative of y with respect to x must be the same for the two, (8) must be true for the given curve and is the relation sought for this case. Similarly, for fourth order of contact the relation (9) also must be satisfied by the given curve.

Ex. 1. Shew that the curve $y = x^3 - 6x^2 + 11x - 6$ has contact of the second order (only) with its circle of curvature at the point (1, 0).

[We here find $d^3y/dx^3 = 6$ while, when $x=1$, $3gg'^2/(1+g^2) = 216/5$. The equation of the circle of curvature is found to be $(x-\frac{8}{3})^2 + (y+\frac{5}{6})^2 = \frac{125}{36}$. The student should draw the graphs of the cubic parabola and the circle of curvature in the neighbourhood of $x=1$.]

Ex. 2. *The conics of third order contact and the conic of fourth order contact at a point of a curve.* An equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \qquad \text{.......(10)}$$

of the second degree in x, y, can be made to represent any conic in the plane of the coordinate axes by appropriate choice of the ratios of the coefficients $a, h, b, \ldots$. On differentiating (10), we get

$$ax + hy + g + (hx + by + f)\frac{dy}{dx} = 0. \qquad \text{.......(11)}$$

If the conic passes through the origin and touches the axis of x there, the

equations (10), (11) shew that $c=0$ and $g=0$. On dividing by f the equation (10) then takes the form

$$2y=ax^2+2hxy+by^2, \qquad \text{(12)}$$

if we write a for $-a/f$, and so on.

Let $y=f(x)$ be the equation of the given curve referred to axes along the tangent and normal at a point P_1 on it, and take (12) as the equation of a conic which touches it there. For the conic we have

$$\left.\begin{aligned}
\frac{dy}{dx}&=ax+hy+(hx+by)\frac{dy}{dx},\\
\frac{d^2y}{dx^2}&=a+h\frac{dy}{dx}+\left(h+b\frac{dy}{dx}\right)\frac{dy}{dx}+(hx+by)\frac{d^2y}{dx^2},\\
\frac{d^3y}{dx^3}&=3h\frac{d^2y}{dx^2}+3b\frac{dy}{dx}\frac{d^2y}{dx^2}+(hx+by)\frac{d^3y}{dx^3},\\
\frac{d^4y}{dx^4}&=4\left(h+b\frac{dy}{dx}\right)\frac{d^3y}{dx^3}+3b\left(\frac{d^2y}{dx^2}\right)^2+(hx+by)\frac{d^4y}{dx^4}.
\end{aligned}\right\} \qquad \text{(13)}$$

At P_1, where x, y, dy/dx all vanish, we therefore have

$$\left(\frac{d^2y}{dx^2}\right)_1=a,\quad \left(\frac{d^3y}{dx^3}\right)_1=3h\left(\frac{d^2y}{dx^2}\right)_1,\quad \left(\frac{d^4y}{dx^4}\right)_1=4h\left(\frac{d^3y}{dx^3}\right)_1+3b\left(\frac{d^2y}{dx^2}\right)_1^2. \qquad \text{(14)}$$

For the given curve we have from (5), (6), (7), remembering that at P_1, $f(0)=0$, $f^{(1)}(0)=0$,

$$\kappa_1=f^{(2)}(0),\quad \left(\frac{d\kappa}{dx}\right)_1=f^{(3)}(0),\quad \left(\frac{d^2\kappa}{dx^2}\right)_1=f^{(4)}(0)-3\kappa_1^3. \qquad \text{(15)}$$

Hence, for third order contact, we have, from (14) and (15),

$$a=\kappa_1,\quad 3h\kappa_1=(d\kappa/dx)_1, \qquad \text{(16)}$$

and for fourth order contact we have, in addition,

$$3b\kappa_1^2=\left(\frac{d^2\kappa}{dx^2}\right)_1+3\kappa_1^3-\frac{4}{3\kappa_1}\left(\frac{d\kappa}{dx}\right)_1^2. \qquad \text{(17)}$$

The equations (16) determine the values of a and h, and by varying b arbitrarily, we get a family of conics each of which has third order contact with the given curve at the point considered. If, however, the contact is of the fourth order, the constant b is given by equation (17) and the conic is then completely determined.

It is easily shewn that the centres of the family of conics having third order contact lie on a straight line through P_1. In fact, the centre (ξ, η) of the conic of constants a, b, h is given by

$$a\xi+h\eta=0,\quad h\xi+b\eta=1. \qquad \text{(18)}$$

These equations give

$$(ab-h^2)\xi=-h,\quad (ab-h^2)\eta=a, \qquad \text{(19)}$$

whence we find

$$\frac{\xi}{\eta}=-\frac{h}{a}=-\frac{1}{3\kappa_1^2}\left(\frac{d\kappa}{dx}\right)_1=\frac{1}{3}\left\{\frac{d}{dx}\left(\frac{1}{\kappa}\right)\right\}_1, \qquad \text{(20)}$$

which proves the statement.

The formulae (19) give the centre of the conic having fourth order contact, the constants a, b, h being given by (16) and (17).

137. Curvature in Oblique Coordinates.

Let the curve AB (Fig. 109) be referred to oblique axes Ox, Oy, of inclination ω, and let P_1, (x_1, y_1), be the point at which the curvature is to be investigated. The gradient at a neighbouring point P, (x, y), is, as in Art. 80, given by

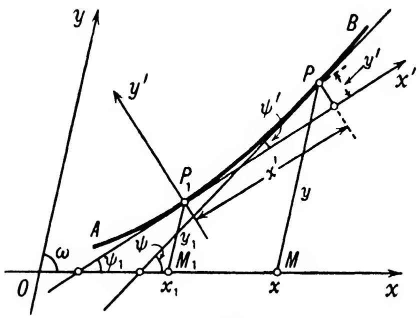

FIG. 109.

$$g = \tan\psi = \frac{\sin\omega \dfrac{dy}{dx}}{1 + \cos\omega \dfrac{dy}{dx}}. \tag{1}$$

From this we obtain the results

$$\frac{dg}{dx} = \frac{\sin\omega \dfrac{d^2y}{dx^2}}{\left(1 + \cos\omega \dfrac{dy}{dx}\right)^2}, \quad \text{.................................(2)}$$

$$1 + g^2 = \frac{1 + 2\cos\omega \dfrac{dy}{dx} + \left(\dfrac{dy}{dx}\right)^2}{\left(1 + \cos\omega \dfrac{dy}{dx}\right)^2}. \quad \text{.........................(3)}$$

Now the curvature κ_1 at P_1 is given by

$$\kappa_1 = (dg'/dx')_{x=x_1}, \quad \text{....................................(4)}$$

where (x', y') are the coordinates of P relative to the axes P_1x', P_1y' along the tangent and normal at P_1, as shewn, and $g' = (g - g_1)/(1 + gg_1)$. At P_1 we thus have

$$\left(\frac{dg'}{dx'}\right)_{x_1} = \frac{\left(\dfrac{dg}{dx}\right)_{x_1}}{1 + g_1^2}\left(\frac{dx}{dx'}\right)_{x_1} = \left\{\frac{\sin\omega \dfrac{d^2y}{dx^2}}{1 + 2\cos\omega \dfrac{dy}{dx} + \left(\dfrac{dy}{dx}\right)^2}\right\}_{x_1} \left(\frac{dx}{dx'}\right)_{x_1}. \quad \text{.....(5)}$$

Also, from equation (12), Ex. 1, Art. 80, p. 155, we have, for the case illustrated,

$$\left(\frac{dx'}{dx}\right)_{x_1} = \sqrt{\left\{1 + 2\cos\omega \left(\frac{dy}{dx}\right)_{x_1} + \left(\frac{dy}{dx}\right)_{x_1}^2\right\}}, \quad \text{..............(6)}$$

and hence

$$\kappa_1 = \frac{\sin\omega \left(\dfrac{d^2y}{dx^2}\right)_{x_1}}{\left\{1 + 2\cos\omega \left(\dfrac{dy}{dx}\right)_{x_1} + \left(\dfrac{dy}{dx}\right)_{x_1}^2\right\}^{3/2}}. \quad \text{............(7)}$$

This reduces to the formula (8) of Art. 133 when $\omega = \frac{1}{2}\pi$.

The general formula

$$\kappa = \frac{\sin\omega \dfrac{d^2y}{dx^2}}{\left\{1 + 2\cos\omega \dfrac{dy}{dx} + \left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

gives the curvature at any point under the first convention of Art. 133. 1, being positive (negative) if the curve is concave (convex) upwards.

EXAMPLES XXIV

1. Find the curvature of the curve $y = a^3/(a^2 + x^2)$ at the point where the tangent is parallel to Ox. [$-2/a$.]

2. Two points P_1, P are taken on a curve and the chord P_1P is drawn. The normal to the curve at P_1 meets the line drawn through P at right-angles to the chord P_1P in the point Q. Prove that ultimately, when P converges to P_1, $P_1Q = |2/\kappa_1|$, where κ_1 is the curvature at P_1.

[Take axes along the tangent and normal at P_1. If P is the point (x, y), then $P_1Q = (x^2/y) + y$.]

3. A curve of the form $y = a + bx + cx^2 + dx^3$ cuts Ox at $x = 1$ and has slope $\frac{1}{4}\pi$ there; it has zero curvature where $x = 2$ and slope $\frac{1}{3}\pi$ there. Find the constants a, b, c, d.

4. A curve whose ordinate y is expressed as a polynomial of the fifth degree in x touches Ox at O and has zero curvature there. The curve also passes through the point $(1, 1)$ and has gradient $\frac{3}{4}$ and curvature 1 there. Find the coefficients of the polynomial and determine the points of contraflexure of the curve.

5. Find the points of the curve $y = 1/x^2$ where the curvature is greatest. [$x = \pm 5^{1/6}$.]

6. Find the curvature of the curve whose ordinates are obtained by a linear combination of the ordinates of a set of curves for the same abscissa.

$\left[\text{If } y = \Sigma A_r y_r,\ \kappa = \dfrac{\Sigma A_r \kappa_r (1 + g_r{}^2)^{3/2}}{\{1 + (\Sigma A_r g_r)^2\}^{3/2}}.\right]$

7. If the equation of a curve is transformed from $y = f(x)$ to $y' = \varphi(x')$ by a change of rectangular axes, prove that $\dfrac{dg/dx}{(1 + g^2)^{3/2}} = \pm \dfrac{dg'/dx'}{(1 + g'^2)^{3/2}}$.

[It is sufficient to suppose the axes merely rotated, since a change of origin alone does not affect the values of g and dg/dx. The result proves the invariance of the formula for the curvature under a rectangular transformation.]

8. Find the curvature at any point of the rectangular hyperbola

$$x = (t^2 + 1)/(t^2 - 1), \quad y = 2t/(t^2 - 1).$$

[$\pm(t^2 - 1)^3/(t^4 + 6t^2 + 1)^{3/2}$ according as $|t| \gtrless 1$.]

9. Find the magnitude of the curvature at any point of the hyperbola $ax^2 + 2hxy - ay^2 = 1$. $[(a^2+h^2)^{-1/2}(x^2+y^2)^{-3/2}.]$

10. Find the coordinates of the centre of curvature of the semi-cubical parabola $9y^2 = 4x^3/a$, and hence find the locus of this centre.

[The locus of the centre of curvature of a curve is called the *evolute* of the curve. Eliminate x between the equations

$$\xi = -x(1+2x/a), \quad \eta^2 = \tfrac{4}{9}\,ax(3+4x/a)^2.]$$

11. Find the curvature at the ends of the greatest and least ordinates of the conic $ax^2 + 2hxy + by^2 = c$. $\left[\pm\sqrt{\left\{\dfrac{a^3}{c(ab-h^2)}\right\}}.\right]$

12. Shew that the curvature at the point $(\tfrac{3}{2}a, \tfrac{3}{2}a)$ on the curve $x^3 + y^3 = 3axy$ is $-8\sqrt{2}/(3a)$.

13. Find the curvature of the curve $y = ax + bx^2 + \ldots + kx^n$ at the origin. $[2b/(1+a^2)^{3/2}.]$

14. Find the conics having third order contact at the origin with the curve $y = ax^2 + bx^3 + \ldots + kx^n$. $[a^2x^2 + bxy + \lambda y^2 - ay = 0.]$

15. Find the points of the ellipse $x^2/a^2 + y^2/b^2 = 1$ at which the circle of curvature has third order contact.

16. Shew that if the circle $x^2 + y^2 - 2yh = 0$ has second order contact with the curve $y = f(x)$ at the origin, where $f'(x) = 0$, then $h = 1/\kappa_0$.

17. At two points P_1, P on a curve straight lines P_1Q_1, PQ are drawn making equal angles α with the corresponding tangents and on the same sides of them. Shew that if a circle of diameter $|1/\kappa_1|$ is drawn to touch the curve at P_1, with its centre on the same side of the curve as the line P_1Q_1, the ultimate point of intersection of the lines P_1Q_1, PQ as P moves up to P_1 is the point of intersection of the circle with P_1Q_1. [Compare Ex. 9, Set XVIII.]

18. If in the preceding Example α is a variable function artan μ of the position of P on the curve, show that the coordinates (ξ, η) of the ultimate point of intersection are given by

$$\xi = \frac{\mu_1}{\left(\dfrac{d\mu}{dx}\right)_1 + (1+\mu_1^2)\kappa_1}, \quad \eta = \frac{\mu_1^2}{\left(\dfrac{d\mu}{dx}\right)_1 + (1+\mu_1^2)\kappa_1},$$

where the axis of x is taken along the tangent at P_1.

19. A particle moves along a plane curve which touches the axis of x at the origin. Prove that if the coordinates (x, y) of the particle are expressed as functions of the time t, the acceleration along Oy when the particle is at O is given by $d^2y/dt^2 = \kappa(dx/dt)^2$, where κ is the curvature at the origin.

[Use the result (45), Ex. 7, Art. 133.]

20. A particle moves in a plane, the component velocities u, v at any instant t parallel to the axes Ox, Oy being given as functions of the position of the particle. Shew that the curvature of the path of the particle in the position (x, y) is of magnitude $\left|\dfrac{\partial v}{\partial x}u^2 + \left(\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial x}\right)uv - \dfrac{\partial u}{\partial y}v^2\right| \Big/ (u^2+v^2)^{3/2}$.

21. Shew that if a curve is referred to axes along the tangent and normal, as in Fig. 104, the Taylor polynomial of the fifth degree for y in terms of x is expressed by

$$y \approx \frac{1}{2!}\kappa_1 x^2 + \frac{1}{3!}\left(\frac{d\kappa}{dx}\right)_1 x^3 + \frac{1}{4!}\left(\frac{d^2\kappa}{dx^2} + 3\kappa^3\right)_1 x^4 + \frac{1}{5!}\left(\frac{d^3\kappa}{dx^3} + 18\kappa^2\frac{d\kappa}{dx}\right)_1 x^5.$$

22. An ellipse of axes $2a$, $2b$ always touches the rectangular axes Ox, Oy. It is required to find the greatest and least distances from O of the points of contact when the ellipse is rotated. If the ellipse touches Ox at A and $OA(=r)$ is an extreme distance, shew that $\kappa r = \cot\alpha$, where κ is the curvature of the ellipse at A and α is the inclination to Ox of the radius from O to its centre. Shew that the coordinates (x, y) of the centre of this ellipse are to be found from $3y^6 - 2(a^2+b^2)y^4 - a^2b^2y^2 + a^2b^2(a^2+b^2) = 0$, $x^2+y^2 = a^2+b^2$, and find the relation between r and y.

23. Two arcs of circles, each of radius r, touch Ox at the points $x = c$, $x = -c$, and have their concavities in opposite senses, that of the former being upwards. A transition curve of the form $y = a_1x + a_3x^3$ is to connect the arcs, having the same gradient and curvature as the arcs at the points of contact. Shew how to find the coefficients a_1, a_3, the coordinates (h, k), $(-h, -k)$ of the points of contact and the gradient at the points.

Shew that if c/r is small compared with 1, $h \approx c\sqrt{3}$, $k \approx (2-\sqrt{3})c^2/r$, $a_1 \approx -\frac{1}{2}(2-\sqrt{3})c/r$, $a_3 \approx \sqrt{3}/(18cr)$.

24. Find the curvature at the origin of each of the two branches of the curve $x = 1 - t^2$, $y = t - t^3$. [$\mp \dfrac{1}{2\sqrt{2}}$ when $t = \pm 1$.]

25. A uniform beam of length l is slightly bent by a weight W applied at one end, the other end being held horizontally. Find the deflection at any point.

[The bending moment and hence the curvature is proportional to the distance from the loaded end. The deflection at distance x from the fixed end is given by $6Ky = Wx^2(3l - x)$.]

26. Shew that the equation of the circle of curvature at the point (ξ, η) of a curve is

$$\begin{vmatrix} m^2 - \mu^2 & x - \xi & y - \eta \\ \dfrac{d\mu^2}{dt} & \dfrac{d\xi}{dt} & \dfrac{d\eta}{dt} \\ \dfrac{d^2\mu^2}{dt^2} & \dfrac{d^2\xi}{dt^2} & \dfrac{d^2\eta}{dt^2} \end{vmatrix} = 0,$$

where $m^2 = x^2 + y^2$, $\mu^2 = \xi^2 + \eta^2$ and t is a parameter specifying the coordinates of the curve.

27. Each point of a curve $y = f(x)$ is transformed by the substitution $\xi = \alpha x$, $\eta = \beta y$. Shew that the radius of curvature is multiplied by $(\alpha^2\cos^2\psi + \beta^2\sin^2\psi)^{3/2}/(\alpha\beta)$, where ψ is the slope of the original curve at (x, y).

If the original curve is the circle $x^2 + y^2 = a^2$, employ this result to shew that the radius of curvature at any point of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is

$$(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}/(ab),$$

θ being the eccentric angle of the point.

28. A transformation of coordinates is effected by the change from axes Ox, Oy, of obliquity ω, to axes Ox', Oy', of obliquity ω', where $\angle xOx' = \theta$. Find the connection between the coordinates (x, y), (x', y') of a point P in the two systems in terms of $\cot\frac{1}{2}\theta$, and shew that if the equation $y = f(x)$ of a curve transforms to $y' = \varphi(x')$, then

$$\frac{\sin\omega \dfrac{d^2y}{dx^2}}{\left\{1 + 2\cos\omega \dfrac{dy}{dx} + \left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}} = \pm \frac{\sin\omega' \dfrac{d^2y'}{dx'^2}}{\left\{1 + 2\cos\omega' \dfrac{dy'}{dx'} + \left(\dfrac{dy'}{dx'}\right)^2\right\}^{3/2}}.$$

29. If a curve is specified by the equations $x = f(t)$, $y = \varphi(t)$ relative to axes of obliquity ω, shew that the curvature at any point is given by

$$\kappa = \pm \sin\omega \frac{f'(t)\varphi''(t) - f''(t)\varphi'(t)}{[\{f'(t)\}^2 + 2\cos\omega f'(t)\varphi'(t) + \{\varphi'(t)\}^2]^{3/2}}.$$

CHAPTER VI

THE TRIGONOMETRICAL FUNCTIONS

138. Introductory Remarks.

We now proceed to consider the application of the Differential Calculus to the Trigonometrical Functions, which constitute the most familiar examples of transcendental functions. The connection between the trigonometrical functions of Analysis and the trigonometrical ratios as defined geometrically has been explained in Chap. I, Art. 15, and the continuity of the functions has been demonstrated. We suppose that the student is acquainted with their fundamental properties as arising geometrically, and here give merely a short summary of such properties and formulae as are required below.

139. Summary of Properties and Formulae.

139. 1. *Bounds of the functions in the range* $[-\infty, \infty]$. The functions $\sin x$, $\cos x$ each have -1 as lower bound and 1 as upper bound, or symbolically, $|\sin x| \leqslant 1$, $|\cos x| \leqslant 1$. The functions $\tan x$, $\cot x$ are unbounded above and below, while the values of $\sec x$, $\operatorname{cosec} x$ are each restricted to the ranges $[-\infty, -1)$, $(1, \infty]$.

139. 2. *Even and odd functions.* The functions $\cos x$, $\sec x$ are even, while $\sin x$, $\tan x$, $\operatorname{cosec} x$, $\cot x$ are odd; for example, $\cos(-x) = \cos x$, $\sin(-x) = -\sin x$, $\tan(-x) = -\tan x$.

139. 3. *Periodicity and quasi-periodicity.* The functions $\sin x$, $\cos x$, $\operatorname{cosec} x$, $\sec x$ are periodic each with period 2π, while $\tan x$, $\cot x$ are periodic each with period π; for example, $\sin(x + 2n\pi) = \sin x$, $\cos(x + 2n\pi) = \cos x$, $\tan(x + n\pi) = \tan x$, $\cot(x + n\pi) = \cot x$, where n is any integer.

For $\sin x$, $\cos x$, $\operatorname{cosec} x$, $\sec x$ the effect of changing the argument by an integral multiple of π is to reproduce the magnitude of the function, but if the multiple is *odd* there is a change of sign; for example, $\sin(x+\pi) = -\sin x$, $\cos(x+\pi) = -\cos x$. This property is described by saying that the functions mentioned have the *quasi-period* π.

139. 4. *Supplementary and complementary arguments.* We have the following relations.

(i) *Supplementary arguments* :

$$\left.\begin{array}{lll}\sin(\pi-x)=\sin x, & \cos(\pi-x)=-\cos x, & \tan(\pi-x)=-\tan x,\\ \operatorname{cosec}(\pi-x)=\operatorname{cosec} x, & \sec(\pi-x)=-\sec x, & \cot(\pi-x)=-\cot x.\end{array}\right\} \quad (1)$$

(ii) *Complementary arguments* :

$$\left.\begin{array}{lll}\sin(\tfrac{1}{2}\pi-x)=\cos x, & \cos(\tfrac{1}{2}\pi-x)=\sin x, & \tan(\tfrac{1}{2}\pi-x)=\cot x,\\ \operatorname{cosec}(\tfrac{1}{2}\pi-x)=\sec x, & \sec(\tfrac{1}{2}\pi-x)=\operatorname{cosec} x, & \cot(\tfrac{1}{2}\pi-x)=\tan x.\end{array}\right\} \quad (2)$$

The relations

$$\left.\begin{array}{ll}\sin(x+\tfrac{1}{2}\pi)=\cos x, & \cos(x+\tfrac{1}{2}\pi)=-\sin x,\\ \tan(x+\tfrac{1}{2}\pi)=-\cot x, & \cot(x+\tfrac{1}{2}\pi)=-\tan x,\end{array}\right\} \quad \ldots\ldots\ldots\ldots(3)$$

corresponding to an increase of amount $\frac{1}{2}\pi$ in the argument, follow at once from the above relations.

139. 5. *Image properties.* From the relations (2), (3) we deduce the relations

$$\sin(\tfrac{1}{2}\pi-x)=\sin(\tfrac{1}{2}\pi+x), \quad \cos(\tfrac{1}{2}\pi-x)=-\cos(\tfrac{1}{2}\pi+x). \ldots\ldots(4)$$

The former of these shews that $\sin x$ has equal values for values of the argument as much below as above the value $\frac{1}{2}\pi$, a property which is described by saying that $\sin x$ has a *positive image* in the point $x=\frac{1}{2}\pi$. In the case of $\cos x$ there is a *negative image* in the point $x=\frac{1}{2}\pi$.

Other image points for the above and the remaining trigonometrical functions are seen at once from their graphs.

139. 6. *The Pythagorean relations.* These are expressed by

$$\cos^2 x+\sin^2 x=1, \quad 1+\tan^2 x=\sec^2 x, \quad \cot^2 x+1=\operatorname{cosec}^2 x. \ldots(5)$$

139. 7. *Addition Theorems.* These are expressed by

$$\sin(x_1\pm x_2)=\sin x_1\cos x_2\pm\cos x_1\sin x_2, \ldots\ldots\ldots\ldots\ldots(6)$$

$$\cos(x_1\pm x_2)=\cos x_1\cos x_2\mp\sin x_1\sin x_2, \ldots\ldots\ldots\ldots\ldots(7)$$

$$\tan(x_1\pm x_2)=\frac{\tan x_1\pm\tan x_2}{1\mp\tan x_1\tan x_2}. \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

(i) *Special cases and extensions.* We have, as special cases,

$$\sin 2x=2\sin x\cos x, \quad \cos 2x=\cos^2 x-\sin^2 x, \quad \tan 2x=\frac{2\tan x}{1-\tan^2 x}. \quad (9)$$

From the second of these, we obtain, with the aid of the Pythagorean relations, the important results

$$\cos^2 x=\tfrac{1}{2}(1+\cos 2x), \quad \sin^2 x=\tfrac{1}{2}(1-\cos 2x), \ldots\ldots\ldots(10)$$

which are constantly used in the forms

$$1+\cos x=2\cos^2\tfrac{1}{2}x, \quad 1-\cos x=2\sin^2\tfrac{1}{2}x. \qquad \text{(11)}$$

(ii) *Factorization of sums of sines and cosines.* We have

$$\sin x_1+\sin x_2=2\sin\tfrac{1}{2}(x_1+x_2)\cos\tfrac{1}{2}(x_1-x_2), \qquad \text{(12)}$$

$$\sin x_1-\sin x_2=2\cos\tfrac{1}{2}(x_1+x_2)\sin\tfrac{1}{2}(x_1-x_2), \qquad \text{(13)}$$

$$\cos x_1+\cos x_2=2\cos\tfrac{1}{2}(x_1+x_2)\cos\tfrac{1}{2}(x_1-x_2), \qquad \text{(14)}$$

$$\cos x_1-\cos x_2=-2\sin\tfrac{1}{2}(x_1+x_2)\sin\tfrac{1}{2}(x_1-x_2). \qquad \text{(15)}$$

Ex. 1. Prove the results

$$\sin 3x=3\sin x-4\sin^3 x, \quad \cos 3x=4\cos^3 x-3\cos x. \qquad \text{(16)}$$

139. 8. *Expressions for* $\sin x$, $\cos x$, $\tan x$ *in terms of* $\tan\frac{1}{2}x$. We have

$$\sin x=\frac{2\tan\frac{1}{2}x}{1+\tan^2\frac{1}{2}x}, \quad \cos x=\frac{1-\tan^2\frac{1}{2}x}{1+\tan^2\frac{1}{2}x}, \quad \tan x=\frac{2\tan\frac{1}{2}x}{1-\tan^2\frac{1}{2}x}. \qquad \text{(17)}$$

140. Sinoidal Functions. Extension of the Term 'Sine Function'.

From the physical point of view, the graph of the function $\sin x$ (or $\cos x$) may be regarded as representing an infinite train of waves in which the form repeats itself after a distance 2π. This distance, which is the wave-length of the train, is often called the *wave-length of the graph*, or the *wave-length of the function*. The *amplitude* of the wave is its maximum height above the mean level, and this term is also applied to the graph or to the function, whose *amplitude* is therefore 1 in the present case.

If, now, we consider the graph of the function[1] $a\sin(mx+\alpha)$, where a, m, α are constants, we see that it may be derived from that of the function $\sin x$ merely by adjusting the scales along the axes and shifting the origin along the axis of x. The points of intersection of the graph with the axis Ox are given by $\sin(mx+\alpha)=0$, that is, by $mx+\alpha=n\pi$, or

$$x=\frac{1}{m}(n\pi-\alpha), \qquad \text{(1)}$$

where n is an integer. The form of the curve repeats itself when $mx+\alpha$ changes by an amount 2π, that is, when x increases (or decreases) by an amount $2\pi/|m|$, and this is accordingly the period for x, or the *wave-length* of the graph. The maximum ordinate is $|a|$, which is the *amplitude* of the graph or of the function. The constant α is often called the *phase-constant*. It is clear that the

[1] The notation here follows the general rule for a function of a function, thus $\sin(mx+\alpha)=\sin u$, where $u=mx+\alpha$.

constants a, m can always be taken positive by adjusting the value of the phase-constant α. Further, the function $a\cos(mx+\alpha)$ can be included in the same form, for it can be written $a\sin(mx+\alpha+\frac{1}{2}\pi)$.

Functions of the form $a\sin(mx+\alpha)$ are called *sinusoidal*, or more simply, *sinoidal functions of x*, but when no confusion can arise it is common to refer to them as *sine functions of x* by an extension of the meaning of that term. Such a function is completely defined by its amplitude, wave-length and phase-constant. If these are a, λ, α, respectively, the sine function is $a\sin\left(\frac{2\pi}{\lambda}x+\alpha\right)$.

Ex. 1. Express the function $a\sin x+b\cos x$ as a sinoidal function, and hence shew that $|a\sin x+b\cos x|\leqslant\sqrt{(a^2+b^2)}$.

We can always find constants R, α, $(R>0)$, such that

$$a=R\cos\alpha,\quad b=R\sin\alpha, \quad\text{..........................(2)}$$

for if we look upon (a, b) as the coordinates of a point P in a plane, R is simply the length of the radius OP and α its inclination to the axis of abscissae. From (2) we find

$$R^2=a^2+b^2,\quad \tan\alpha=b/a. \quad\text{..........................(3)}$$

We can therefore write the given function in the form

$$\sqrt{(a^2+b^2)}(\cos\alpha\sin x+\sin\alpha\cos x),\quad\text{or}\quad \sqrt{(a^2+b^2)}\,.\,\sin(x+\alpha). \quad\text{...(4)}$$

It follows that its maximum value is $\sqrt{(a^2+b^2)}$ and it occurs when $x+\alpha=2n\pi+\frac{1}{2}\pi$, ($n$ any integer), that is, when $\tan x=\cot\alpha=a/b$.

141. Potential Continuity of the Function $\dfrac{\sin x}{x}$ at $x=0$.

In proceeding to the derivatives of the trigonometrical functions we require the following theorem :

The function $(\sin x)/x$ *is potentially continuous at* $x=0$, *and its completing value there is* 1.

In the language of limits, the theorem asserts that $(\sin x)/x$ has the definite limit 1 as x converges to zero, and is represented symbolically by

$$\lim_{x\to 0}\frac{\sin x}{x}=1. \quad\text{...............................(1)}$$

We shall assume that the student is familiar with the theorem in this form. It is proved geometrically in books on Trigonometry. An analytical proof would, at this stage, be rather too elaborate.[1]

[1] One such method follows closely that used for the exponential function in Art. 158. See Ex. 17, Set XXVI.

The ε-condition for the potential continuity of the function in question when $x=0$ is expressed by

$$\left|\frac{\sin x}{x}-1\right|<\varepsilon,\ (\varepsilon \text{ arbitrary}),\ \text{for } |x|<h,\ (h \text{ appropriate}).^1 \quad \ldots(2)$$

If we denote the completed function by $\left(\frac{\sin x}{x}\right)^*$, we have $\left(\frac{\sin x}{x}\right)^*_0=1.$

A characteristic portion of the graph of the function $\left(\frac{\sin x}{x}\right)^*$ is shewn in Fig. 110, the range of x being from about -2π to about 2π.

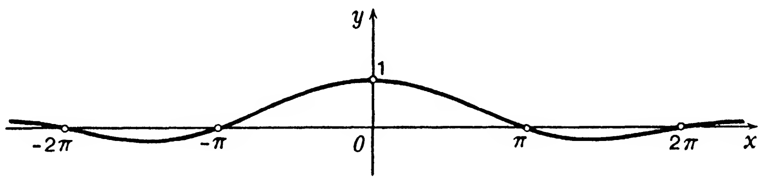

FIG. 110.

$$y=\frac{\sin x}{x}.$$

The completing value of the function $\frac{\sin^\dagger x}{x}$ at $x=0$, (where $\sin^\dagger x$ is the function whose value at x is the sine of x degrees, as in Art. 15), is easily deduced. We have

$$\frac{\sin^\dagger x}{x}=\frac{\sin\dfrac{\pi x}{180}}{\dfrac{\pi x}{180}}\cdot\frac{\pi}{180}. \quad \ldots\ldots(3)$$

Now when x converges to zero, so also does $\pi x/180$, and therefore

$$\lim_{x\to 0}\frac{\sin\dfrac{\pi x}{180}}{\dfrac{\pi x}{180}}=1. \quad \ldots\ldots(4)$$

Therefore

$$\lim_{x\to 0}\frac{\sin^\dagger x}{x}=\frac{\pi}{180}, \quad \ldots\ldots(5)$$

and thus $\pi/180$ is the required completing value.

142. Derivatives of the Trigonometrical Functions.

It is sufficient to investigate the derivative of the sine function alone from first principles, the derivatives of the remaining functions being obtained immediately by the use of the general theorems

[1] Actually we shall shew, Ex. 1, Art. 146, p. 337, that $\left|\frac{\sin x}{x}-1\right|\leqslant \frac{1}{6}x^2$, and therefore $\left|\frac{\sin x}{x}-1\right|<\varepsilon$ if $|x|<\sqrt{(6\varepsilon)}$, so that $\sqrt{(6\varepsilon)}$ is an appropriate value of h.

established in Chap. II for the derivatives of combinations of functions.

142. 1. *Derivative of* $\sin x$. Taking x_1 as starting point and Δx as the increment of argument, y_1 and Δy being the corresponding starting value and increment of function, we have

$$y_1 = \sin x_1, \quad y_1 + \Delta y = \sin(x_1 + \Delta x), \quad \text{.................(1)}$$

so that

$$\frac{\Delta y}{\Delta x} = \frac{\sin(x_1 + \Delta x) - \sin x_1}{\Delta x}$$

$$= \frac{2\cos(x_1 + \tfrac{1}{2}\Delta x)\sin\tfrac{1}{2}\Delta x}{\Delta x} = \cos(x_1 + \tfrac{1}{2}\Delta x)\frac{\sin\tfrac{1}{2}\Delta x}{\tfrac{1}{2}\Delta x}. \quad \text{.......(2)}$$

Now $\cos(x_1 + \frac{1}{2}\Delta x)$, that is, $\cos\frac{1}{2}(x_1 + x)$, is a continuous function of x, having the value $\cos x_1$ when $x = x_1$, and $\dfrac{\sin\frac{1}{2}\Delta x}{\frac{1}{2}\Delta x}$ is potentially continuous at the value x_1 of x (where $\Delta x = 0$), the completing value being 1. The right-hand side of (2) is therefore potentially continuous for the same value of x and its completing value is the product of $\cos x_1$ and 1, that is, $\cos x_1$. We have thus proved that $\Delta y/\Delta x$ is potentially continuous and has the completing value $\cos x_1$. The function $\sin x$ therefore has a definite derivative at x_1 and its value is $\cos x_1$. Since x_1 is any value of x, we may drop the suffix and write

$$\frac{d\sin x}{dx} = \cos x, \quad \text{...................................(3)}$$

or

$$\frac{d\sin x}{dx} = \sin(x + \tfrac{1}{2}\pi). \quad \text{..........................(4)}$$

The second form of the derivative is very convenient in the discussion of higher derivatives.

142. 2. *Derivative of* $\cos x$. Writing $\cos x = \sin(x + \frac{1}{2}\pi) = \sin u$, where $u = x + \frac{1}{2}\pi$, we have, on employing the rule for the derivative of a function of a function,

$$\frac{d\cos x}{dx} = \frac{d\sin(x + \frac{1}{2}\pi)}{dx} = \frac{d\sin u}{dx}$$

$$= \frac{d\sin u}{du}\frac{du}{dx}$$

$$= \cos u \times 1$$

$$= \cos(x + \tfrac{1}{2}\pi) = -\sin x. \quad \text{...........(5)}$$

Ex. 1. Establish the following results, where a, m, α are constants :

$$\left.\begin{aligned} &\text{(i) } \frac{d}{dx} a \sin x = a \cos x, && \text{(iii) } \frac{d}{dx} a \sin(mx+\alpha) = ma \cos(mx+\alpha), \\ &\text{(ii) } \frac{d}{dx} \sin mx = m \cos mx, && \text{(iv) } \frac{d}{dx} a \cos(mx+\alpha) = -ma \sin(mx+\alpha). \end{aligned}\right\} \quad \ldots(6)$$

142. 3. *Derivatives of the remaining trigonometrical functions.* The derivatives of the secondary trigonometrical functions can now be written down. We have the following results :

$$\text{(i) } \frac{d \tan x}{dx} = \frac{d}{dx} \frac{\sin x}{\cos x} = \frac{\cos x \dfrac{d \sin x}{dx} - \sin x \dfrac{d \cos x}{dx}}{\cos^2 x}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x. \quad (7)$$

$$\text{(ii) } \frac{d \operatorname{cosec} x}{dx} = \frac{d}{dx} \frac{1}{\sin x} = -\frac{1}{\sin^2 x} \cos x = -\operatorname{cosec} x \cot x. \quad \ldots\ldots\ldots\ldots(8)$$

$$\text{(iii) } \frac{d \sec x}{dx} = \frac{d}{dx} \frac{1}{\cos x} = -\frac{1}{\cos^2 x}(-\sin x) = \sec x \tan x. \quad \ldots\ldots\ldots\ldots(9)$$

$$\text{(iv) } \frac{d \cot x}{dx} = \frac{d}{dx} \frac{1}{\tan x} = -\frac{1}{\tan^2 x} \sec^2 x = -\operatorname{cosec}^2 x. \quad \ldots\ldots\ldots\ldots(10)$$

The equivalent forms

$$\frac{d \tan x}{dx} = 1 + \tan^2 x, \quad \frac{d \cot x}{dx} = -(1 + \cot^2 x) \quad \ldots\ldots\ldots\ldots(11)$$

should also be noted.

In cases (i), (iii), the values $\frac{2n+1}{2}\pi$ of x must be excluded, and in cases (ii), (iv), the values $n\pi$, (n an integer), since for these values of x the functions concerned are undefined (infinite).

Ex. 2. *The function* $(\sin x)/x$, $(x \neq 0)$. For the derivative we have

$$\frac{d}{dx} \frac{\sin x}{x} = \frac{\cos x}{x} - \frac{\sin x}{x^2} = \frac{x \cos x - \sin x}{x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

The derivative of the completed function $\left(\dfrac{\sin x}{x}\right)^*$ at $x = 0$ is investigated in Ex. 3, Art. 146, p. 338.

Ex. 3. *The functions* $\sin \frac{1}{x}$, $x \sin \frac{1}{x}$. Writing $u = 1/x$, we have

$$\frac{d \sin \frac{1}{x}}{dx} = \frac{d \sin u}{dx} = \frac{du}{dx} \frac{d \sin u}{du} = \left(-\frac{1}{x^2}\right) \cos u = -\frac{1}{x^2} \cos \frac{1}{x}, \quad \ldots\ldots\ldots\ldots(13)$$

and

$$\frac{d}{dx} x \sin \frac{1}{x} = \sin \frac{1}{x} + x \frac{d \sin \frac{1}{x}}{dx} = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

The former of the given functions is undefined when $x=0$, but, as in Art. 55, the latter is potentially continuous there, the completing value being 0. The derivative of $x \sin \frac{1}{x}$ does not, however, converge to a finite limit as $x \to 0$. Portions of the graphs of the two functions are shewn on p. 89.

Ex. 4. *The function* $y=\sec^m(x \tan \tfrac{1}{2}x)$. Writing $x \tan \tfrac{1}{2}x = v$, we have $y=\sec^m v$, which becomes $y=u^m$ on putting $\sec v = u$. Then $dy/du = mu^{m-1}$, $du/dv = \sec v \tan v$, $dv/dx = \tan \tfrac{1}{2}x + \tfrac{1}{2}x \sec^2 \tfrac{1}{2}x$.

The extension of the rule for the derivative of a function of a function now gives

$$\frac{dy}{dx}=\frac{dy}{du}\,\frac{du}{dv}\,\frac{dv}{dx}=mu^{m-1}\,.\sec v \tan v\,.(\tan \tfrac{1}{2}x + \tfrac{1}{2}x \sec^2 \tfrac{1}{2}x)$$

$$=m \sec^m (x \tan \tfrac{1}{2}x)\,.\tan (x \tan \tfrac{1}{2}x)\,.(\tan \tfrac{1}{2}x + \tfrac{1}{2}x \sec^2 \tfrac{1}{2}x). \quad \ldots\ldots\ldots(15)$$

Ex. 5. Find approximately the abscissae of the points of intersection of the line $y=x+6$ with the curve $y=2\sin x$.

At a point of intersection we have $x+6=2\sin x$. The distribution of the roots of this equation may, at first, be indicated graphically. In Fig. 111, portions of the graphs of the functions $x+6$, $2\sin x$ are drawn, and there are seen to be three points of intersection P_1, P_2, P_3, whose abscissae are approximately $-4{\cdot}2$, $-6{\cdot}6$ and $-8{\cdot}0$. In applying theoretical considerations to the investigation of the roots, we first make use of Rolle's Theorem. Writing $f(x)=x+6-2\sin x$, we have $f'(x)=1-2\cos x$, which vanishes when $\cos x = \tfrac{1}{2}$, or when $x=2n\pi \pm \tfrac{1}{3}\pi$, ($n$ an integer).

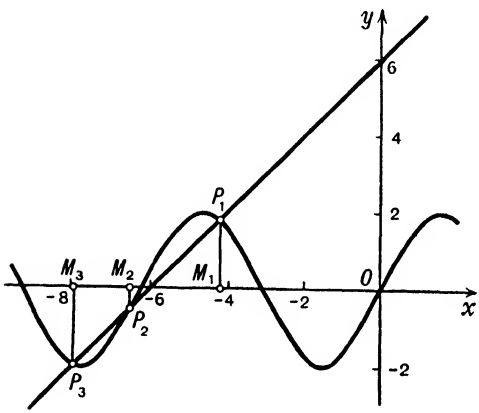

FIG. 111.

Since $|\sin x| \leqslant 1$, $f(x)$ is positive if $x > -4$ and negative if $x < -8$. The roots are thus to be sought in the intervals $[-8, -\tfrac{7}{3}\pi)$, $(-\tfrac{7}{3}\pi, -\tfrac{5}{3}\pi)$, $(-\tfrac{5}{3}\pi, -4]$. The following scheme shews that there are three roots.

x	-8	$-\tfrac{7}{3}\pi$	$-\tfrac{5}{3}\pi$	-4
$f(x)$	$-$	$+$	$-$	$+$

Approximations to these roots may be readily found with the aid of a set of tables in which the values of the sine are tabulated for angles expressed in radians. A large table in this connection is that of Hayashi,[1] in which, for a wide range of values of x, there are tabulated, among other functions,

[1] Hayashi, *Siebenstellige Tafeln der Kreis- und Hyperbelfunktionen*, Berlin, 1926. In using the Tables the student should be on his guard against misprints, especially in leading figures.

(i) the degree measure of x radians, (ii) $\sin x$, (iii) $\cos x$, (iv) $\tan x$, (v) $\text{arsin}\, x$, (vi) $\text{arcos}\, x$, (vii) $\text{artan}\, x$, all in parallel columns. The process of using such a table for the purpose described is as follows. Suppose that we consider the root in the interval $(-\frac{3}{2}\pi, -4)$ and start with the value $-4{\cdot}1$ of x. The value of $f(x)$ is then $-4{\cdot}1+6-2\times 0{\cdot}818$, or $0{\cdot}264$. If next we try $x=-4{\cdot}2$, we get $f(x)=0{\cdot}056$, while for $x=-4{\cdot}23$, we get $f(x)=-0{\cdot}0018$. The root is thus slightly greater than $-4{\cdot}23$.

Approximations to the remaining roots, determined in the same tentative way, are $-6{\cdot}58$, $-7{\cdot}98$. Closer approximations can now be obtained in each case by employing Newton's method, as described in Art. 117.

Ex. 6. Find the angles at which the curves $y=x^2$, $y=\cos x$ intersect.

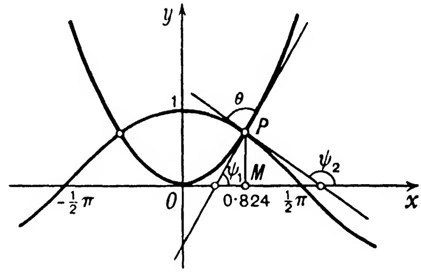

FIG. 112.

At a point of intersection we have $x^2=\cos x$, and on account of the symmetry, it is sufficient to consider the point for which x, y are both positive. An approximation to the root can be made in the tentative way described above if we employ Barlow's *Tables of Squares*, for example, in conjunction with Hayashi. We find, after some trials,

$$x=0{\cdot}824, \quad x^2=0{\cdot}6790, \quad \cos x=0{\cdot}6793, \quad \text{..................(16)}$$

and the value $0{\cdot}824$ of x is a sufficiently close approximation for the present purpose.

Referring to Fig. 112, we have $\theta=\psi_2-\psi_1$, where θ is the angle sought and $\tan\psi_1$, $\tan\psi_2$ are the gradients of the respective curves $y=x^2$, $y=\cos x$ at the point considered. Now

$$\tan\psi_1=2x=1{\cdot}648, \quad \tan\psi_2=-\sin x=-0{\cdot}734, \quad \text{..............(17)}$$

and, restricting ψ_1, ψ_2 to the range $(0, \pi)$, we have

$$\psi_1=\text{artan}\ 1{\cdot}648=1{\cdot}025, \quad \psi_2=\pi-\text{artan}\ 0{\cdot}734=2{\cdot}508, \quad \text{........(18)}$$

and hence $\theta=1{\cdot}483$ radians $=84^\circ\ 58'$, nearly.

Ex. 7. A rod AB, of length a, is pinned to a fixed point at A and is free to rotate in a vertical plane. To the end B is pinned a second rod BC, of length l (greater than a), and the end C is constrained to move along the horizontal line AC. If AB is made to rotate with constant angular velocity ω, find the velocity of C when AB makes an angle θ with the horizontal.

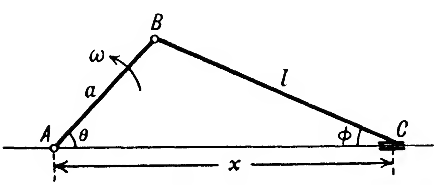

FIG. 113.

Denoting the distance AC (Fig. 113) by x and the angle ACB by φ, we have $x=a\cos\theta+l\cos\varphi$, and therefore

$$\frac{dx}{dt}=-a\sin\theta\frac{d\theta}{dt}-l\sin\varphi\frac{d\varphi}{dt}. \quad \text{........................(19)}$$

Also $a\sin\theta = l\sin\varphi$, so that

$$a\cos\theta\frac{d\theta}{dt} = l\cos\varphi\frac{d\varphi}{dt}. \qquad (20)$$

On eliminating $d\varphi/dt$, we get

$$\frac{dx}{dt} = -a\left(\sin\theta + \frac{\sin\varphi\cos\theta}{\cos\varphi}\right)\frac{d\theta}{dt} = -\omega a\frac{\sin(\theta+\varphi)}{\cos\varphi}. \qquad (21)$$

Again, since $\sin\varphi = \frac{a}{l}\sin\theta$, we have $\cos\varphi = \frac{1}{l}\sqrt{(l^2 - a^2\sin^2\theta)}$, and hence

$$\frac{dx}{dt} = -\omega a\sin\theta\left\{1 + \frac{a\cos\theta}{\sqrt{(l^2 - a^2\sin^2\theta)}}\right\}$$

$$= -\omega a\sin\theta\left\{1 + \frac{a}{l}\cos\theta\left(1 - \frac{a^2}{l^2}\sin^2\theta\right)^{-1/2}\right\}. \qquad (22)$$

To a first approximation, in which we treat a/l as negligible compared with 1, we have

$$\frac{dx}{dt} \approx -\omega a\sin\theta. \qquad (23)$$

If, next, we omit terms involving powers of a/l greater than the third, we may write

$$\left(1 - \frac{a^2}{l^2}\sin^2\theta\right)^{-1/2} \approx 1 + \frac{a^2}{2l^2}\sin^2\theta, \qquad (24)$$

and to this order of approximation we have

$$\frac{dx}{dt} \approx -\omega a\sin\theta\left\{1 + \frac{a}{l}\cos\theta\left(1 + \frac{a^2}{2l^2}\sin^2\theta\right)\right\}. \qquad (25)$$

If we now neglect a^3/l^3 as being small compared with 1, we get the approximation

$$\frac{dx}{dt} \approx -\omega a\sin\theta\left(1 + \frac{a}{l}\cos\theta\right) = -\omega a\left(\sin\theta + \frac{a}{2l}\sin 2\theta\right). \qquad (26)$$

This Example finds a practical application in the ordinary crank connecting-rod mechanism in a reciprocating engine, AB corresponding to the crank, BC to the connecting rod, and C to the piston.

143. Derivative of $\sin^\dagger x$ with respect to x.

Using the notation $\sin^\dagger x$ for the analytical function whose value is equal to the sine of an angle of x degrees, we have $\sin^\dagger x = \sin\frac{\pi x}{180}$, and therefore

$$\frac{d\sin^\dagger x}{dx} = \frac{d\sin\frac{\pi x}{180}}{dx} = \frac{\pi}{180}\cos\frac{\pi x}{180} = \frac{\pi}{180}\cos^\dagger x. \qquad (1)$$

The same awkward factor $\pi/180$ appears in the derivatives of each of the functions $\cos^\dagger x$, $\tan^\dagger x$, $\operatorname{cosec}^\dagger x$, $\sec^\dagger x$, $\cot^\dagger x$. It is avoided, as we have seen, when the analytical trigonometrical functions are defined in terms of the trigonometrical ratios of angles measured in *radians*, and the radian measure is introduced into Trigonometry for this reason.

144. Further Formulae for the Curvature of a Curve.

With the aid of the derivatives of the trigonometrical functions we can obtain some simple transformations of the ordinary formula for the curvature of a curve, as given in Chap. V. Starting with

$$\kappa=\frac{dg/dx}{(1+g^2)^{3/2}}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and introducing $\tan\psi$ for g, we get

$$\kappa=\frac{\sec^2\psi\dfrac{d\psi}{dx}}{(1+\tan^2\psi)^{3/2}}=\frac{\sec^2\psi\dfrac{d\psi}{dx}}{\pm\sec^3\psi}=\pm\cos\psi\frac{d\psi}{dx}=\pm\frac{d\sin\psi}{dx}, \quad\ldots\ldots(2)$$

where the positive or negative sign is to be taken according as ψ lies in the range $(0, \frac{1}{2}\pi)$ or $(\frac{1}{2}\pi, \pi)$. This is equivalent to the result $\kappa=\dfrac{d}{dx}\dfrac{g}{\sqrt{(1+g^2)}}$, given in Ex. 1, Art. 133. 1, p. 299.

Again, we have

$$\frac{d\cos\psi}{dy}=-\sin\psi\frac{d\psi}{dy}=-\sin\psi\cot\psi\frac{d\psi}{dx}=-\cos\psi\frac{d\psi}{dx}=-\frac{d\sin\psi}{dx}, \ldots(3)$$

and we may therefore also write

$$\kappa=\mp\frac{d\cos\psi}{dy}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

where the negative or positive sign is to be taken according as ψ is less or greater than $\frac{1}{2}\pi$. This is equivalent to the formula

$$\kappa=-\frac{d}{dy}\frac{1}{\sqrt{(1+g^2)}}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

We shall subsequently introduce a more general convention for the measurement of the slope ψ and with it a convention for the measurement of the curvature, whereby the formulae

$$\kappa=\frac{d\sin\psi}{dx}, \quad \kappa=-\frac{d\cos\psi}{dy}, \quad\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

hold in all cases; see Chap. XII, Art. 284.

The above simplified formulae were referred to in Art. 133. 1, Chap. V.

Ex. 1. *The parabola* $y^2=4ax$, $(y>0)$. We have $y\dfrac{dy}{dx}=2a$, whence

$$y=2a\cot\psi. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Differentiating with respect to y we get

$$1 = -2a \operatorname{cosec}^2 \psi \frac{d\psi}{dy} = -2a \operatorname{cosec}^3 \psi \sin \psi \frac{d\psi}{dy} = 2a \operatorname{cosec}^3 \psi \frac{d \cos \psi}{dy}, \quad \ldots\ldots(8)$$

and since ψ lies in the range $(0, \frac{1}{2}\pi)$, the formula (4) gives

$$\kappa = -\frac{\sin^3 \psi}{2a}. \quad \ldots\ldots(9)$$

145. Successive Derivatives of the Trigonometrical Functions.

145. 1. *Successive derivatives of* $\sin x$. Writing $f(x) = \sin x$, we have

$$f^{(1)}(x) = \cos x = \sin (x + \tfrac{1}{2}\pi), \quad \ldots\ldots(1)$$

as in Art. 142. 1, and, on repeating the process, we have

$$f^{(2)}(x) = \cos (x + \tfrac{1}{2}\pi) = \sin (x + \tfrac{2}{2}\pi). \quad \ldots\ldots(2)$$

It is plain that the successive derivatives are obtained by increasing the argument of the sine by $\frac{1}{2}\pi$ at each stage, and the general derivative is therefore given by

$$f^{(n)}(x) = \sin (x + \tfrac{1}{2}n\pi). \quad \ldots\ldots(3)$$

145. 2. *Successive derivatives of* $\cos x$. Writing $f(x) = \cos x$, we have, as in Art. 142. 2,

$$f^{(1)}(x) = -\sin x = \cos (x + \tfrac{1}{2}\pi), \quad \ldots\ldots(4)$$

and, as in the case of $\sin x$, the general derivative is found to be given by

$$f^{(n)}(x) = \cos (x + \tfrac{1}{2}n\pi). \quad \ldots\ldots(5)$$

145. 3. *Successive derivatives of the secondary trigonometrical functions.* No such simple formulae as the above exist for the general derivatives of the other trigonometrical functions. If derivatives of fairly low order only are required, it is usually simplest to carry out the successive differentiations directly, but otherwise it is more convenient to employ 'recurrence' formulae giving the successive derivatives linearly in terms of preceding derivatives.

(i) *Case of* $\tan x$. Writing $y = \tan x$, we can put $y \cos x = \sin x$, and, on equating the nth derivatives of the two sides of this equation, employing Leibniz's Theorem for the product on the left, we get

$$y^{(n)} \cos x + ny^{(n-1)} \cos (x + \tfrac{1}{2}\pi) + \frac{n(n-1)}{2!} y^{(n-2)} \cos (x + \tfrac{2}{2}\pi) + \ldots$$
$$+ y \cos (x + \tfrac{1}{2}n\pi) = \sin (x + \tfrac{1}{2}n\pi). \quad \ldots\ldots(6)$$

A second recurrence formula is obtained by starting with the equation $dy/dx = 1 + \tan^2 x$, or

$$y^{(1)} = 1 + yy, \quad \text{.................................(7)}$$

and equating the nth derivatives of each side, using Leibniz's Theorem for the product yy. We get

$$y^{(n+1)} = y^{(n)}y + ny^{(n-1)}y^{(1)} + \frac{n(n-1)}{2!}y^{(n-2)}y^{(2)} + \dots + ny^{(1)}y^{(n-1)} + yy^{(n)}. \quad (8)$$

By giving to n the values 1, 2, ... in turn, the successive derivatives are readily obtained from either of these formulae.

(ii) *Case of* $\sec x$. Writing $y = \sec x$, $z = \tan x$, we have

$$y^{(1)} = \sec x \tan x = yz, \quad \text{.........................(9)}$$

and on equating the nth derivatives, employing Leibniz's Theorem, we have

$$y^{(n+1)} = y^{(n)}z + ny^{(n-1)}z^{(1)} + \frac{n(n-1)}{2!}y^{(n-2)}z^{(2)} + \dots + ny^{(1)}z^{(n-1)} + yz^{(n)}, \quad (10)$$

from which the successive derivatives of $\sec x$ can be written down when those of $\tan x$ are known.

(iii) *Cases of* $\operatorname{cosec} x$, $\cot x$. Since $\operatorname{cosec} x = -\sec(x + \frac{1}{2}\pi)$ and $\cot x = -\tan(x + \frac{1}{2}\pi)$, the successive derivatives of the functions considered can be written down as soon as those of $\sec x$, $\tan x$ are known. Thus

$$\frac{d^n \operatorname{cosec} x}{dx^n} = -\frac{d^n \sec(x + \frac{1}{2}\pi)}{dx^n} = -\frac{d^n \sec u}{du^n}, \quad \frac{d^n \cot x}{dx^n} = -\frac{d^n \tan u}{du^n}, \quad (11)$$

where $u = x + \frac{1}{2}\pi$.

Ex. 1. Verify that the first five derivatives of $\tan x$ ($\equiv t$) are:

$$1 + t^2, \quad 2t + 2t^3, \quad 2 + 8t^2 + 6t^4, \quad 16t + 40t^3 + 24t^5, \quad 16 + 136t^2 + 240t^4 + 120t^6. \ \dots(12)$$

Ex. 2. *The function* $\cos^2 x$. Here we have

$$y^{(1)} = -\sin 2x = \cos(2x + \tfrac{1}{2}\pi), \quad y^{(2)} = -2\sin(2x + \tfrac{1}{2}\pi) = 2\cos(2x + \tfrac{2}{2}\pi), \ \dots\dots(13)$$

and the general derivative is plainly given by

$$y^{(n)} = 2^{n-1}\cos(2x + \tfrac{1}{2}n\pi). \quad \text{............................(14)}$$

Ex. 3. *The circle* $x^2 + y^2 = a^2$. Introducing a parameter θ such that $x = a\cos\theta$, $y = a\sin\theta$, as in Ex. 1, Art. 39, p. 57, we have

$$\frac{dx}{d\theta} = -a\sin\theta, \quad \frac{dy}{d\theta} = a\cos\theta, \quad \text{..........................(15)}$$

and hence

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{\cos\theta}{\sin\theta} = -\cot\theta. \quad \text{..........................(16)}$$

Also

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\frac{dy}{dx}=\frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}=\operatorname{cosec}^2\theta\,\frac{1}{-a\sin\theta}=-\frac{1}{a}\operatorname{cosec}^3\theta. \quad \text{.......(17)}$$

If P, (x, y), is a point on the circle (Fig. 114), the parameter θ is the measure of the angle xOP. If ψ is the slope at P, we have $\tan\psi = -\cot\theta$, so that

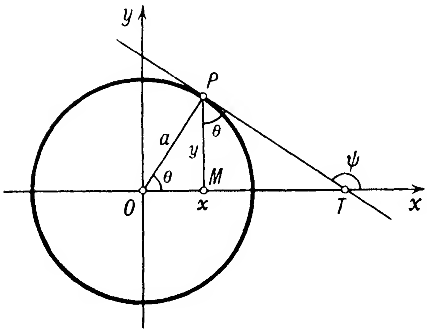

FIG. 114.

$$\psi=\theta+\tfrac{1}{2}\pi+n\pi, \quad (n \text{ an integer}). \quad (18)$$

The student should shew that (16), (17) lead to the formula $\kappa = \mp 1/a$ for the curvature, where the upper or lower sign is to be taken according as $\sin\theta$, or y, is positive or negative.

Ex. 4. Shew that at the point θ on the ellipse $x = a\cos\theta$, $y = b\sin\theta$, we have

$$\frac{dy}{dx}=-\frac{b}{a}\cot\theta, \quad \frac{d^2y}{dx^2}=-\frac{b}{a^2}\operatorname{cosec}^3\theta. \quad \text{.......................(19)}$$

[The parameter θ is the measure of the *eccentric angle* of the point concerned. The ellipse is obtained from the circle $x^2+y^2=a^2$ by orthogonal projection, as in Ex. 6, Art. 83. 4, p. 164, and this circle is called the *auxiliary circle*. If Q, on the circle, is the point of which P, on the ellipse, is the projection, the eccentric angle xOQ is equal to θ.]

Ex. 5. *The cycloid.* This is the path traced out by a carried point on the circumference of a circle which rolls along a straight line. Let Ox (Fig. 115) be the given line and P, (x, y), the position of the point when the circle has

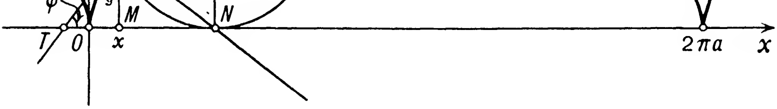

FIG. 115.

$x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$.

turned through an angle θ, P being initially at O. If C is the centre of the circle and NC is the ordinate to C, we have $ON = \text{arc } PN = a\theta$, the distance rolled over on the line Ox being equal to the arc rolled over on the circle. Hence

$$\left.\begin{aligned} x &= OM = ON - MN = ON - PL = a\theta - a\sin\theta, \\ y &= MP = NC - LC = a - a\cos\theta. \end{aligned}\right\} \quad \text{.............(20)}$$

Thus, for any position of P we have

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta), \quad \text{.......................(21)}$$

which are therefore parametric equations of the curve. The derivative is given by

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \cot\tfrac{1}{2}\theta. \quad \text{.......................(22)}$$

If ψ is the slope of the tangent PT at P, we therefore have $\tan\psi = \cot\frac{1}{2}\theta$, so that

$$\psi = \tfrac{1}{2}\pi - \tfrac{1}{2}\theta. \quad \text{....................................(23)}$$

It follows that if we join PN, the angle TPN is equal to $\frac{1}{2}\pi$, since the angle PNC is equal to $\frac{1}{2}\pi - \frac{1}{2}\theta$. Therefore TP is perpendicular to PN, that is, PN is the normal at P.

Again we have

$$\frac{d^2y}{dx^2} = \frac{1}{dx/d\theta}\frac{d}{d\theta}\frac{dy}{dx} = \frac{1}{dx/d\theta}\frac{d}{d\theta}\cot\tfrac{1}{2}\theta$$

$$= \frac{1}{a(1-\cos\theta)}\left(-\tfrac{1}{2}\operatorname{cosec}^2\tfrac{1}{2}\theta\right) = -\frac{1}{4a}\operatorname{cosec}^4\tfrac{1}{2}\theta. \quad \text{..............(24)}$$

From the formulae (22), (24) we find for the curvature at any point in the range $(0, 2\pi)$ of θ,

$$\kappa = \frac{d^2y/dx^2}{\{1+(dy/dx)^2\}^{3/2}} = \frac{-\frac{1}{4a}\operatorname{cosec}^4\frac{1}{2}\theta}{(1+\cot^2\frac{1}{2}\theta)^{3/2}} = -\frac{1}{4a}\operatorname{cosec}\tfrac{1}{2}\theta. \quad \text{.........(25)}$$

Hence $$\rho = 1/|\kappa| = 4a\sin\tfrac{1}{2}\theta. \quad \text{..............................(26)}$$

In the above Figure, PN is the normal to the cycloid at P and we have $PN = 2a\sin\frac{1}{2}\theta$. Hence, if I is the centre of curvature, I lies on PN, produced, and is such that $PI = 2PN$.

Ex. 6. *Simple harmonic motion.* When the independent variable of a sine function is taken to be the time t elapsed since a certain instant, the function is commonly said to have a *simple harmonic* variation, or to be a simple harmonic function of the time. Suppose, in particular, that the sine function gives the displacement x, from a certain origin, of a particle moving along a straight line, so that we may write

$$x = a\sin(\omega t + \gamma), \quad \text{................................(27)}$$

where ω, γ take the place of m, α in Art. 140. The particle is then said to have a *simple harmonic motion* (S.H.M.) in the line. Since, as t increases, $\sin(\omega t + \gamma)$ oscillates between -1 and $+1$, its values recurring after a time $2\pi/\omega$, the particle oscillates between $x = -a$ and $x = +a$ and resumes its position after that time. The S.H.M. is therefore also described as a S.H. oscillation of *amplitude* a (the half-range), *period* $2\pi/\omega$, and *phase-constant* γ. The *frequency*, that is, the number of oscillations per unit time, is the reciprocal, $\omega/2\pi$, of the period. The phase-constant gives the position x_0 of the particle when $t = 0$, namely $x_0 = a\sin\gamma$, and the times $-\gamma/\omega + n\pi/\omega$ at which the particle passes through the origin $x = 0$. The particle is at $x = a$ when $t = -\gamma/\omega + \frac{1}{2}\pi/\omega + 2n\pi/\omega$, and at $x = -a$ when $t = -\gamma/\omega - \frac{1}{2}\pi/\omega + 2n\pi/\omega$.

On differentiating equation (27), we get

$$\frac{dx}{dt} = \omega a \cos(\omega t + \gamma), \quad \frac{d^2x}{dt^2} = -\omega^2 a \sin(\omega t + \gamma) = -\omega^2 x. \quad \text{.........(28)}$$

Thus the velocity and acceleration have S.H. variations of amplitudes ωa, $\omega^2 a$, respectively. The acceleration is proportional to the displacement and of the opposite sign. It will be shewn later, (Chap. XX), that the converse statement is also true, that is, if the acceleration of a particle is proportional to its displacement and of the opposite sign, then the motion is S.H. The importance of S.H.M. in Physics depends on this result.

Ex. 7. *Connection of S.H.M. and uniform motion in a circle.* The equation (27) for S.H.M. in a straight line Ox may be written

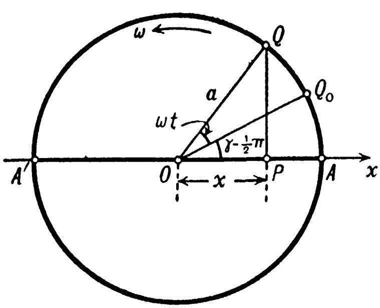

FIG. 116.

$$x = a \cos(\omega t + \gamma - \tfrac{1}{2}\pi). \quad \text{......(29)}$$

Suppose, now, that a radius OQ, of length a, rotates in a fixed plane through Ox and with uniform angular velocity ω (Fig. 116). Then if $\gamma - \frac{1}{2}\pi$ is the inclination of the radius to Ox when $t = 0$, the inclination at time t is $\omega t + \gamma - \frac{1}{2}\pi$, and hence the point P on Ox, for which $x = a \cos(\omega t + \gamma - \frac{1}{2}\pi)$, is the orthogonal projection of Q on that line. It follows that a S.H.M. may be defined as the motion of the projection on a fixed line of a point which moves uniformly in a circle. The velocity of P at any time is, according to (28), proportional to the ordinate PQ.

Ex. 8. *Simple harmonic motion produced by a rotating plate.*

(*a*) Two fixed planes LOx, MOx intersect along the line Ox (Fig. 117) and are inclined at an angle α. A radius OP, of length a, rotates with uniform angular velocity ω in the plane LOx so that its inclination to Ox at time t is $\omega t + \gamma$. The perpendicular PQ to the plane LOx intersects the plane MOx in Q. Shew that PQ has a S.H. vibration of period $2\pi/\omega$ and of amplitude $a \tan \alpha$.

Let the plane through PQ perpendicular to Ox meet that line in N. Then $\angle PNQ = \alpha$ and

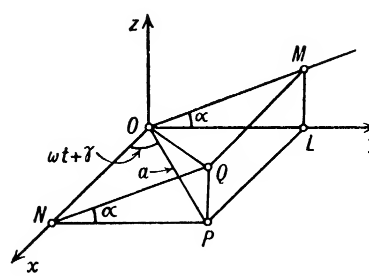

FIG. 117.

$$\left.\begin{aligned} PQ &= PN \tan \alpha \\ &= OP \sin POx \tan \alpha \\ &= a \tan \alpha \sin(\omega t + \gamma) \end{aligned}\right\} \quad \text{. ....(30)}$$

The last expression for PQ shews that its variation possesses the properties stated.

(*b*) If the planes LOx, MOx rotate together with uniform angular velocity ω about a fixed line Oz perpendicular to the plane LOx, while the points P, Q, in the respective planes, remain in a parallel fixed line, the length of PQ is

given by the same expression as in (*a*). Hence, since P is now a fixed point, Q performs a S.H.M. in a fixed line, the amplitude being $a \tan \alpha$ and the period $2\pi/\omega$.

(*c*) In the piston-slant (crankless) mechanism the geometry of (*a*) or (*b*) is utilized. The plane MOx is materialized as a plate (slant) which rotates uniformly about Oz, and PQ is a piston-rod which is made to oscillate in a fixed line by constraining the point Q of it to remain on the plane face of the slant. The motion of the point Q (or of any other point of the piston-rod) is then S.H.M.

(*d*) In (*a*) the line PQ lies always on a circular cylinder of radius a, on which LOx traces a normal section and MOx an oblique section. If this cylinder is developed into a plane, the normal section develops into a straight line and the oblique section into a sine-curve. Let $P'Q'$, in the developed cylinder, correspond to PQ. Then the motion of Q' is prescribed by making P' move with uniform velocity $a\omega$ along the straight line on which it lies while the point Q', which lies on a perpendicular line, moves along the sine-curve. These results follow at once from the facts that the distance of P' from an appropriate origin on its line of motion is $a(\omega t + \gamma)$, or x' say, and that

$$P'Q' = PQ = a \tan \alpha \sin(\omega t + \gamma) = a \tan \alpha \sin(x'/a). \quad \text{...........(31)}$$

146. Application of the General Intermediate Derivative Theorem to the Primary Trigonometrical Functions. Taylor's Theorem for sin x and cos x.

Taking 0 as the starting point and any finite value of x as the end point, the general statement of the $I.D_n.T.$ is

$$f(x) = f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n}{n!}f^{(n)}(\theta x), \quad \text{....(1)}$$

where $0 < \theta < 1$.

146. 1. *The* $I.D_n.T.$ *for* $\sin x$. Quoting from Art. 145. 1, we have

$$f^{(r)}(x) = \sin(x + \tfrac{1}{2}r\pi), \quad f^{(r)}(0) = \sin \tfrac{1}{2}r\pi, \quad \text{...............(2)}$$

and the theorem (1) thus becomes, in the present case,

$$\sin x = \sin 0 + x \sin \tfrac{1}{2}\pi + \frac{x^2}{2!}\sin \tfrac{2}{2}\pi + \ldots$$

$$+ \frac{x^{n-1}}{(n-1)!}\sin\frac{n-1}{2}\pi + \frac{x^n}{n!}\sin(\theta x + \tfrac{1}{2}n\pi). \quad \text{......(3)}$$

Two cases thus arise according as n is even or odd.

(i) *n even*. We here have

$$\sin\frac{n-1}{2}\pi = (-1)^{(n-2)/2}, \quad \sin(\theta x + \tfrac{1}{2}n\pi) = (-1)^{n/2}\sin\theta x. \quad \text{....(4)}$$

On omitting the vanishing terms, the formula (3) now becomes

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^{(n-2)/2} \frac{x^{n-1}}{(n-1)!} + (-1)^{n/2} \frac{x^n}{n!} \sin \theta x. \quad \ldots(5)$$

(ii) n *odd*. We now have

$$\left.\begin{array}{l} \sin \frac{n-2}{2}\pi = (-1)^{(n-3)/2}, \quad \sin \frac{n-1}{2}\pi = 0, \\ \qquad\qquad \sin(\theta x + \frac{1}{2}n\pi) = (-1)^{(n-1)/2} \cos \theta x, \end{array}\right\} \ldots(6)$$

and the formula (3) becomes

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^{(n-3)/2} \frac{x^{n-2}}{(n-2)!} + (-1)^{(n-1)/2} \frac{x^n}{n!} \cos \theta x. \quad \ldots(7)$$

It appears that odd powers only of x occur in the expansion, apart from the remainder term. The Taylor polynomials for $\sin x$ are thus all of odd degree, that of degree $2p-1$, in which there are p (non-vanishing) terms, being

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^{p-1} \frac{x^{2p-1}}{(2p-1)!}. \quad \ldots\ldots\ldots\ldots(8)$$

Equation (7) shews that the excess of $\sin x$ over this polynomial, that is, the remainder term in Taylor's Theorem with Remainder for $\sin x$, may be taken to be given by

$$R_{2p+1}(x) = (-1)^p \frac{x^{2p+1}}{(2p+1)!} \cos \theta x. \quad \ldots\ldots\ldots\ldots(9)$$

Now $|\cos \theta x| \leqslant 1$, and therefore

$$|R_{2p+1}(x)| \leqslant \frac{|x|^{2p+1}}{(2p+1)!}. \quad \ldots\ldots\ldots\ldots(10)$$

Thus the error made in using the polynomial (8) as an approximation to $\sin x$ is not greater than the magnitude of the last term in the Taylor polynomial for $\sin x$ of degree $2p+1$.

146. 2. *Taylor's Theorem for* $\sin x$. We now shew that these Taylor polynomials satisfy the condition for arbitrarily close approximation to the function $\sin x$, in other words, that the condition for the validity of Taylor's Theorem is satisfied. We consider the upper barrier to the remainder given by (10) and shew that a positive integer, P say, can be found [1] such that

$$\frac{|x|^{2p+1}}{(2p+1)!} < \varepsilon, \text{ (ε arbitrary), provided } p \geqslant P. \quad \ldots\ldots\ldots(11)$$

[1] The investigation given is intended merely to demonstrate the existence of suitable values of P, not to find the smallest value.

If $|x|<1$, it is plainly sufficient to choose P so that

$$|x|^{2P+1}<\varepsilon. \quad \text{.............................(12)}$$

If $|x|=1$, it is sufficient to choose P so that

$$\frac{1}{(2P+1)!}<\varepsilon. \quad \text{.............................(13)}$$

If $|x|>1$, let r be a number between 0 and 1 (for example $\frac{1}{2}$), and let k be the smallest positive integer such that $|x|/k \leqslant r$. Then if $2p+1 \geqslant k$, we have

$$\begin{aligned}\frac{|x|^{2p+1}}{(2p+1)!}&=\frac{|x|^{k-1}}{(k-1)!}\frac{|x|}{k}\frac{|x|}{k+1}\cdots\frac{|x|}{2p+1}\\&<\frac{|x|^{k-1}}{(k-1)!}\left\{\frac{|x|}{k}\right\}^{2p+2-k}\\&\leqslant\frac{|x|^{k-1}}{(k-1)!}r^{2p+2-k}, \quad \text{..........................(14)}\end{aligned}$$

where the factor $|x|^{k-1}/(k-1)!$ is independent of p. Since $r<1$, we can always choose P such that

$$r^{2P+2-k}<\frac{(k-1)!}{|x|^{k-1}}\varepsilon, \quad (\text{and } 2P+1>k), \quad \text{.........(15)}$$

and then if $p \geqslant P$, we have

$$\frac{|x|^{2p+1}}{(2p+1)!}<\frac{|x|^{k-1}}{(k-1)!}r^{2P+2-k}<\varepsilon. \quad \text{..................(16)}$$

Thus the conditions for Taylor's Theorem are satisfied by $\sin x$ for all values of x, and we have

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots+(-1)^{p-1}\frac{x^{2p-1}}{(2p-1)!}+\ldots. \quad \text{.......(17)}$$

The closeness of approximation of the earlier Taylor polynomials to $\sin x$ is illustrated in Fig. 118, which plainly shews the increase of the range within which a given degree of approximation is attained as the degree of the polynomial is increased.

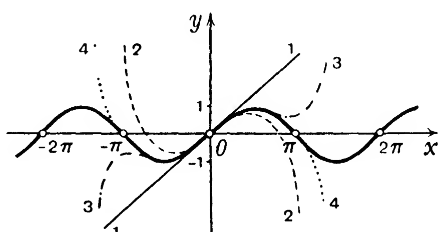

FIG. 118.

146. 3. *The I.D$_n$.T. for* $\cos x$. Quoting from Art. 145. 2, we have

$$f^{(r)}(x)=\cos(x+\tfrac{1}{2}r\pi), \quad f^{(r)}(0)=\cos\tfrac{1}{2}r\pi, \quad \text{.............(18)}$$

and the $I.D_n.T.$ for $\cos x$ is thus expressed by

$$\cos x = \cos 0 + x \cos \tfrac{1}{2}\pi + \frac{x^2}{2!}\cos \tfrac{2}{2}\pi + \dots + \frac{x^{n-1}}{(n-1)!}\cos \frac{n-1}{2}\pi + \frac{x^n}{n!}\cos(\theta x + \tfrac{1}{2}n\pi). \quad \text{......(19)}$$

Two cases again arise, according as n is even or odd.

(i) *n even.* In this case we have

$$\left.\begin{array}{l} \cos \dfrac{n-2}{2}\pi = (-1)^{(n-2)/2}, \quad \cos \dfrac{n-1}{2}\pi = 0, \\ \qquad\qquad \cos(\theta x + \tfrac{1}{2}n\pi) = (-1)^{n/2}\cos \theta x, \end{array}\right\} \quad \text{......(20)}$$

and the theorem becomes, on omitting the vanishing terms,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^{(n-2)/2}\frac{x^{n-2}}{(n-2)!} + (-1)^{n/2}\frac{x^n}{n!}\cos \theta x. \quad \text{...(21)}$$

(ii) *n odd.* We now have

$$\cos \frac{n-1}{2}\pi = (-1)^{(n-1)/2}, \quad \cos(\theta x + \tfrac{1}{2}n\pi) = (-1)^{(n+1)/2}\sin \theta x, \quad \text{...(22)}$$

and the theorem becomes

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^{(n-1)/2}\frac{x^{n-1}}{(n-1)!} + (-1)^{(n+1)/2}\frac{x^n}{n!}\sin \theta x. \quad \text{...(23)}$$

It thus appears that, apart from the remainder term, even powers only of x appear in the expansion, and the Taylor polynomial for $\cos x$ of degree $2p-2$, in which there are p (non-vanishing) terms, is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^{p-1}\frac{x^{2p-2}}{(2p-2)!}. \quad \text{...............(24)}$$

Equation (21) shews that the excess of $\cos x$ over this polynomial, that is, the remainder term, may be taken to be given by

$$R_{2p}(x) = (-1)^p \frac{x^{2p}}{(2p)!}\cos \theta x. \quad \text{....................(25)}$$

The magnitude of this term cannot exceed $x^{2p}/(2p)!$. It follows that the error made in using the polynomial (24) as an approximation to $\cos x$ is not greater than the magnitude of the last term in the Taylor polynomial for $\cos x$ of degree $2p$.

146. 4. *Taylor's Theorem for* $\cos x$. As in the case of $\sin x$, the conditions for Taylor's Theorem are satisfied by the function $\cos x$ for all values of x, and we have

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^{p-1}\frac{x^{2p-2}}{(2p-2)!} + \dots. \quad \text{.......(26)}$$

146. 5. The series (17), (26), as functions of x, may be taken as analytical definitions of the functions $\sin x$, $\cos x$. The use of infinite series for the purpose of *defining* a function and the deduction of the fundamental properties of the functions $\sin x$, $\cos x$ when defined in this way, cannot, however, be dealt with at the present stage ; compare the remarks of Art. 52.

146. 6. *Polynomial barriers for* $\sin x$, $\cos x$. Consider the above equation (7), taking x positive. Since $\cos \theta x$ lies in the range $(-1, 1)$, if we take $(n-1)/2$ to be an even integer, $2m$ say, we find that

$$\sin x \leqslant x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{x^{4m+1}}{(4m+1)!}, \quad \text{.....................}(27)$$

and if we take $(n-1)/2$ to be an odd integer, $2m+1$ say, we find that

$$\sin x \geqslant x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots - \frac{x^{4m-3}}{(4m-3)!}. \quad \text{.....................}(28)$$

If we use Cor. 1, Art. 98. 5, starting with the positive function $1 - \cos x$, the alternative sign of equality in these results is avoided.

In particular, if in (27) we put $m = 1$, we now find

$$\sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}, \quad \text{..............................}(29)$$

and if in (28) we put $m = 0$, we find

$$\sin x > x - \frac{x^3}{3!}. \quad \text{..................................}(30)$$

In the inequalities (27)-(30) the signs $>$, $<$ are plainly interchanged if x is negative.

Now consider equation (21). If we take $n/2$ to be an even integer, $2m$ say, we find

$$\cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{x^{4m}}{(4m)!}, \quad \text{..........................}(31)$$

and if we take $n/2$ to be an odd integer, $2m+1$ say, we find

$$\cos x > 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots - \frac{x^{4m+2}}{(4m+2)!}. \quad \text{.....................}(32)$$

In particular, if in (31) we put $m = 1$, we find

$$\cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!}, \quad \text{............................}(33)$$

and if in (32) we put $m = 0$, we find

$$\cos x > 1 - \frac{x^2}{2!}. \quad \text{..................................}(34)$$

Ex. 1. Shew that $\left|\dfrac{\sin x}{x} - 1\right| < \frac{1}{6}x^2$.

If we put $n=3$ in (7), we find

$$\sin x = x - \tfrac{1}{6}x^3 \cos \theta x, \quad \text{.............................(35)}$$

and therefore

$$\left| \frac{\sin x}{x} - 1 \right| = \tfrac{1}{6}x^2 \,|\cos \theta x\,| \leqslant \tfrac{1}{6}x^2. \quad \text{.....................(36)}$$

As in Sub-article 6, the alternative sign of equality can be omitted.

Ex. 2. *Approximate evaluation of* $\sin x$ *and* $\cos x$. On account of the periodic properties of the functions, it is obviously unnecessary to calculate $\sin x$, $\cos x$ for any value of x outside the range $(0, \tfrac{1}{2}\pi)$. Further, since $\sin x = \cos(\tfrac{1}{2}\pi - x)$, $\cos x = \sin(\tfrac{1}{2}\pi - x)$, the values of the functions can be determined from a knowledge of their values in the range $(0, \tfrac{1}{4}\pi)$. If we take x as great as 1, corresponding to an angle of about $57° \, 17'$, the error made in calculating sin 1 by the use of the Taylor polynomial of the ninth degree, $(p=5)$, is, according to (10), not greater than $1/11!$, or about[1] $2{\cdot}5 \times 10^{-8}$. Writing

$$\sin 1 \approx 1 - 1/3! + 1/5! - 1/7! + 1/9!, \quad \text{.......................(37)}$$

and working to eight decimal places, we find

$$\sin 1 \approx 0{\cdot}841\ 471\ 01, \quad \text{..(38)}$$

in which there may be an accumulated error of 4×10^{-8} due to approximating in the final decimal place in the reciprocals of the factorial terms. The total error cannot therefore exceed $6{\cdot}5 \times 10^{-8}$, or the result (38) cannot be in error by more than 1 unit in the seventh place. It is, in fact, correct to the seventh place.

In the same way, working with the approximation

$$\cos 1 \approx 1 - 1/2! + 1/4! - 1/6! + 1/8! - 1/10!, \quad \text{.................(39)}$$

we find

$$\cos 1 \approx 0{\cdot}540\ 302\ 3, \quad \text{...(40)}$$

correct to seven decimal places. Here the error due to the omission of terms other than those written is less than $1/12!$, that is, 2×10^{-9}.

Ex. 3. *Differentiation of* $(\sin x)/x$. We have

$$\frac{d}{dx}\frac{\sin x}{x} = \frac{\cos x}{x} - \frac{\sin x}{x^2}, \quad \text{..............................(41)}$$

which, with the aid of the $I.D_4.T.$ for $\cos x$ and the $I.D_5.T.$ for $\sin x$, can be written in the form

$$\frac{d}{dx}\frac{\sin x}{x} = \frac{1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 \cos \theta x}{x} - \frac{x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 \cos \theta' x}{x^2}$$

$$= -\tfrac{1}{3}x + \tfrac{1}{24}x^3(\cos \theta x - \tfrac{1}{5}\cos \theta' x). \quad \text{..........................(42)}$$

The derivative (41) is continuous except at $x=0$, where it is undefined, but (42) shews that it is potentially continuous there, its completing value being 0. This is also the derivative of the completed function $\left(\dfrac{\sin x}{x}\right)^*$ at the same point. For, from equation (35) we have

$$\frac{\left(\dfrac{\sin x}{x}\right)^* - 1}{x} = -\tfrac{1}{6}x \cos \theta x, \quad \text{..........................(43)}$$

[1] A table of $1/n!$ up to $n=15$ is given in Appendix II, p. 1072.

and since $|\frac{1}{6}x\cos\theta x| < \varepsilon$ if $|x| < 6\varepsilon$, (ε arbitrary), the function on the left-hand side of (43) is potentially continuous at $x=0$, the completing value there being 0.

Ex. 4. Shew that $\left(\dfrac{\sin x}{x}\right)^3 > \cos x$ if $|x| < 3$.

Since $(\sin x)/x$, $\cos x$ are both even functions, it is sufficient to consider only positive values of x. We then have $\dfrac{\sin x}{x} > 1 - \frac{1}{6}x^2$, $\cos x < 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$, and therefore

$$\begin{aligned}\left(\frac{\sin x}{x}\right)^3 - \cos x &> (1 - \tfrac{1}{6}x^2)^3 - (1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4)\\ &= \tfrac{1}{24}x \;\; - \tfrac{1}{216}x^6,\\ &> 0 \text{ if } x^2 < 9. \quad \text{.................................(44)}\end{aligned}$$

Actually, the first positive root of the equation $\left(\dfrac{\sin x}{x}\right)^3 = \cos x$ occurs in the neighbourhood of $x = 4{\cdot}61$, and the inequality stated thus holds for values of $|x|$ up to this magnitude.

Ex. 5. If the function y is defined in the neighbourhood of $x=0$ by the relation $y + \sin y = 2x$, so that $y=0$ when $x=0$, find the Taylor polynomial of the fifth degree in x for it.

This equation defines x as an up-way differentiable function of y in a neighbourhood of $x=0$, $y=0$, and dx/dy does not vanish when $y=0$. We have $1+\cos y = 2dx/dy$ and hence $dy/dx = 2/(1+\cos y)$. The derivatives of y with respect to x can then be found by differentiating the last equation repeatedly with respect to x and inserting at each step the value of dy/dx in terms of y. The values so found will be finite in the range with which we are concerned, since $1+\cos y$ does not vanish in the neighbourhood of $y=0$. There is therefore a Taylor polynomial of any finite degree for y with $x=0$ as starting point. Further, it is clear that all the terms in this polynomial will be odd powers of x, since each term in the given equation is odd.

We proceed to obtain the earlier Taylor polynomials by the method of *successive substitutions*, (compare Art. 129). Replacing $\sin y$ by its Taylor polynomial of the first degree, we get the equation

$$y + y = 2x, \quad \text{...(45)}$$

so that, to a first approximation, we have $y \approx x$.

If, next, we replace $\sin y$ by $y - \frac{1}{6}y^3$, we get

$$y + y = 2x + \tfrac{1}{6}y^3 \approx 2x + \tfrac{1}{6}x^3, \quad \text{...........................(46)}$$

so that, to a second approximation, we have $y \approx x + \frac{1}{12}x^3$.

Finally, replacing $\sin y$ by $y - \frac{1}{6}y^3 + \frac{1}{120}y^5$, we get

$$y + y = 2x + \tfrac{1}{6}y^3 - \tfrac{1}{120}y^5 \approx 2x + \tfrac{1}{6}(x + \tfrac{1}{12}x^3)^3 - \tfrac{1}{120}(x + \tfrac{1}{12}x^3)^5, \quad \text{........(47)}$$

giving, to a third approximation,

$$y \approx x + \tfrac{1}{12}x^3 + \tfrac{1}{60}x^5, \quad \text{...............................(48)}$$

on rejecting powers of x higher than the fifth.

146. 7. *Starting point different from zero.* The Taylor expansions for $\sin(x_1+h)$, $\cos(x_1+h)$ are brought back to those for $\sin x$, $\cos x$ with the aid of the Addition Theorems for $\sin x$, $\cos x$. Thus we have

$$\begin{aligned}\sin(x_1+h) &= \sin x_1 \cos h + \cos x_1 \sin h \\ &= \sin x_1\left\{1-\frac{h^2}{2!}+\frac{h^4}{4!}-\dots+(-1)^{p-1}\frac{h^{2p-2}}{(2p-2)!}+\dots\right\} \\ &\quad + \cos x_1\left\{h-\frac{h^3}{3!}+\frac{h^5}{5!}-\dots+(-1)^{p-1}\frac{h^{2p-1}}{(2p-1)!}+\dots\right\}, \quad (49)\end{aligned}$$

and similarly for $\cos(x_1+h)$. The student should verify that the above expression is the same as that obtained by the direct application of Taylor's Theorem to $\sin x$ with x_1 as starting point and h as increment.

147. Taylor Polynomials for the Secondary Trigonometrical Functions. The simplicity of the treatment of the Taylor polynomials and Taylor's Theorem for $\sin x$ and $\cos x$ depends on the existence of simple formulae for the general derivatives of these functions. In the case of the other trigonometrical functions, not only is there no simple formula for the derivative of any order n for a general value of x, as noted in Art. 145. 3, but there is no simple formula for the general derivative for the particular value 0 of x. It is only by introducing notations especially for the purpose [1] that we can give simple formulae for the nth polynomials in these cases, and although such notations can be used in the expression of the corresponding Taylor Series, they do not help us to discuss the remainders. For these reasons we confine ourselves here to shewing how to find the successive derivatives at the starting point 0 by means of recurrence formulae, and write down the earlier Taylor polynomials.

It may be observed that from the point of view of calculation, the discussion of series for the secondary functions is not urgent, seeing that their values can be obtained from those for $\sin x$ and $\cos x$. The earlier polynomials are, however, of value for the purpose of small corrections.

(i) *Case of* $\tan x$. The formula (6) of Art. 145. 3 gives, on putting $x=0$,

$$y_0^{(n)}-\frac{n(n-1)}{2!}y_0^{(n-2)}+\frac{n(n-1)(n-2)(n-3)}{4!}y_0^{(n-4)}-\dots=\sin\tfrac{1}{2}n\pi, \quad \dots(1)$$

where $y_0^{(r)}$ means the value of d^ry/dx^r when $x=0$. This equation

[1] This refers to the special sets of numbers known as the *Bernoulli* and *Euler Numbers*.

shews that all the even derivatives vanish when $x=0$, for, if n is even, $\sin \frac{1}{2}n\pi$ vanishes, while all the derivatives on the left-hand side are even. Giving to n the successive odd values 1, 3, 5, ... , we find

$$y_0^{(1)}=1, \quad y_0^{(3)}=2, \quad y_0^{(5)}=16, \quad y_0^{(7)}=272, \quad y_0^{(9)}=7936, \; \ldots\ldots\ldots(2)$$

and so on. Thus the Taylor polynomial for $\tan x$ of the ninth degree is

$$x+\tfrac{1}{3}x^3+\tfrac{2}{15}x^5+\tfrac{17}{315}x^7+\tfrac{62}{2835}x^9. \; \ldots\ldots\ldots(3)$$

The investigation of a remainder range even for a polynomial of this low degree is necessarily troublesome and is not carried out here.

From the point of view of *systematic* treatment of the Taylor series and polynomials of the trigonometrical functions, the case of $\tan x$ should come *after* that of $\cot x$, use being made of the formula

$$\tan x=\cot x-2\cot 2x \; \ldots\ldots\ldots(4)$$

to obtain the series for the former function. This procedure has the further advantage of leading immediately to the series for $\operatorname{cosec} x$, for we have the identity

$$\operatorname{cosec} x=\cot \tfrac{1}{2}x-\cot x. \; \ldots\ldots\ldots(5)$$

The case of $\sec x$, however, requires an independent discussion. The reason for the order actually adopted is that $\tan x$ and $\sec x$ satisfy the conditions for the Taylor expansion at $x=0$, whereas $\operatorname{cosec} x$ and $\cot x$ become infinite at that starting point, and so require special treatment.

(ii) *Case of* $\sec x$. We make use of the formula (10) of Art. 145. 3, where $y=\sec x$, $z=\tan x$. Putting $x=0$ and noting that $z_0^{(2r)}=0$, we get

$$y_0^{(n+1)}=ny_0^{(n-1)}z_0^{(1)}+\frac{n(n-1)(n-2)}{3!}y_0^{(n-3)}z_0^{(3)}+\ldots+y_0z_0^{(n)}. \; \ldots(6)$$

Since $y^{(1)}=yz$, so that $y_0^{(1)}=0$, this formula shews that all the odd derivatives of $\sec x$ vanish when $x=0$. For the first few even derivatives we find

$$y_0^{(2)}=1, \quad y_0^{(4)}=5, \quad y_0^{(6)}=61, \quad y_0^{(8)}=1385, \; \ldots\ldots\ldots(7)$$

and the Taylor polynomial of the eighth degree is therefore

$$1+\frac{1}{2!}x^2+\frac{5}{4!}x^4+\frac{61}{6!}x^6+\frac{1385}{8!}x^8. \; \ldots\ldots\ldots(8)$$

As in the case of $\tan x$, the investigation of a remainder range even for a polynomial of this degree is troublesome.

(iii) *Cases of* $\operatorname{cosec} x$, $\cot x$. As remarked above, we cannot take 0 as the starting point for Taylor's Theorem in these cases. However, the functions $x \operatorname{cosec} x$, $x \cot x$, the value of each of which we

define[1] to be 1 when $x=0$, can be expanded in powers of x provided that x lies in the range $[-\pi, \pi]$. The same is true of the functions $\operatorname{cosec} x - 1/x$, $\cot x - 1/x$, the value of each of which we define to be 0 when $x=0$.

To obtain a recurrence formula in the first case, write $y = x \operatorname{cosec} x$, which gives $y \sin x = x$ together with $y_0 = 1$. Differentiating n times, with the aid of Leibniz's Theorem, and putting $x=0$, we find

$$ny_0^{(n-1)} - \frac{n(n-1)(n-2)}{3!} y_0^{(n-3)} + \ldots + y_0 \sin \tfrac{1}{2}n\pi = 0. \quad \ldots\ldots\ldots(9)$$

All the odd derivatives plainly vanish when $x=0$, the first few even ones being given by $y_0^{(2)} = 1/3$, $y_0^{(4)} = 7/15$, $y_0^{(6)} = 31/21$, $y_0^{(8)} = 127/15$. We thus have the approximations

$$x \operatorname{cosec} x \approx 1 + \frac{x^2}{3!} + \frac{7}{3}\frac{x^4}{5!} + \frac{31}{3}\frac{x^6}{7!} + \frac{381}{5}\frac{x^8}{9!}, \quad \ldots\ldots\ldots(10)$$

$$\operatorname{cosec} x - \frac{1}{x} \approx \frac{x}{3!} + \frac{7}{3}\frac{x^3}{5!} + \frac{31}{3}\frac{x^5}{7!} + \frac{381}{5}\frac{x^7}{9!}. \quad \ldots\ldots\ldots(11)$$

In the case of $x \cot x$, it is convenient to write $y = x \cot x$, $z = \tan x$, so that $yz = x$ together with $y_0 = 1$, $z_0 = 0$. Differentiating n times, putting $x=0$ and remembering that $z_0^{(2r)} = 0$, we get

$$ny_0^{(n-1)} z_0^{(1)} + \frac{n(n-1)(n-2)}{3!} y_0^{(n-3)} z_0^{(3)} + \ldots = 0. \quad \ldots\ldots\ldots(12)$$

All the odd derivatives of y vanish when $x=0$, the first few even ones being given by $y_0^{(2)} = -2/3$, $y_0^{(4)} = -8/15$, $y_0^{(6)} = -32/21$, $y_0^{(8)} = -128/15$. We thus have the approximations

$$x \cot x \approx 1 - 2\frac{x^2}{3!} - \frac{8}{3}\frac{x^4}{5!} - \frac{32}{3}\frac{x^6}{7!} - \frac{384}{5}\frac{x^8}{9!}, \quad \ldots\ldots\ldots(13)$$

$$\cot x - \frac{1}{x} \approx -2\frac{x}{3!} - \frac{8}{3}\frac{x^3}{5!} - \frac{32}{3}\frac{x^5}{7!} - \frac{384}{5}\frac{x^7}{9!}. \quad \ldots\ldots\ldots(14)$$

147. 1. *Polynomial lower barrier for* $\tan x$, $(0 < x < \frac{1}{2}\pi)$. Consider the successive derivatives of $\tan x$. We have

$$\left.\begin{aligned} &y^{(1)} = \sec^2 x = 1 + y^2, \quad y^{(2)} = 2yy^{(1)} = 2y + 2y^3, \\ &y^{(3)} = (2+6y^2)y^{(1)} = (2+6y^2)(1+y^2) = 2 + 8y^2 + 6y^4, \end{aligned}\right\} \quad \ldots\ldots\ldots(15)$$

and so on. It is clear that the terms on the right-hand sides all have the positive sign so that the successive derivatives are all positive, since x has been taken to lie between 0 and $\frac{1}{2}\pi$ and y is therefore positive. It follows that

[1] The functions are potentially continuous at $x=0$ and have the completing value 1.

the Lagrange remainder $\dfrac{x^n}{n!}f^{(n)}(\theta x)$ in Taylor's Theorem for $\tan x$ is positive for all values of n and hence that $\tan x$ is greater than the Taylor polynomial of degree $n-1$. In particular, taking $n=10$ we have, from (3),

$$\tan x > x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \tfrac{17}{315}x^7 + \tfrac{62}{2835}x^9. \quad\text{.....................(16)}$$

The simpler result

$$\tan x > x + \tfrac{1}{3}x^3, \quad\text{...................................(17)}$$

which is an evident consequence of (16), is frequently employed.

Ex. 1. Obtain the Taylor polynomial of the seventh degree for $\tan x$ (i) from those for $\sin x$ and $\cos x$ by division, (ii) by the method of disposable coefficients.

[According to Theorem III, Art. 129, the nth polynomial for u/v is obtained by dividing the nth polynomial for u by that for v and rejecting terms in the quotient involving powers of x above the nth. In (i) we should divide the 7th degree polynomial for $\sin x$ by the 6th degree polynomial for $\cos x$ (the term in x^7 here vanishing). To apply the method (ii), assume a form $\alpha x + \beta x^3 + \gamma x^5 + \delta x^7$, and equate coefficients of powers of x on the two sides of the equation

$$(\alpha x + \beta x^3 + \gamma x^5 + \delta x^7)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}, \quad\text{..........(18)}$$

after rejecting powers of x above the 7th on the left-hand side.]

Ex. 2. Shew that $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$ if $0 < x < \tfrac{1}{2}\pi$.

For the range stated we have $\tan x > x + \tfrac{1}{3}x^3$, $\sin x > x - \tfrac{1}{6}x^3$, and hence $\dfrac{\tan x}{x} > 1 + \tfrac{1}{3}x^2$, $\dfrac{x}{\sin x} < \dfrac{1}{1 - \frac{1}{6}x^2}$. Therefore

$$\frac{\tan x}{x} - \frac{x}{\sin x} > 1 + \tfrac{1}{3}x^2 - \frac{1}{1 - \frac{1}{6}x^2} > \frac{\frac{1}{6}x^2 - \frac{1}{18}x^4}{1 - \frac{1}{6}x^2}. \quad\text{.................(19)}$$

The last expression written is positive if $x^2 < 3$, and the inequality is therefore proved when x lies in the range stated. See also Ex. 1, p. 345.

Ex. 3. *Taylor polynomials for* $\tan x$, $x \operatorname{cosec} x$ *deduced from those for* $x \cot x$. Making use of the above identities (4), (5), and employing polynomials of the sixth degree, for example, we have

$$x \cot x - 2x \cot 2x \approx \left(1 - \frac{x^2}{3} - \frac{x^4}{45} - \frac{2x^6}{945}\right) - \left(1 - \frac{4x^2}{3} - \frac{16x^4}{45} - \frac{128x^6}{945}\right)$$

$$= x^2 + \frac{1}{3}x^4 + \frac{2}{15}x^6, \quad\text{...(20)}$$

and division by x gives the Taylor polynomial for $\tan x$ of the fifth degree.

Again

$$2(\tfrac{1}{2}x \cot \tfrac{1}{2}x) - x \cot x \approx 2\left(1 - \frac{1}{3}\frac{x^2}{4} - \frac{1}{45}\frac{x^4}{16} - \frac{1}{945}\frac{x^6}{32}\right) - \left(1 - \frac{x^2}{3} - \frac{x^4}{45} - \frac{2x^6}{945}\right)$$

$$= 1 + \frac{x^2}{3!} + \frac{7}{3}\frac{x^4}{5!} + \frac{31}{3}\frac{x^6}{7!}, \quad\text{.................................(21)}$$

which is the Taylor polynomial of the sixth degree for $x \operatorname{cosec} x$.

148. Application of the Method of Proportional Parts to Tables of Trigonometrical Functions.

In the largest tables of trigonometrical functions which are employed for ordinary purposes, the values of the functions are given to seven places of decimals. The argument is most often in degree measure, the range being from 0° to 90°, and the common difference is not greater than 1 minute. We shall see that for the primary trigonometrical functions a smaller common difference than 1 minute is not required for the satisfactory application of the method of proportional parts for the purpose of interpolating in a seven-figure table, but this is true of the secondary functions only in part of the range. The consideration of tables of logarithms of the trigonometrical functions comes later.

For analytical purposes it is very convenient to employ tables in which the functions $\sin x$, $\cos x$ are given with x as argument. In the *Smithsonian Tables*, such tables are given to five places of decimals with a common difference of argument 0·0001 for the range (0, 0·1000) and with a common difference 0·001 for the range (0·100, 1·600). Tables of the same kind are given by Hayashi [1] with various ranges and to many places of decimals.

Occasionally a table in degree measure is made more convenient for analytical purposes by the addition of the equivalent of the arguments in radians, as in Lodge's edition of Bremiker's *Logarithms*. In applying the theory of proportional parts we can, of course, always imagine this done, the corresponding common difference being then the fraction $\pi/180$ of that for the argument in degrees.

148. 1. *Tables of sines and cosines.* Considering first the Smithsonian table for $\sin x$, mentioned above, the formula $\frac{1}{8}h^2\,|\,f''(x)\,|_{max}$ for the uncertainty of an interpolated value becomes

$$\tfrac{1}{8}h^2\,|\sin x\,|_{max}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

which, for the first part of the table, where $h=10^{-4}$, becomes $\frac{1}{8}\times 10^{-8}\,|\sin x\,|_{max}$. The uncertainty is therefore less than $\frac{1}{8}\times 10^{-8}$ in this part, and is negligible in a five-figure table. For the second part of the table, in which $h=10^{-3}$, the uncertainty is less than $\frac{1}{8}\times 10^{-6}$ and is still negligible in a five-figure table. The same conclusions apply to the table of $\cos x$.

In the tables of Hayashi, the values of $\sin x$ (and $\cos x$) are tabulated for values of x ranging from 0 to 50. In the part of the

[1] See footnote on p. 324.

tables extending from 0·100 to 2·999, the constant difference of argument is 0·001 and the values of the functions are given to twelve places of decimals. The *M.P.P.* cannot be here employed to interpolate correctly to this degree of accuracy and special interpolation formulae are necessary. See, for example, p. 277 of Hayashi.

Taking in the next place a table in which the argument is in degree measure, the common difference being 1 minute, the equivalent difference in a table of $\sin x$ (or $\cos x$) is $\pi/(180\times 60)$, or 0·0002909, nearly. The uncertainty in this case is therefore not greater than

$$\frac{1}{8}\left(\frac{\pi}{180\times 60}\right)^2 \approx 10^{-8}, \quad \text{.........................(2)}$$

and is negligible in a seven-figure table.

148. 2. *Table of tangents.* It will be sufficient to consider a table in degree measure with a common difference of 1 minute, corresponding to a difference of $\pi/(180\times 60)$ in an equivalent table of $\tan x$. Here we have $f(x)=\tan x$, $f''(x)=2\tan x\,(1+\tan^2 x)$, and the uncertainty of interpolation is given by

$$\frac{1}{8}\left(\frac{\pi}{180\times 60}\right)^2 \times 2\tan x_1(1+\tan^2 x_1), \quad \text{................(3)}$$

where x_1 is the upper terminal of the sub-range in which the interpolation is made. Since $\tan x$ increases indefinitely when x increases to $\frac{1}{2}\pi$, the expression (3) also increases indefinitely, and the *M.P.P.* is then not applicable.

If $x_1 \leqslant \frac{1}{4}\pi$, so that $\tan x_1 \leqslant 1$, the uncertainty is not greater than $\frac{1}{2}\left(\frac{\pi}{180\times 60}\right)^2$, or about $4{\cdot}23\times 10^{-8}$, which is just negligible in a seven-figure table (apart from accumulation of errors).

The details for the other secondary trigonometrical functions are left to the student as exercises.

149. Miscellaneous Examples involving Derivatives of the Direct Trigonometrical Functions.

149. 1. *Inequalities.* *Ex.* 1. Shew that $\dfrac{\tan x}{x} - \dfrac{x}{\sin x} > 0$ if $0 < x < \frac{1}{2}\pi$.

We here give an alternative proof to that of Ex. 2, Art. 147, p. 343, making use of Cor. I to Art. 98. 5. It will be observed that it is sufficient to demonstrate the inequality $\sec x - \cos x - x^2 > 0$ in the range stated. Denoting this latter function by $f(x)$, we have the results

$$\left.\begin{aligned} f(0)=0,\quad f^{(1)}(x) &= \sec x\tan x + \sin x - 2x,\\ f^{(2)}(x) &= \sec x\tan^2 x + \sec^2 x + \cos x - 2,\\ f^{(3)}(x) &= \sec x\tan^3 x + 5\sec^3 x\tan x - \sin x.\end{aligned}\right\} \quad \text{............(1)}$$

The signs of $f^{(1)}(x)$, $f^{(2)}(x)$ in the range $[0, \frac{1}{2}\pi]$ are not immediately decided by inspection, but we observe that $f^{(3)}(x)$ is positive in the range, for it can be written in the form

$$f^{(3)}(x) = \frac{\sin x\,(\sin^2 x + 5 - \cos^4 x)}{\cos^4 x}. \qquad\qquad (2)$$

Since $f^{(2)}(0) = 0$, we conclude that $f^{(2)}(x) > 0$ in the range $[0, \frac{1}{2}\pi]$, and since $f^{(1)}(0) = 0$, we conclude similarly that $f^{(1)}(x) > 0$ in the same range. A final application of the Corollary leads to the desired result.

Ex. 2. Shew in the same way that the functions $\left(\frac{\sin x}{x}\right)^2 - \cos x$, $\left(\frac{\sin x}{x}\right)^3 - \cos x$ are positive, but that $\left(\frac{\sin x}{x}\right)^4 - \cos x$ is negative in neighbourhoods of $x = 0$. Shew that for the first function the neighbourhood is approximately $(-4{\cdot}75, 4{\cdot}75)$, and for the third, $(-1{\cdot}19, 1{\cdot}19)$.

149. 2. *Small corrections.* *Ex.* 3. The sides a, b, c of a triangle ABC are measured and the angles calculated. Find the error in the angle A due to errors δa, δb, δc in the sides.

According to the statement of Art. 104. 1, the whole error in A is obtained by calculating the errors which would arise on the assumption that only one measurement at a time was subject to error, and adding. Here we employ the formula $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$. If we regard b, c as correct, the contribution to δA arising from δa is given by

$$\frac{dA}{da}\delta a = \frac{a}{bc \sin A}\delta a = \frac{a}{2S}\delta a, \qquad\qquad (3)$$

where S is the area. Regarding c, a as correct, the contribution arising from δb is given by

$$\frac{dA}{db}\delta b = -\frac{a^2 + b^2 - c^2}{2b^2 c \sin A}\delta b = -\frac{a \cos C}{bc \sin A}\delta b = -\frac{a \cos C}{2S}\delta b. \qquad\qquad (4)$$

Similarly, regarding a, b as correct, the contribution arising from δc is $-\frac{a \cos B}{2S}\delta c$. We therefore have the result

$$\delta A \approx \frac{a}{2S}(\delta a - \cos C\,\delta b - \cos B\,\delta c). \qquad\qquad (5)$$

If the magnitudes of the errors δa, δb, δc do not exceed ε, the greatest possible error in A is given by

$$|\,\delta A\,| \approx \frac{a}{2S}(1 + |\cos C\,| + |\cos B\,|)\varepsilon, \qquad\qquad (6)$$

where, on the right-hand side, we have added the greatest magnitudes of the errors due to the three sides.

149. 3. *Maxima and minima.* *Ex.* 4. Investigate the extreme values of the function $\tan x - 2 \sin x$.

Denoting the function by $f(x)$, we have

$$f'(x) = \sec^2 x - 2\cos x, \quad f''(x) = 2\sec^2 x \tan x + 2\sin x = 2\sin x\,(\sec^3 x + 1). \qquad (7)$$

The stationary condition $f'(x)=0$ gives $\cos^3 x = 1/2$, or $x = 2n\pi \pm 0{\cdot}6539$. When $\sec^3 x = 2$, $f''(x) = 6\sin x$, which is positive when $x = 2n\pi + 0{\cdot}6539$ and negative when $x = 2n\pi - 0{\cdot}6539$. The former values of x are thus associated with minimum values of $f(x)$, and the latter with maximum values.

Further we may write $f(x) = \sin x\,(\sec x - 2)$, and at the extreme values we have $f(x) = \sin x\,(2^{1/3} - 2) = -0{\cdot}7401 \sin x$. The maximum values are thus each equal to $-0{\cdot}7401\sin(-0{\cdot}6539)$, i.e. to $0{\cdot}7401 \times 0{\cdot}6083$, or to $0{\cdot}4502$, while the minimum values are each equal to $-0{\cdot}4502$.

Ex. 5. Two points A, B and a line CD are coplanar, the points being on opposite sides of the line. If P is a variable point on the line, find the position of P when $AP + \mu PB$ is a minimum, μ being a given positive constant.

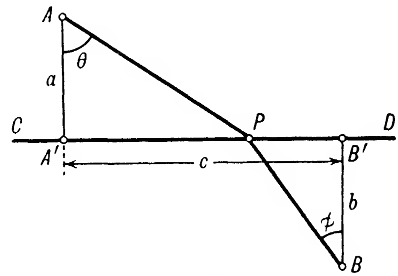

FIG. 119.

Referring to Fig. 119, we have, on denoting the function in question by l,

$$l = a\sec\theta + \mu b \sec\varphi, \quad \ldots\ldots(8)$$

and also

$$c = a\tan\theta + b\tan\varphi, \quad \ldots\ldots(9)$$

where c is the (fixed) distance $A'B'$. From these we get, on differentiating with respect to θ,

$$\frac{dl}{d\theta} = a\sec\theta\tan\theta + \mu b\sec\varphi\tan\varphi\frac{d\varphi}{d\theta}, \quad a\sec^2\theta + b\sec^2\varphi\frac{d\varphi}{d\theta} = 0, \ldots\ldots(10)$$

and, on eliminating $d\varphi/d\theta$, we obtain

$$\frac{dl}{d\theta} = a\sec^2\theta\,(\sin\theta - \mu\sin\varphi). \quad \ldots\ldots(11)$$

Also

$$\frac{d^2 l}{d\theta^2} = 2a\sec^2\theta\tan\theta\,.\,(\sin\theta - \mu\sin\varphi) + a\sec^2\theta\left(\cos\theta - \mu\cos\varphi\frac{d\varphi}{d\theta}\right). \quad \ldots(12)$$

When $dl/d\theta = 0$ we have $\sin\theta = \mu\sin\varphi$, and since $d\varphi/d\theta < 0$, it follows that then $d^2l/d\theta^2 > 0$. There is thus a minimum value of l when $\dfrac{\sin\theta}{\sin\varphi} = \mu$.

This result has a practical application in connection with the refraction of light at the interface between two media of different optical densities. It is known in Optics as *Snell's Law*.

149. 4. *Concavity, contact.* *Ex.* 6. Determine the general form and the points of contraflexure of the curve $y = x - \sin 2x$.

We have $f^{(1)}(x) = 1 - 2\cos 2x$, $f^{(2)}(x) = 4\sin 2x$, $f^{(3)}(x) = 8\cos 2x$. The second derivative is positive whenever $\sin 2x$ is positive, that is, when x lies in any one of the ranges given by $[n\pi, (n+\frac{1}{2})\pi]$, and in these intervals the curve is concave upwards. In the remaining intervals it is convex upwards, and there are points of contraflexure where $x = \frac{1}{2}n\pi$, since for these values of x we have $f^{(2)}(x) = 0$, $f^{(3)}(x) \neq 0$.

To find the gradients at the points of contraflexure, we observe that if n is even, $\cos n\pi = 1$, so that $f^{(1)}(x) = 1 - 2 = -1$, while if n is odd, $\cos n\pi = -1$ and then $f^{(1)}(x) = 3$. Portion of the curve is shewn in Fig. 120, the broken lines being the curves $y = x$, $y = -\sin 2x$, the addition of the (algebraic) ordinates of which yields those of the required curve.

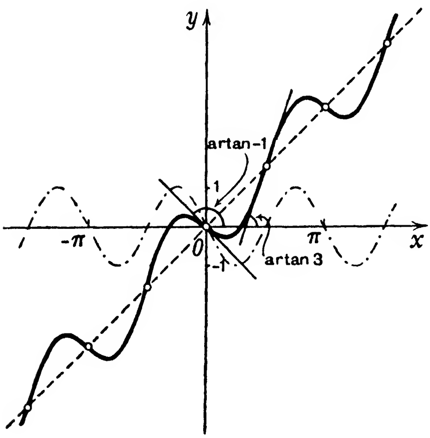

FIG. 120.
$y = x - \sin 2x$.

Ex. 7. Shew that the contact of the parabola $(x - a\pi)^2 - 8a(2a - y) = 0$ with the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ at the point for which $\theta = \pi$ is of the third order.

[If there is contact of the third order, the first three derivatives for the two curves will be respectively equal at the point, but the fourth derivatives will differ. It will be observed that the curves do not cross at the point of contact.]

149. 5. *Approximate solution of an equation by Newton's method.* *Ex.* 8. Investigate the roots of the equation $\tan x = x$.

One root is evidently $x = 0$. To determine the distribution of the other roots we may employ Rolle's Theorem. Writing $f(x) = \tan x - x$, we have $f'(x) = \sec^2 x - 1 = \tan^2 x$, and this vanishes when $x = n\pi$. Since $f(x)$ has infinite discontinuities when $x = \frac{2n+1}{2}\pi$, we are led to seek the roots in the sub-ranges given by $\left[\frac{2n-1}{2}\pi, n\pi\right]$, $\left[n\pi, \frac{2n+1}{2}\pi\right]$.

Now as x converges to $\frac{2n-1}{2}\pi$ from above, $f(x) \to -\infty$, while as x converges to $\frac{2n+1}{2}\pi$ from below, $f(x) \to +\infty$. Further, $f(n\pi) = -n\pi$, and hence there is only one root in a range given by $\left[\frac{2n-1}{2}\pi, \frac{2n+1}{2}\pi\right]$, and it occurs in the upper (lower) sub-range if n is positive (negative).

Consider now the root lying in the range $(\pi, \frac{3}{2}\pi]$, that is, between 3·14 and 4·71. With the aid of the tables we find $\tan 4{\cdot}49 = 4{\cdot}4223$, $\tan 4{\cdot}50 = 4{\cdot}6377$, and so the root lies between 4·49 and 4·50. Now $f''(x) = 2\sec^2 x \tan x$, and for the values of x considered, the sign of this is positive. Since $f(4{\cdot}50)$ is also positive, we may take the approximation $x_1 = 4{\cdot}50$ to the root as starting point in the application of Newton's method of approximating to a root, a procedure which, according to Art. 118, ensures success of the iteration process from the beginning. A second approximation to the root is now given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 4{\cdot}50 - \frac{\tan 4{\cdot}50 - 4{\cdot}50}{\tan^2 4{\cdot}50} = 4{\cdot}50 - \frac{0{\cdot}1377}{(4{\cdot}6377)^2}$$

$$= 4{\cdot}50 - 0{\cdot}0064 = 4{\cdot}4936. \quad \text{.............................(13)}$$

Now repeat the process, taking 4·4936 as starting point. We have

$$\left.\begin{aligned}\tan 4{\cdot}4936 &= \tan(4{\cdot}4936-\pi)=\tan 1{\cdot}352007=4{\cdot}497457,\\ f(x)&=0{\cdot}003857,\quad f'(x)=\tan^2 x=20{\cdot}23,\end{aligned}\right\}\quad\ldots\ldots\ldots(14)$$

and the Newton correction is $-0{\cdot}003857/20{\cdot}23$, or $-0{\cdot}00019$, so that

$$x_3=4{\cdot}49341. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

Finally, taking 4·49341 as starting point, we get

$$\tan x=4{\cdot}4934205,\quad f(x)=0{\cdot}0000105,\quad f'(x)=20{\cdot}19,\ \ldots\ldots\ldots\ldots(16)$$

and the Newton correction is $-0{\cdot}0000105/20{\cdot}19$, or $-0{\cdot}0000005$, so that

$$x_4=4{\cdot}4934095. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17)$$

Now, according to Art. 119, the error involved in the use of the Newton correction at the last step is at most of order $\frac{1}{2}\frac{|f''(x)|}{|f'(x)|^3}y^2$, that is, $\frac{\tan x(1+\tan^2 x)}{(\tan^2 x)^3}y^2$, which is here equal to $\frac{4{\cdot}49\times 21{\cdot}19}{(20{\cdot}19)^3}\times 10^{-10}$, or about $\frac{1}{8}\times 10^{-10}$. Thus the value of x_4 is the required root correct to seven places of decimals.

The first sixteen (positive) roots of the given equation are quoted in Jahnke-Emde, *Tables of Functions*, p. 30. Silberstein gives these and nineteen more in his *Synopsis of Applicable Mathematics*. The negative roots are equal in magnitude to the positive roots.

The given equation and the more general form $\tan x = mx$ are of importance in connection with the theory of the Conduction of Heat.

The distribution of the roots of the equation $\tan x = x$ may be investigated graphically by finding the abscissae of the points of intersection of the graphs $y=\tan x$, $y=x$. Portions of these curves are shewn in Fig. 121, where it is seen that for increasing values of $|x|$ the roots approximate more and more closely to half-integral multiples of π.

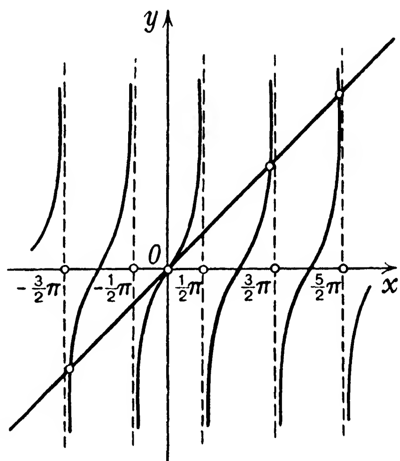

FIG. 121.

150. The Inverse Trigonometrical Functions.

The inversion of the trigonometrical functions has been considered in Art. 37. 1, Chap. I, as an illustration of the general theory of inversion. Starting, for example, with the relation $y=\sin x$, and confining ourselves to the range $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ of x, we have inverted it into the form $x=\operatorname{arsin} y$, where $\operatorname{arsin} y$ is a single-valued function of y for the range $(-1, 1)$ of that variable. We have also seen that there is adjoined to the function $\operatorname{arsin} y$ a series of functions defined

by $n\pi+(-1)^n \operatorname{arsin} y$, for integral values of n, each of which corresponds to the inversion of the relation $y=\sin x$ in a range $\left(\frac{2n-1}{2}\pi, \frac{2n+1}{2}\pi\right)$.

It is clear that in the range $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ of x the relations $y=\sin x$, $x=\operatorname{arsin} y$ are equivalent, each of the functions direct sine and inverse sine being the inverse of the other. The rôle of the variables can, of course, be interchanged, so that x is used as the independent variable in the inverse function, and then $y=\operatorname{arsin} x$, $x=\sin y$ form the pair of equivalent relations in the range $(-1, 1)$ of x. Similar remarks apply to the other inverse trigonometrical functions.

We now recall the definitions of the inverse trigonometrical functions and the salient facts concerning them, using the standard notation.

150. 1. *The inverse sine.* We have $y=\operatorname{arsin} x$, $x=\sin y$, the range of x being $(-1, 1)$ and that of y being $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. The adjoined functions are defined by $n\pi+(-1)^n \operatorname{arsin} x$. The graph of the function $\operatorname{arsin} x$ is most easily obtained by taking the geometrical image of the graph $y=\sin x$, for the range $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ of x, in the line bisecting the angle xOy, as described in Art. 36; see Fig. 122, in which $A'B'$ is the graph of $\operatorname{arsin} x$ and AB that of $\sin x$ in the range stated.

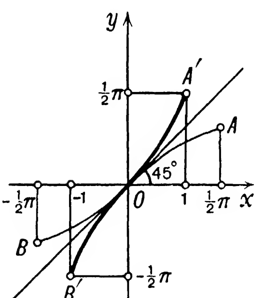

FIG. 122.
$y=\operatorname{arsin} x$.

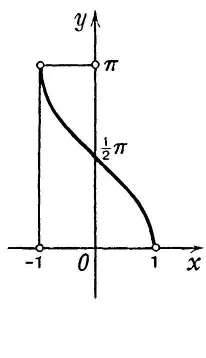

FIG. 123.
$y=\operatorname{arcos} x$.

150. 2. *The inverse cosine.* Here we have $y=\operatorname{arcos} x$, $x=\cos y$, the range of x being $(-1, 1)$ and that of y being $(0, \pi)$. The adjoined functions are defined by $2n\pi\pm\operatorname{arcos} x$. The graph is shewn in Fig. 123.

150. 3. *The inverse tangent.* Here we have $y=\operatorname{artan} x$, $x=\tan y$, the range of x being $[-\infty, \infty]$ and that of y being $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$. The adjoined functions are defined by $n\pi+\operatorname{artan} x$. The graph is shewn in Fig. 124.

150. 4. *The inverse cosecant.* We have the relations $y=\operatorname{arcosec} x$, $x=\operatorname{cosec} y$, the ranges for x being $[-\infty, -1)$, $(1, \infty]$ and the cor-

responding ranges for y being $[0, -\frac{1}{2}\pi)$, $(\frac{1}{2}\pi, 0]$. The adjoined functions are defined by $n\pi + (-1)^n$ arcosec x. The graph is shewn in Fig. 125.

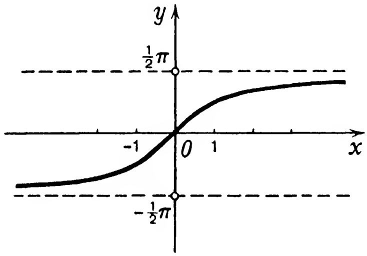

FIG. 124.
$y = \text{artan}\, x$.

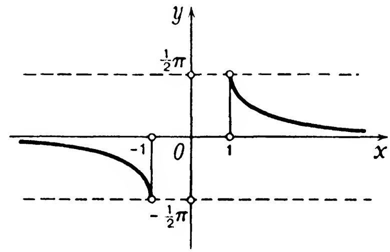

FIG. 125.
$y = \text{arcosec}\, x$.

150. 5. *The inverse secant.* We have $y = \text{arsec}\, x$, $x = \sec y$, the ranges for x being $[-\infty, -1)$, $(1, \infty]$ and the corresponding ranges for y being $[\frac{1}{2}\pi, \pi)$, $(0, \frac{1}{2}\pi]$. The adjoined functions are defined by $2n\pi \pm \text{arsec}\, x$. The graph is shewn in Fig. 126.

150. 6. *The inverse cotangent.* Here $y = \text{arcot}\, x$, $x = \cot y$. The range of x is $[-\infty, \infty]$; corresponding to the range $[-\infty, 0]$ of x, y has the range $[0, -\frac{1}{2}\pi]$, and corresponding to the range $(0, \infty]$ of x, y has the range $(\frac{1}{2}\pi, 0]$. The adjoined functions are given by $n\pi + \text{arcot}\, x$. The graph is shewn in Fig. 127.

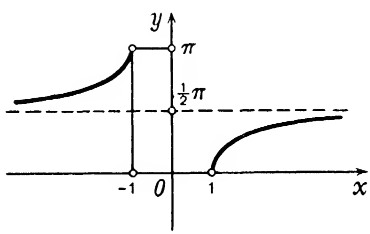

FIG. 126.
$y = \text{arsec}\, x$.

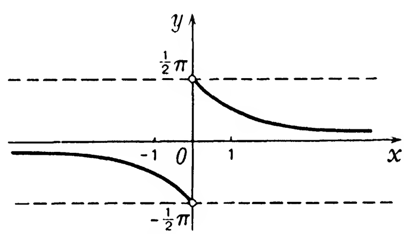

FIG. 127.
$y = \text{arcot}\, x$.

150. 7. *Relations between the inverse functions.*[1] The relations which we now state are inverse forms of relations already stated for the direct functions. Analytical proofs are supplied in three cases. The student should investigate proofs of the other results on similar lines.

$$\text{(i) arcosec}\, x = \text{arsin}\,\frac{1}{x}, \quad \text{arsec}\, x = \text{arcos}\,\frac{1}{x}, \quad \text{arcot}\, x = \text{artan}\,\frac{1}{x}. \quad \ldots(1)$$

These are inverse forms of the relations cosec $x = 1/\sin x$, etc.

[1] The student should observe that there are restrictions on the various arguments introduced, corresponding to the restricted ranges of definition of the inverse functions.

To prove the first relation, observe that if $-1 \leqslant 1/x \leqslant 1$, we can put $1/x = \sin\theta$, where $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$, and hence $x = \operatorname{cosec}\theta$. Then $\operatorname{arcosec} x = \theta = \operatorname{arsin}(1/x)$.

(ii) $$\operatorname{arsin} x + \operatorname{arcos} x = \tfrac{1}{2}\pi. \qquad \text{.........................(2)}$$

This is the inverse form of the relation $\sin(\frac{1}{2}\pi - x) = \cos x$.

In proof, observe that since $-1 \leqslant x \leqslant 1$, we can put $x = \sin\theta$, where $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$, and hence $x = \cos(\frac{1}{2}\pi - \theta)$, where $0 \leqslant \frac{1}{2}\pi - \theta \leqslant \pi$. Then we have

$$\operatorname{arsin} x = \theta = \tfrac{1}{2}\pi - (\tfrac{1}{2}\pi - \theta) = \tfrac{1}{2}\pi - \operatorname{arcos} x. \qquad \text{...................(3)}$$

On replacing x by $1/x$, we have the equivalent relation

$$\operatorname{arcosec} x + \operatorname{arsec} x = \tfrac{1}{2}\pi. \qquad \text{.......................(4)}$$

(iii) $$\operatorname{artan} x + \operatorname{arcot} x = \pm\tfrac{1}{2}\pi, \text{ according as } x \gtrless 0. \qquad \text{...........(5)}$$

This is the inverse form of the relation $\tan(\frac{1}{2}\pi - x) = \cot x$.

(iv) $$\operatorname{arcos} x = \begin{cases} \operatorname{arsin}\sqrt{(1-x^2)}, & (x \geqslant 0), \\ \pi - \operatorname{arsin}\sqrt{(1-x^2)}, & (x \leqslant 0). \end{cases} \qquad \text{..............(6)}$$

These are inverse forms of the relation $\sin x = \pm\sqrt{(1-\cos^2 x)}$.

(v) $$\operatorname{arsin} x = \operatorname{artan}\frac{x}{\sqrt{(1-x^2)}}. \qquad \text{.......................(7)}$$

This is the inverse form of the relation $\tan x = \dfrac{\sin x}{\pm\sqrt{(1-\sin^2 x)}}$.

(vi) $$\operatorname{arcos} x = \begin{cases} \operatorname{artan}\dfrac{\sqrt{(1-x^2)}}{x}, & (x > 0), \\ \pi + \operatorname{artan}\dfrac{\sqrt{(1-x^2)}}{x}, & (x < 0). \end{cases} \qquad \text{.............(8)}$$

These are inverse forms of the relation $\tan x = \dfrac{\pm\sqrt{(1-\cos^2 x)}}{\cos x}$.

(vii) *Inverse forms of the Addition Theorems.*

(a) $$\operatorname{arsin} x_1 \pm \operatorname{arsin} x_2 = n\pi + (-1)^n \operatorname{arsin}\{x_1\sqrt{(1-x_2^2)} \pm x_2\sqrt{(1-x_1^2)}\}, \qquad \text{......(9)}$$

where n has the value -1, 0, or 1 according as $\operatorname{arsin} x_1 \pm \operatorname{arsin} x_2$ lies in the range $(-\pi, -\frac{1}{2}\pi)$, $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, or $(\frac{1}{2}\pi, \pi)$, respectively. This is the inverse form of the theorem

$$\sin(x_1 \pm x_2) = \sin x_1 \cos x_2 \pm \cos x_1 \sin x_2. \qquad \text{...........(10)}$$

We may put $x_1 = \sin\theta_1$, $\sqrt{(1-x_1^2)} = \cos\theta_1$, where $-\frac{1}{2}\pi \leqslant \theta_1 \leqslant \frac{1}{2}\pi$, and $x_2 = \sin\theta_2$, $\sqrt{(1-x_2^2)} = \cos\theta_2$, where $-\frac{1}{2}\pi \leqslant \theta_2 \leqslant \frac{1}{2}\pi$, and therefore $x_1\sqrt{(1-x_2^2)} + x_2\sqrt{(1-x_1^2)} = \sin(\theta_1 + \theta_2)$, where $-\pi \leqslant (\theta_1 + \theta_2) \leqslant \pi$.

If $-\frac{1}{2}\pi \leqslant (\theta_1 + \theta_2) \leqslant \frac{1}{2}\pi$, we therefore have

$$\operatorname{arsin} x_1 + \operatorname{arsin} x_2 = \theta_1 + \theta_2 = \operatorname{arsin}\{x_1\sqrt{(1-x_2^2)} + x_2\sqrt{(1-x_1^2)}\}; \qquad \text{......(11)}$$

if $-\pi \leqslant (\theta_1+\theta_2) \leqslant -\frac{1}{2}\pi$, we have $-\frac{1}{2}\pi \leqslant (-\theta_1-\theta_2-\pi) \leqslant 0$, and since $\sin(\theta_1+\theta_2)=\sin(-\theta_1-\theta_2-\pi)$, we therefore have

$$\begin{aligned}\operatorname{arsin} x_1 + \operatorname{arsin} x_2 &= \theta_1+\theta_2 = -\pi-(-\theta_1-\theta_2-\pi)\\ &= -\pi - \operatorname{arsin}\{x_1\sqrt{(1-x_2^2)}+x_2\sqrt{(1-x_1^2)}\}; \quad \ldots(12)\end{aligned}$$

if $\frac{1}{2}\pi \leqslant (\theta_1+\theta_2) \leqslant \pi$, we have $0 \leqslant (\pi-\theta_1-\theta_2) \leqslant \frac{1}{2}\pi$, and since $\sin(\theta_1+\theta_2) = \sin(\pi-\theta_1-\theta_2)$, we therefore have

$$\begin{aligned}\operatorname{arsin} x_1 + \operatorname{arsin} x_2 &= \theta_1+\theta_2 = \pi-(\pi-\theta_1-\theta_2)\\ &= \pi - \operatorname{arsin}\{x_1\sqrt{(1-x_2^2)}-x_2\sqrt{(1-x_1^2)}\}. \quad \ldots\ldots(13)\end{aligned}$$

The formulae for the difference $\operatorname{arsin} x_1 - \operatorname{arsin} x_2$ are obtained by putting $-x_2$ for x_2.

(*b*) $\arccos x_1 + \arccos x_2$

$$= 2n\pi \pm \arccos\{x_1x_2 - \sqrt{(1-x_1^2)}\sqrt{(1-x_2^2)}\}, \quad \ldots\ldots(14)$$

where $n=0$ and the upper alternative sign applies if $\arccos x_1 + \arccos x_2$ lies in the range $(0, \pi)$, while $n=1$ and the lower sign applies if the same sum lies in the range $(\pi, 2\pi)$;

$$\arccos x_1 - \arccos x_2 = \pm\arccos\{x_1x_2 + \sqrt{(1-x_1^2)}\sqrt{(1-x_2^2)}\}, \quad \ldots(15)$$

where the upper or lower sign applies according as $\arccos x_1 - \arccos x_2$ lies in the range $(0, \pi)$ or $(-\pi, 0)$. These are inverse forms of the theorem

$$\cos(x_1 \pm x_2) = \cos x_1 \cos x_2 \mp \sin x_1 \sin x_2. \quad \ldots\ldots(16)$$

(*c*) $$\operatorname{artan} x_1 \pm \operatorname{artan} x_2 = n\pi + \operatorname{artan}\frac{x_1 \pm x_2}{1 \mp x_1x_2}, \quad \ldots\ldots(17)$$

where n has the value -1, 0, or 1 according as $\operatorname{artan} x_1 \pm \operatorname{artan} x_2$ lies in the range $[-\pi, -\frac{1}{2}\pi]$, $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$, or $[\frac{1}{2}\pi, \pi]$, respectively. This is the inverse form of the theorem

$$\tan(x_1 \pm x_2) = (\tan x_1 \pm \tan x_2)/(1 \mp \tan x_1 \tan x_2). \quad \ldots\ldots(18)$$

Ex. 1. Prove the results (i) $2\operatorname{artan}\frac{1}{5} = \operatorname{artan}\frac{5}{12}$, (ii) $4\operatorname{artan}\frac{1}{5} = \operatorname{artan}\frac{120}{119}$, (iii) $\operatorname{artan}\frac{1}{5} = 2\operatorname{artan}\frac{1}{10} - \operatorname{artan}\frac{1}{515}$.

Ex. 2. Prove the result $4\operatorname{artan}\frac{1}{5} - \operatorname{artan}\frac{1}{239} = \frac{1}{4}\pi$.

Employing the result (ii) of the preceding Example, we have

$$\begin{aligned}4\operatorname{artan}\tfrac{1}{5} - \tfrac{1}{4}\pi &= \operatorname{artan}\tfrac{120}{119} - \operatorname{artan} 1\\ &= \operatorname{artan}\frac{120/119-1}{1+120/119} = \operatorname{artan}\tfrac{1}{239}. \quad \ldots\ldots(19)\end{aligned}$$

With the aid of the above result (iii), we now find

$$8\operatorname{artan}\tfrac{1}{10} - 4\operatorname{artan}\tfrac{1}{515} - \operatorname{artan}\tfrac{1}{239} = \tfrac{1}{4}\pi. \quad \ldots\ldots(20)$$

It is shewn below, Art. 152. 3, that this result can be used conveniently to find $\frac{1}{4}\pi$ by means of the series for $\operatorname{artan} x$, the number of terms to be taken into account being small because of the smallness of the arguments 1/10, 1/515, 1/239.

151. Derivatives of the Inverse Trigonometrical Functions.

The derived function of each of the inverse trigonometrical functions is, as we shall now demonstrate, an algebraic function of the independent variable, and it is on account of this property that the inverse trigonometrical functions are of importance in Analysis. The actual determination of the derivatives can be carried out in more than one way. We may, for example, proceed as in the usual manner of defining dy/dx, by forming the quotient $\Delta y/\Delta x$ and applying considerations of continuity to the expression so obtained. The simplest procedure, however, is to employ the rule connecting the derivative of an inverse function with that of the direct function, which is expressed by $\dfrac{dy}{dx}=\dfrac{1}{dx/dy}$.

151. 1. *Derivative of* arsin x, ($|x|\leqslant 1$). Writing $y=\operatorname{arsin} x$, where y lies in the range $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, we have $x=\sin y$. Therefore $dx/dy=\cos y$, and, by the rule quoted, we have

$$\frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\sqrt{(1-\sin^2 y)}}=\frac{1}{\sqrt{(1-x^2)}}, \quad \text{.................(1)}$$

the positive sign for the radical being taken since, in the range of y mentioned, $\cos y$ is positive. We thus have the formula

$$\frac{d \operatorname{arsin} x}{dx}=\frac{1}{\sqrt{(1-x^2)}}. \quad \text{..........................(2)}$$

The derivative is positive throughout the range $[-1, 1]$ of x. In Fig. 123, it is seen that the gradient of $A'B'$ is positive throughout, consistently with the analytical result. Further, we observe that as $|x| \to 1$, $dy/dx \to +\infty$, while, in the Figure, the tangent tends to parallelism with Oy. There is a positive infinite lower derivative when $x=1$, and a positive infinite upper derivative when $x=-1$.

151. 2. *Derivative of* arcos x, ($|x|\leqslant 1$). From the result (2) of the preceding Article, when written in the form $\operatorname{arcos} x=\frac{1}{2}\pi-\operatorname{arsin} x$, we have

$$\frac{d \operatorname{arcos} x}{dx}=-\frac{d \operatorname{arsin} x}{dx}=-\frac{1}{\sqrt{(1-x^2)}}. \quad \text{................(3)}$$

151. 3. *Derivative of* artan x. Proceeding as in the case of arsin x, we write $y=\operatorname{artan} x$, where y is in the range $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$, which gives $x=\tan y$. Therefore $dx/dy=\sec^2 y$, and by the rule referred to, we have

$$\frac{dy}{dx}=\frac{1}{\sec^2 y}=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2}. \quad \text{....................(4)}$$

We thus have the formula

$$\frac{d \operatorname{artan} x}{dx} = \frac{1}{1+x^2}. \quad \text{.............................(5)}$$

Ex. 1. *The function* $\operatorname{arsin}\frac{x}{a}$, $(|x| \leqslant |a|)$. We write $y = \operatorname{arsin} u$, where $u = x/a$, which gives $dy/du = 1/\sqrt{(1-u^2)}$, $du/dx = 1/a$. Then we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{\sqrt{(1-u^2)}} \cdot \frac{1}{a} = \sqrt{\left(\frac{a^2}{a^2-x^2}\right)} \cdot \frac{1}{a} = \pm \frac{1}{\sqrt{(a^2-x^2)}}, \quad \text{...........(6)}$$

where the positive (negative) sign is to be taken if a is positive (negative).

Ex. 2. Shew that if $|f(x)| \leqslant 1$,

$$\frac{d \operatorname{arsin} f(x)}{dx} = \frac{f'(x)}{\sqrt{[1-\{f(x)\}^2]}}. \quad \text{...........................(7)}$$

Ex. 3. Shew that if $|x| \leqslant |a|$,

$$\frac{d}{dx} \operatorname{arcos} \frac{x}{a} = \mp \frac{1}{\sqrt{(a^2-x^2)}}, \quad \text{.............................(8)}$$

where the upper (lower) sign is to be taken if a is positive (negative).

Ex. 4. Shew that

$$\frac{d}{dx} \operatorname{artan} \frac{x}{a} = \frac{a}{a^2+x^2}. \quad \text{................................(9)}$$

151. 4. *Derivative of* $\operatorname{arcosec} x$, $(|x| \geqslant 1)$. We have the relation $\operatorname{arcosec} x = \operatorname{arsin}\frac{1}{x}$, and hence, writing $f(x) = 1/x$ in the above result (7), we get

$$\frac{d \operatorname{arcosec} x}{dx} = \frac{d \operatorname{arsin}\frac{1}{x}}{dx} = \frac{-\frac{1}{x^2}}{\sqrt{\left(1-\frac{1}{x^2}\right)}} = -\frac{1}{x^2\sqrt{\left(1-\frac{1}{x^2}\right)}}. \quad \text{......(10)}$$

Since $x^2\sqrt{\left(1-\frac{1}{x^2}\right)} = |x|\sqrt{(x^2-1)}$, we may write

$$\frac{d \operatorname{arcosec} x}{dx} = -\frac{1}{|x|\sqrt{(x^2-1)}}. \quad \text{.....................(11)}$$

The derivative is thus negative throughout.

151. 5. *Derivative of* $\operatorname{arsec} x$, $(|x| \geqslant 1)$. From the relation (3) of the preceding Article, we deduce the result

$$\frac{d \operatorname{arsec} x}{dx} = -\frac{d \operatorname{arcosec} x}{dx} = \frac{1}{|x|\sqrt{(x^2-1)}}. \quad \text{............(12)}$$

The same result can also be obtained by starting with the relation $\operatorname{arsec} x = \operatorname{arcos}\frac{1}{x}$, or by using the method of the first Sub-article.

151. 6. *Derivative of* arcot x. From the relation (4) of the preceding Article, we deduce the result

$$\frac{d \operatorname{arcot} x}{dx} = -\frac{d \operatorname{artan} x}{dx} = -\frac{1}{1+x^2}. \qquad \text{.................(13)}$$

Ex. 5. Establish the results

$$\left.\begin{aligned} &\text{(i) } \frac{d}{dx}\operatorname{arcosec}\frac{x}{a} = -\frac{a}{|x|\sqrt{(x^2-a^2)}}, \quad \text{(ii) } \frac{d}{dx}\operatorname{arsec}\frac{x}{a} = \frac{a}{|x|\sqrt{(x^2-a^2)}}, \\ &\qquad\qquad \text{(iii) } \frac{d}{dx}\operatorname{arcot}\frac{x}{a} = -\frac{a}{a^2+x^2}, \end{aligned}\right\} \text{......(14)}$$

where in cases (i), (ii), we have $|x| \geqslant |a|$.

Ex. 6. Find the derivative of the function $\operatorname{arsin}^2 x$.

[Write $y = u^2$, where $u = \operatorname{arsin} x$.]

Ex. 7. Find the derivative of $(1+x^2)^{m/2} \sin\{m(\operatorname{artan} x + \frac{1}{2}\pi)\}$.[1]

We have

$$\frac{dy}{dx} = \frac{d(1+x^2)^{m/2}}{dx}\sin\{m(\operatorname{artan} x + \tfrac{1}{2}\pi)\} + (1+x^2)^{m/2}\frac{d \sin\{m(\operatorname{artan} x + \frac{1}{2}\pi)\}}{dx}. \text{...(15)}$$

Now

$$\frac{d}{dx}(1+x^2)^{\frac{m}{2}} = \frac{m}{2}(1+x^2)^{\frac{m}{2}-1}\cdot\frac{d}{dx}(1+x^2) = mx(1+x^2)^{\frac{m}{2}-1}, \text{..........(16)}$$

and, writing $m(\operatorname{artan} x + \frac{1}{2}\pi) = u$, we have

$$\frac{d}{dx}\sin\{m(\operatorname{artan} x + \tfrac{1}{2}\pi)\} = \frac{d \sin u}{dx} = \frac{d \sin u}{du}\cdot\frac{du}{dx} = \cos u\,.\,\frac{m}{1+x^2}$$

$$= \frac{m}{1+x^2}\cos\{m(\operatorname{artan} x + \tfrac{1}{2}\pi)\}. \text{...............(17)}$$

Therefore

$$\frac{dy}{dx} = mx(1+x^2)^{\frac{m}{2}-1}\sin\{m(\operatorname{artan} x + \tfrac{1}{2}\pi)\} + (1+x^2)^{\frac{m}{2}}\frac{m}{1+x^2}\cos\{m(\operatorname{artan} x + \tfrac{1}{2}\pi)\}$$

$$= m(1+x^2)^{\frac{m}{2}-1}[x \sin\{m(\operatorname{artan} x + \tfrac{1}{2}\pi)\} + \cos\{m(\operatorname{artan} x + \tfrac{1}{2}\pi)\}]. \text{...........(18)}$$

We can simplify this result by putting

$$x = \tan\{(\operatorname{artan} x + \tfrac{1}{2}\pi) - \tfrac{1}{2}\pi\} = -\cot(\operatorname{artan} x + \tfrac{1}{2}\pi), \text{...........(19)}$$

which gives, on putting φ temporarily for $\operatorname{artan} x + \frac{1}{2}\pi$,

$$\frac{dy}{dx} = m(1+x^2)^{\frac{m}{2}-1}(-\cot\varphi \sin m\varphi + \cos m\varphi) = m(1+x^2)^{\frac{m}{2}-1}\left\{-\frac{\sin(m-1)\varphi}{\sin\varphi}\right\}$$

$$= -m(1+x^2)^{\frac{m}{2}-1}\frac{\sin\{(m-1)(\operatorname{artan} x + \frac{1}{2}\pi)\}}{\cos(\operatorname{artan} x)}. \text{............................(20)}$$

[1] The results of this Example are employed in the next Article. They appear naturally in finding the nth derivative of $1/(a^2+x^2)$ by the use of complex numbers, but this method is outside our scope.

Now $\cos(\operatorname{artan} x) = (1+x^2)^{-1/2}$, the positive sign of the radical being taken since $\operatorname{artan} x$ lies in the range $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, and therefore

$$\frac{dy}{dx} = -m(1+x^2)^{(m-1)/2}\sin\{(m-1)(\operatorname{artan} x + \tfrac{1}{2}\pi)\}. \quad \text{............(21)}$$

On writing $-n$ for m, we have the result

$$\frac{d}{dx}\left[\frac{\sin\{n(\operatorname{artan} x + \frac{1}{2}\pi)\}}{(1+x^2)^{n/2}}\right] = \frac{n\sin\{(n+1)(\operatorname{artan} x + \frac{1}{2}\pi)\}}{(1+x^2)^{(n+1)/2}}, \quad \text{.........(22)}$$

the derivative in this case being obtained from the function by changing n into $n+1$ and then multiplying by n.

152. Higher Derivatives and Taylor's Theorem for the Inverse Trigonometrical Functions.

Since the derivative of each of the inverse functions is an algebraic function, the methods already given for the differentiation of such functions can be employed to determine the higher derivatives. Repeated differentiation, however, soon becomes laborious, and special methods have been developed which avoid this increasing complexity. We note here only the cases of the inverse sine and tangent.

152. 1. *Case of* $\operatorname{arsin} x$, $(|x| \leqslant 1)$. We find

$$\left.\begin{aligned} f^{(1)}(x) &= \frac{1}{(1-x^2)^{1/2}}, & f^{(2)}(x) &= \frac{x}{(1-x^2)^{3/2}}, \\ f^{(3)}(x) &= \frac{1+2x^2}{(1-x^2)^{5/2}}, & f^{(4)}(x) &= \frac{3x(3+2x^2)}{(1-x^2)^{7/2}}, \end{aligned}\right\} \quad \text{.............(1)}$$

and so on. A recurrence formula for the successive derivatives may be obtained as follows. From the first and second of the above results we have

$$(1-x^2)f^{(2)}(x) = xf^{(1)}(x), \quad \text{.........................(2)}$$

and, applying Leibniz's Theorem, we find, after a little reduction,

$$(1-x^2)f^{(n+2)}(x) - (2n+1)xf^{(n+1)}(x) - n^2 f^{(n)}(x) = 0. \quad \text{.........(3)}$$

This expresses the derivative of order $n+2$ linearly in terms of those of orders $n+1$ and n. The successive derivatives can thus be found for any value of n by a step-by-step process from a knowledge of the first and second derivatives only.

This method gives readily the values of the successive derivatives when $x=0$. We get the relation

$$f^{(n+2)}(0) = n^2 f^{(n)}(0), \quad \text{..........................(4)}$$

and since $f^{(2)}(0)=0$, it follows, by Induction, that all the even derivatives vanish when $x=0$. Further,

$$\left.\begin{aligned} f^{(1)}(0)=1,\quad f^{(3)}(0)=1^2f^{(1)}(0)=1,\quad f^{(5)}(0)=3^2f^{(3)}(0)=3^2,\\ f^{(7)}(0)=5^2f^{(5)}(0)=3^2\,.\,5^2, \end{aligned}\right\} \quad\text{.....(5)}$$

and generally,

$$f^{(2n+1)}(0)=3^2\,.\,5^2\ldots(2n-3)^2(2n-1)^2. \quad\text{................(6)}$$

The Taylor polynomial for arsin x of degree $2n+1$ is thus

$$x+\frac{x^3}{3!}+3^2\frac{x^5}{5!}+\ldots+\{3^2\,.\,5^2\ldots(2n-1)^2\}\frac{x^{2n+1}}{(2n+1)!}. \quad\text{.........(7)}$$

We cannot enter on a satisfactory discussion of the remainder at this stage, since there is no suitable simple formula for the derivative of any order n for a general value of x. We may state, however, that the remainder does converge to zero as n increases indefinitely provided that x lies in the range $(-1,\,1)$. See Ex. 2, Set LXX, p. 937.

152. 2. *Case of* artan x. Denoting the function by y, or $f(x)$, we have

$$y^{(1)}=\frac{1}{1+x^2},\quad y^{(2)}=-\frac{2x}{(1+x^2)^2},\quad y^3=-\frac{2(1-3x^2)}{(1+x^2)^3},\quad y^{(4)}=\frac{24x(1-x^2)}{(1+x^2)^4},\ \text{(8)}$$

and so on. A compact form for the general derivative is obtained as follows. We have $x=\tan y$, and the first of the above results gives

$$y^{(1)}=\frac{1}{1+x^2}=\frac{1}{1+\tan^2 y}=\cos^2 y. \quad\text{.....................(9)}$$

Differentiating this relation with respect to x, we obtain

$$y^{(2)}=-\sin 2y\,.\,y^{(1)}=-\sin 2y\cos^2 y=\cos^2 y\sin\{2(y+\tfrac{1}{2}\pi)\},\ \text{...(10)}$$

and again,

$$\begin{aligned} y^{(3)}&=[\cos^2 y\,.\,2\cos\{2(y+\tfrac{1}{2}\pi)\}-\sin 2y\sin\{2(y+\tfrac{1}{2}\pi)\}]\,y^{(1)}\\ &=2\cos^3 y\,[\cos y\cos\{2(y+\tfrac{1}{2}\pi)\}-\sin y\sin\{2(y+\tfrac{1}{2}\pi)\}]\\ &=2\cos^3 y\cos(3y+\tfrac{3}{2}\pi)=2\cos^3 y\sin\{3(y+\tfrac{1}{2}\pi)\}. \quad\text{............(11)} \end{aligned}$$

These results suggest the general form [1]

$$y^{(n)}=(n-1)!\cos^n y\sin\{n(y+\tfrac{1}{2}\pi)\}, \quad\text{................................(12)}$$

or, since $\cos y=(1+x^2)^{-1/2}$,

$$y^{(n)}=(n-1)!\,\frac{\sin\{n(\operatorname{artan}x+\tfrac{1}{2}\pi)\}}{(1+x^2)^{n/2}} \quad\text{................................(13)}$$

[1] The expression in (13) is the compact form for the $(n-1)$th derivative of $1/(1+x^2)$, referred to in Ex. 2, Art. 88.

That (13) is the correct form follows by Induction. For we have shewn in Ex. 7 of the preceding Article, equation (22), that the derivative of $y^{(n)}$ is obtained by changing n into $n+1$, and the above formula (13) has been shewn to be true for the values 1, 2, 3 of n.

If we put $x=0$ in (13), we get

$$f^{(n)}(0)=(n-1)!\sin\tfrac{1}{2}n\pi, \qquad\text{.......................(14)}$$

which shews that the even derivatives all vanish when $x=0$, and which gives, for the odd derivatives, the formula

$$f^{(2r+1)}(0)=(-1)^r(2r)! \qquad\text{.......................(15)}$$

Taylor's Theorem with Remainder for artan x is thus expressed by

$$\operatorname{artan} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + R_{2n+1}(x), \qquad\text{.....(16)}$$

where

$$\begin{aligned} R_{2n+1}(x) &= \frac{x^{2n+1}}{(2n+1)!} f^{(2n+1)}(\theta x), \quad (0<\theta<1), \\ &= \frac{x^{2n+1}}{(2n+1)!}\cdot\frac{(2n)!}{(1+\theta^2x^2)^{n+\frac{1}{2}}}\sin\{(2n+1)(\operatorname{artan}\theta x+\tfrac{1}{2}\pi)\} \\ &= \frac{1}{2n+1}\left(\frac{x^2}{1+\theta^2x^2}\right)^{n+\frac{1}{2}}\sin\{(2n+1)(\operatorname{artan}\theta x+\tfrac{1}{2}\pi)\}, \qquad\text{...(17)} \end{aligned}$$

so that

$$|R_{2n+1}(x)| \leqslant \frac{1}{2n+1}\left(\frac{x^2}{1+\theta^2x^2}\right)^{n+\frac{1}{2}} < \frac{|x|^{2n+1}}{2n+1}. \qquad\text{..........(18)}$$

The magnitude of the error committed in replacing artan x by its Taylor polynomial of degree $2n-1$ is thus not greater than that of the last term in the polynomial of degree $2n+1$. Further, $\dfrac{|x|^{2n+1}}{2n+1}$ decreases indefinitely as $n\to\infty$, provided that $|x|\leqslant 1$, and the conditions for Taylor's Theorem being then satisfied, we have

$$\operatorname{artan} x = x - \frac{x^3}{3} + \dots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \dots. \qquad\text{.........(19)}$$

On the other hand, when $|x|>1$, the Taylor polynomials do not satisfy the necessary condition for the validity of Taylor's Theorem that the general term $x^{2n-1}/(2n-1)$ should converge to zero when $n\to\infty$. For if $u_n=|x|^{2n-1}/(2n-1)$, we have

$$\frac{u_{n+1}}{u_n}=\frac{2n-1}{2n+1}x^2, > 1 \text{ if } n > \frac{x^2+1}{2(x^2-1)}. \qquad\text{..............(20)}$$

Thus after a certain value of n the term u_n increases with n and so cannot converge to zero when $n\to\infty$. The Taylor Theorem as expressed by (19) therefore does not hold if $|x|>1$.

The series (19) is commonly known as *Gregory's series for* artan x. The closeness of approximation of the earlier Taylor polynomials to the function artan x is shewn in Fig. 128, the range of convergence being clearly indicated.

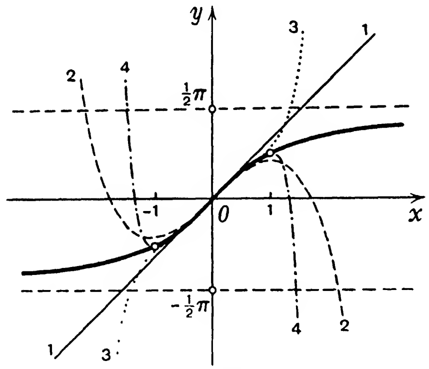

FIG. 128

152. 3. *Use of Gregory's series in the approximate evaluation of* π. Putting $x=1$ in (19), we get a series which may be employed in the evaluation of artan 1, that is, of $\frac{1}{4}\pi$. The convergence, however, is too slow for the series to be of much practical use. The rapidity of the convergence of (19) increases as x decreases, and by means of a sum of multiples of the series for small values of x, such as those given in Ex. 2, Art. 150. 7, we are able to find $\frac{1}{4}\pi$ with great accuracy by the use of only a small number of terms. Thus, taking the result

$$\tfrac{1}{4}\pi = 8 \operatorname{artan} \tfrac{1}{10} - 4 \operatorname{artan} \tfrac{1}{515} - \operatorname{artan} \tfrac{1}{239}, \quad \dots\dots\dots\dots\dots\dots (21)$$

we require only five, two, two terms, respectively, of the Gregory's series for artan $\frac{1}{10}$, artan $\frac{1}{515}$, artan $\frac{1}{239}$ to obtain accuracy to ten places of decimals. The error in calculating $\frac{1}{4}\pi$ in this way is then less than 2×10^{-9}. Denoting the successive inverse tangents to be calculated by α, β, γ, we have

$$\left.\begin{aligned} \alpha &= \frac{1}{10} - \frac{1}{3}\cdot\frac{1}{10^3} + \frac{1}{5}\cdot\frac{1}{10^5} - \frac{1}{7}\cdot\frac{1}{10^7} + \frac{1}{9}\cdot\frac{1}{10^9} - \dots, \\ \beta &= \frac{1}{515} - \frac{1}{3}\cdot\frac{1}{515^3} + \dots, \quad \gamma = \frac{1}{239} - \frac{1}{3}\cdot\frac{1}{239^3} + \dots . \end{aligned}\right\} \quad \dots\dots\dots\dots (22)$$

(i) *Evaluation of* α.[1]

$\frac{1}{10} = 0{\cdot}100\ 000\ 000\ 0$,	$\frac{1}{10} = 0{\cdot}100\ 000\ 000\ 0$,	
$\frac{1}{10^3} = 0{\cdot}001\ 000\ 000\ 0$,	$\frac{1}{3}\cdot\frac{1}{10^3} =$	$0{\cdot}000\ 333\ 333\ 3$,
$\frac{1}{10^5} = 0{\cdot}000\ 010\ 000\ 0$,	$\frac{1}{5}\cdot\frac{1}{10^5} = 0{\cdot}000\ 002\ 000\ 0$,	
$\frac{1}{10^7} = 0{\cdot}000\ 000\ 100\ 0$,	$\frac{1}{7}\cdot\frac{1}{10^7} =$	$0{\cdot}000\ 000\ 014\ 3$,
$\frac{1}{10^9} = 0{\cdot}000\ 000\ 001\ 0$,	$\frac{1}{9}\cdot\frac{1}{10^9} = 0{\cdot}000\ 000\ 000\ 1$,	
	$0{\cdot}100\ 002\ 000\ 1$,	$0{\cdot}000\ 333\ 347\ 6$.

$$\alpha = 0{\cdot}099\ 668\ 652\ 5. \quad \dots\dots\dots\dots\dots\dots\dots\dots (23)$$

[1] The student should note the advantage of the systematic arrangement by means of which each term of the series is obtained from the preceding term.

(ii) *Evaluation of* β.

$\frac{1}{515}=0{\cdot}001\ 941\ 747\ 6,$	$\frac{1}{515}=0{\cdot}001\ 941\ 747\ 6,$
$\frac{1}{515^3}=0{\cdot}000\ 000\ 007\ 3,$	$-\frac{1}{3}\cdot\frac{1}{515^3}=-0{\cdot}000\ 000\ 002\ 4.$

$$\beta=0{\cdot}001\ 941\ 745\ 2. \qquad\qquad (24)$$

(iii) *Evaluation of* γ.

$\frac{1}{239}=0{\cdot}004\ 184\ 100\ 4,$	$\frac{1}{239}=0{\cdot}004\ 184\ 100\ 4,$
$\frac{1}{239^3}=0{\cdot}000\ 000\ 073\ 2,$	$-\frac{1}{3}\cdot\frac{1}{239^3}=-0{\cdot}000\ 000\ 024\ 4.$

$$\gamma=0{\cdot}004\ 184\ 076\ 0. \qquad\qquad (25)$$

We therefore have the results

$$\tfrac{1}{4}\pi=8\alpha-4\beta-\gamma=0{\cdot}785\ 398\ 163\ 2, \quad \pi=3{\cdot}141\ 592\ 652\ 8. \qquad (26)$$

We now examine barriers to the absolute errors in these results. The error in α due to rejecting terms of the series beyond $\frac{1}{9}\cdot\frac{1}{10^9}$ is less than $\frac{1}{11}\cdot\frac{1}{10^{11}}$, and so does not affect the tenth place of decimals. In the cases of β and γ, the errors of the same nature do not exceed $\frac{1}{5}\cdot\frac{1}{515^5}$ and $\frac{1}{5}\cdot\frac{1}{239^5}$, respectively. These are of the respective orders of magnitude $1/(5\times4\times10^{13})$, $1/(5\times8\times10^{11})$, and are also negligible. There only remain errors due to approximating in the last figures written. In the calculation of α two of the terms are accurate in the tenth place, and in the other three terms the error is less than one-half a unit in this place. The error in α is thus less than $1{\cdot}5\times10^{-10}$. For β and γ the corresponding errors are each less than 1×10^{-10}. Hence the error in $8\alpha-4\beta-\gamma$, that is, in $\frac{1}{4}\pi$, is less than 17×10^{-10}, and the error in π is accordingly less than 68×10^{-10}. The calculation thus shews that the value of π lies between $3{\cdot}1415926460$ and $3{\cdot}1415926596$. As a matter of fact, the errors are not as great as the upper barriers stated and are not all in the same direction. The value of π correct to ten decimal places is $3{\cdot}1415926536$.

153. Tables for the Inverse Functions. Application of the Method of Proportional Parts.

Tables giving the inverse trigonometrical functions $\operatorname{arsin} x$, $\operatorname{artan} x$, etc., directly for equidistant values of the argument x are not common. Such tables are, however, given in the collection of Hayashi and are at times very useful.

Usually the inverse functions are found by reading the tables of the direct functions $\sin x$, $\tan x$, etc., backwards. In Art. 125, Chap. IV, we have examined, for the general case of inversion, the uncertainty of a single result taken directly from a table in this

reverse way as well as the uncertainty involved in the application of the *M.P.P.* to interpolation for a value lying between two successive tabulated values.

Suppose that we are concerned with a table of values of $\sin x$ with a common difference h of the argument x. If the table is n-figure, so that the value y_1 tabulated against x_1 is subject to an uncertainty of magnitude $5 \times 10^{-(n+1)}$, this value x_1, taken as an approximation to $\operatorname{arsin} y_1$, must be regarded as subject to an uncertainty of magnitude

$$\frac{5 \times 10^{-(n+1)}}{\left(\dfrac{dy}{dx}\right)_1}, \quad \text{that is,} \quad \frac{5 \times 10^{-(n+1)}}{\cos x_1}, \quad \text{or} \quad \frac{5 \times 10^{-(n+1)}}{\sqrt{(1-y_1^2)}}. \quad \ldots\ldots(1)$$

The uncertainty is thus increased in the ratio $1 : \cos x$. For small values of y_1 (or x_1), this gives the same n-decimal accuracy as y has. If, however, y_1 approaches 1, or x_1 approaches $\frac{1}{2}\pi$, the uncertainty increases; for example, suppose the argument is $x_1 = 1{\cdot}558$ and that $y_1{}^*$, $(0{\cdot}9999181\ldots)$, is the n-figure tabulated value of $\sin x_1$. We then have $\cos x_1 \approx 0{\cdot}0126$, and if we take $1{\cdot}558$ as the value of $\operatorname{arsin} y_1{}^*$, the uncertainty is about $5 \times 10^{-(n+1)} \times 79$, or $4 \times 10^{-(n-1)}$. Thus we cannot expect accuracy to more than $n-2$ places in using this part of the table in reverse.

In the five-figure *Smithsonian Tables*, sin $1{\cdot}558$ is given as $0{\cdot}99992$, while the actual value of arsin $0{\cdot}99992$ is $1{\cdot}55814$.

In the application of the *M.P.P.* to the same reverse process, we have shewn that the uncertainty may be taken as $\frac{1}{8}h^2 \left|\dfrac{d^2y}{dx^2}\right| \Big/ \left|\dfrac{dy}{dx}\right|$, which in the present case becomes $\frac{1}{8}h^2 \tan x$. Thus the uncertainty attaching to the inverse use of the table is again increased, relative to the uncertainty attaching to the direct use of the table, in the ratio $1 : \cos x$. The numerical example just employed serves to shew the increase of the uncertainty as the value of y_1 approaches 1.

If we use a table in which the argument is in degree measure, we must remember that the common difference h in the above formula applies to the argument x of the function $\sin x$, and if h' is the common difference in degree measure, we have $h = \pi h'/180$, where $h' = 1/60$ if the common difference is 1 minute. The uncertainty of an interpolated value is, in this latter case,

$$\frac{1}{8} \cdot \frac{\pi^2}{(180 \times 60)^2} \cdot \tan x. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The condition for completeness of inverse reading of an n-figure table of $\sin x$ is expressed by

$$\tfrac{1}{8}h^2 \tan x < \frac{5 \times 10^{-(n+1)}}{\cos x}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

that is, by $\sin x < 4 \times 10^{-n}/h^2$. This condition is plainly satisfied for all values of x by the *Smithsonian Tables*, for example, where $n=5$ and the maximum value of h is 10^{-3}. For a seven-figure table in which the argument is in degree measure with common difference of 1 minute, the value of h is approximately $2{\cdot}9 \times 10^{-4}$, and the above condition reduces to $\sin x < 4 \times 10^{-7}/(2{\cdot}9 \times 10^{-4})^2$, which is again plainly satisfied for all values of x. Thus an interpolated value of x is given to the same accuracy as the adjacent tabulated values throughout the tables mentioned.

Ex. 1. Shew that if $x < 1{\cdot}5502$, (equivalent to $88°\ 49'$), the uncertainty in x, taken as the value of the inverse function corresponding to a tabulated value of $\sin x$ in a seven-figure table, is less than $2{\cdot}5 \times 10^{-6}$ (corresponding to one-half of a second).

[With the aid of this result we conclude that interpolation to the nearest second in the inverse use of a 7-figure table of sines in which the argument is in degree measure with common difference of 1 minute, can be effected by the *M.P.P.* for values of the argument up to $88°\,49'$.]

Ex. 2. Given sine $27°\ 53' = 0{\cdot}4676727$ correct to seven places of decimals, find the uncertainty in the value $27°\ 53'$ taken as the equivalent in degree measure of arsin $0{\cdot}4676727$.

[The uncertainty is given by $5 \times 10^{-8}/\text{cosine } 27°\ 53'$, that is, $5 \times 10^{-8}/0{\cdot}884$, or $5{\cdot}66 \times 10^{-8}$, equivalent to $0{\cdot}0117$ seconds.]

Ex. 3. A table of tangents is given correct to n places of decimals, the common difference of argument being h. Shew that the condition of completeness for inverse reading is expressed by $\tan x \sec^2 x < 2 \times 10^{-n}/h^2$.

If the argument is given in degrees with common difference of 1 minute, and if the table is 7-figure, shew that the condition of completeness is satisfied if the argument is less than $47°\ 20'$.

EXAMPLES XXV

1. Sketch the graphs of the functions $\sin(2\pi x/l)$, $2\sin(\tfrac{1}{2}x + \tfrac{1}{4}\pi)$, $\cos(3x + \tfrac{1}{6}\pi)$. Explain how the graph of $\sin x$ may be employed as a representation of the graphs of the functions $\sin(x - \tfrac{1}{4}\pi)$, $\cos x$, $\sin x + \cos x$.

2. Sketch the graphs of the functions (i) $\tan x - x/(1-x^2)$, (ii) $\tan x - x + 1/x$, (iii) $\cot x + x - 1/x$, (iv) $\tan x + \cot \tfrac{1}{2}x$. Use the graph in each case to estimate the smallest positive value of x for which the function vanishes.

3. Examine the potential continuity of the following functions when $x=0$ and supply the completing values for those which are potentially continuous there :

(i) $x \operatorname{cosec} x$; (ii) $(\tan x)/x$; (iii) $x \cot x$; (iv) $\dfrac{1-\cos x}{x}$; (v) $\dfrac{x}{1-\cos x}$;

(vi) $\dfrac{1-\cos x}{x^2}$; (vii) $\dfrac{\tan x - \sin x}{x^3}$; (viii) $\dfrac{1-\cos x}{\tan^2 x}$; (ix) $\dfrac{\sin(a+x)-\sin a}{x}$;

(x) $\dfrac{x-\sin x}{x^3}$; (xi) $\dfrac{3\sin x - \sin 3x}{x^3}$, (xii) $\dfrac{1-\frac{1}{2}x^2-\cos x}{x^4}$.

EXAMPLES XXVI

DIFFERENTIATION OF DIRECT TRIGONOMETRICAL FUNCTIONS

1. Find the first derivative of each of the following functions :

(i) $\sin^2 x$; (ii) $\sin^2 x + \cos^2 x$; (iii) $\cos^m x$; (iv) $\sin^n(ax^m+b)$;
(v) $\sin^n(\sin mx)$; (vi) $\operatorname{cosec}^2 x$; (vii) $1/(a^2\cos^2 x + b^2 \sin^2 x)$;
(viii) $x^2\sin(3-2x)$; (ix) $x^m \sin^n x$; (x) $1/(a+b\sin x)$;
(xi) $\sin(m\sin x)$; (xii) $\sin\sqrt{(1-x)}$; (xiii) $(ax^2+b)\sin(cx^2+d)$;
(xiv) $\sin\{\sqrt{(1-x^2)}\tan\frac{3}{2}x\}$; (xv) $\sin|x|$; (xvi) $|\sin x|$;
(xvii) $\sin(\pi\tan x)$; (xviii) $\sin\dfrac{1+x^2}{1+x^3}$; (xix) $\dfrac{\sin 2x - \tan x}{\cos x}$;
(xx) $\dfrac{\sin(\pi x)}{x(1-x)}$; (xxi) $\dfrac{\cos x}{1+\sin x}$; (xxii) $\dfrac{a\cos^2 x + b\sin^2 x}{\alpha\cos^2 x + \beta\sin^2 x}$;
(xxiii) $\sqrt{x}\tan x$; (xxiv) $7x^2\sqrt{(\cot x)}$; (xxv) $\sin x^m$;
(xxvi) $\tan x + \cot\frac{1}{2}x$.

2. Tangents are drawn from the origin to the graph of the function $a\sin(x/c)$. Shew that the abscissae of the points of contact are given by $\tan(x/c)=x/c$. Find the corresponding equation when the tangents are drawn from any point (ξ, η).
[$a\sin(x/c)-\eta=(x-\xi)\dfrac{a}{c}\cos(x/c)$.]

3. Find the values of c for which the line $y=\frac{1}{2}x+c$ is (i) a tangent, (ii) a normal, to the curve $y=3\sin x$.
[(i) $\pm\frac{1}{2}\sqrt{35}-\frac{1}{2}(2n\pi\pm\arccos\frac{1}{6})$, (ii) $\pm\sqrt{5}-\frac{1}{2}(2n\pi\mp\arccos\frac{2}{3})$.]

4. A curve is specified by the equations $x=3\cos\theta-\cos 3\theta$, $y=3\sin\theta-\sin 3\theta$ in terms of the parameter θ. Shew that the gradient at any point is $\tan 2\theta$, and find the value of d^2y/dx^2.

5. Shew that the curves

(i) $y=\pm x$, $y=x\sin(1/x)$, and (ii) $y=x^2$, $y=x^2\sin(1/x)$

touch at their respective points of contact.

6. A rod AB of length $2a$ is suspended horizontally by means of two vertical strings, each of length l, their upper ends being fixed at the same level. The rod is turned about the vertical through its middle point with constant angular velocity ω. Find the vertical velocity of the rod when the angle of twist (measured from the equilibrium position) is θ.
[$\omega a^2\sin\theta/\sqrt{(l^2-4a^2\sin^2\frac{1}{2}\theta)}$.]

7. Find the cosine of the inclination, to Ox, of the tangent at any point of the helix $x = a\cos\theta$, $y = a\sin\theta$, $z = c\theta$.

[Use the result of Ex. 23, Set XII, Chap. II.]

8. Prove that all the successive derivatives of $\tan x$ and of $\sec x$ are positive in the range $(0, \frac{1}{2}\pi)$ of x.

[Use the results (8), (10) of Art. 145 and employ the method of Induction.]

9. Shew that each of the functions $\sin mx$, $\cos mx$, $a\sin mx + b\cos mx$, (a, b any real numbers), satisfies the equation $d^2y/dx^2 + m^2y = 0$.

10. If $y = x\sin x$, show that $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + y(x^2+2) = 0$.

11. If $z = x^2\sin(y/x)$, prove that $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x\partial y} + y^2\dfrac{\partial^2 z}{\partial y^2} = 2x^2\sin(y/x)$.

12. Find the nth derivative of each of the following functions:
(i) $\sin mx$; (ii) $\cos(mx+\alpha)$; (iii) $\sin ax\cos bx$; (iv) $x^2\sin x$;
(v) $x^2\sin^3 x$; (vi) $\cos x\cos 2x\cos 3x$; (vii) $\sin^2 kx$; (viii) $\cos^4 x$;
(ix) $\cos^n x$, (n a positive integer).

[In case (ix) express $\cos^n x$ linearly in terms of cosines of integral multiples of x; see Hobson, *Trigonometry*, Art. 221.]

13. Shew that $\lim\limits_{h\to 0}\dfrac{\Delta^n\sin x}{h^n} = \sin(x + \frac{1}{2}n\pi)$.

[Use the result of Ex. 5, Set IX.]

14. Find the transforms of dy/dx, d^2y/dx^2 when the independent variable x is changed to t by means of the relation $x = \tan t$. Shew that this substitution transforms the equation $(1+x^2)\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} = 0$ into $\dfrac{d^2y}{dt^2} = 0$.

15. If $y = \sin n\theta/\sin\theta$ and $x = \cos\theta$, prove that

$$(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + (n^2-1)y = 0.$$

16. Shew that if a, c are positive, the equation $a + c\tan x = b\sin x$ has two real roots between 0 and $\frac{1}{2}\pi$ if $b^{2/3} > a^{2/3} + c^{2/3}$.

17. Employing the result

$$\sin\frac{rk}{n} - \sin\frac{(r-1)k}{n} = 2\cos\frac{(r-\frac{1}{2})k}{n}\sin\frac{\frac{1}{2}k}{n},\quad (r = 1, 2, \ldots, n),$$

where k is any number in the range $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, prove that if $h = mk/n$, (m, n integers, $m < n$), then

$$\frac{\sin k}{k} < \frac{\sin h}{h} < \frac{\sin k}{k}\sec k.$$

Hence shew that $(\sin h)/h$ converges to a definite positive number when $h \to 0$.

Shew how to find $\sin\dagger(30/2^n)$, $\cos\dagger(30/2^n)$ by the use of the formulae $\sin\frac{1}{2}x = \sqrt{\{\frac{1}{2}(1-\cos x)\}}$, $\cos\frac{1}{2}x = \sqrt{\{\frac{1}{2}(1+\cos x)\}}$. In particular, shew that $\sin\dagger(30/2^5) = 0{\cdot}0163617$, $\cos\dagger(30/2^5) = 0{\cdot}9998661$, $\sec\dagger(30/2^5) = 1{\cdot}000134$, and hence that if $|h| < 30/2^5$, $(\sin\dagger h)/h$ lies between $0{\cdot}017452$ and $0{\cdot}017455$. Deduce that the radian lies between $57^\circ{\cdot}29$ and $57^\circ{\cdot}30$.

[The actual limit is $\pi/180$, or $0{\cdot}017453$, and the radian is equal to $57^\circ{\cdot}2958$.]

EXAMPLES XXVII

EXPANSIONS. APPROXIMATIONS BY TAYLOR POLYNOMIALS

1. Find the value of sin† 48 correct to six decimal places by using the expansions for sin† 3, cos† 3.

2. Calculate the value of the sine of 15° 11′ correct to six decimal places, using the known values of sin† 30, cos† 30.

3. Obtain power series for the following functions:

(i) $(\sin x)/x$; (ii) $(1-\cos x)/x$; (iii) $\sin^2 x$;
(iv) $\cos^2 x$; (v) $\cos^3 x$; (vi) $\sin(x+\frac{1}{4}\pi)\cos x$.

4. Shew that the difference between the length of a straight line and its projection on another is less than 2% of the length if the angle between the lines is less than 10°.

5. If l is the length of a circular arc and a that of the corresponding chord, shew that the error made in replacing l by a is of the third order in x, where x is the angle (in radians) subtended by the arc at the centre. Shew that if the angle is $\pi/18$, an upper barrier to the fractional error incurred in this approximation is 0·0013.

If b is the chord corresponding to one-half the arc, shew that the approximation $l \approx \frac{1}{3}(8b-a)$ is correct to the fourth order in x. Find an upper barrier to the error if $x < \frac{1}{3}\pi$.

6. Shew that the Taylor polynomial for $\dfrac{3\sin x}{2+\cos x} - x$ of the fifth degree in x is $-x^5/180$. Shew that the given function decreases as x increases and is equal to $-0{\cdot}071$ when $x=\frac{1}{2}\pi$. Compare this result with that obtained from the Taylor polynomial found.

7. Obtain the Taylor polynomial of the eighth degree for $\sec x$ from that for $\cos x$ (i) by division of the latter into 1, (ii) by the method of disposable coefficients.

8. If $y = x\cot x$, shew that $x\,dy/dx = y - x^2 - y^2$, and that if $n > 2$,

$$xy^{(n+1)} = -(n-1)y^{(n)} - \{y^{(n)}y + \binom{n}{1}y^{(n-1)}y^{(1)} + \binom{n}{2}y^{(n-2)}y^{(2)} + \dots + yy^{(n)}\}.$$

Hence find the values of the first eight derivatives when $x=0$.

9. Calculate $\tan\frac{1}{10}$ by the use of the Taylor polynomial of the third degree. By considering the remainder term, shew that an upper barrier to the error committed is $1{\cdot}5\times 10^{-6}$.

[The known value of $\tan\frac{1}{10}$ is 0·1003347.]

10. Shew that the Taylor polynomial of the sixth degree for $x^2\cot^2 x$ is $1-\frac{2}{3}x^2+\frac{1}{45}x^4+\frac{2}{189}x^6$. [Use Theorem II, Art. 129.]

11. Employ the method of successive substitutions to find the Taylor polynomial of the fifth degree for y in terms of x when $y-\alpha\sin y=\beta x$, given that $y=0$ when $x=0$. Hence shew that if powers of α higher than the second are neglected,

$$y=\beta x+\alpha(\beta x-\tfrac{1}{6}\beta^3x^3+\tfrac{1}{120}\beta^5x^5)+\alpha^2(\beta x-\tfrac{2}{3}\beta^3x^3+\tfrac{2}{15}\beta^5x^5).$$

12. If $y - \sin y = \frac{1}{6}x^3$, shew that the Taylor polynomial of the fifth degree for y in terms of x is $x + \frac{1}{60}x^3 + \frac{1}{1400}x^5$.

[Assume the form $y = x + \alpha x^3 + \beta x^5 + \gamma x^7$.]

13. Shew graphically that the larger roots of the equation $x = \cot x$ are given approximately by $x = n\pi$. Employ the method of successive substitutions to shew that $x = n\pi + \dfrac{1}{n\pi} - \dfrac{4}{3n^3\pi^3} + \dfrac{53}{15n^5\pi^5}$ is a fourth approximation to the root near $x = n\pi$.

[Use the successive Taylor polynomials for $x \cot x$, and in the iterative process put $\cot x_{r+1} = x_r$.]

14. Establish the following inequalities when x lies in the range $[0, \pi]$:

(i) $\sin x > x \cos x$; (ii) $2(1 - \cos x) > x \sin x$;

(iii) $x(2 + \cos x) > 3 \sin x$; (iv) $x^2 + x \sin x + 4 \cos x - 4 > 0$.

15. Establish the following inequalities when x lies in the range $[0, \frac{1}{2}\pi]$:

(i) $\tan(\frac{1}{2}x + \frac{1}{4}\pi) > 1 + \tan x$; (ii) $1 + \tan x > \sec x$;

(iii) $\tan x + 2 \sin x > 3x$; (iv) $\tan x - 24 \tan \frac{1}{2}x - 4 \sin x + 15x > 0$.

16. In a triangle ABC the sides b, c and the angle A are measured. If there is a small error δA in the measured value of A, find (i) the consequent error δa in the calculated value of a, and (ii) the error $\delta\Delta$ in the calculated value Δ of the area.

If $b = 24$ yards, $c = 39$ yards, $A = 53°\ 17'$, $\delta A = 0°\ 5'$, shew that $\delta a = 0{\cdot}035$ yards and $\delta\Delta = 0{\cdot}406$ square yards.

17. It was intended to set out an equilateral triangle ABC whose sides should be each 100 yards long by laying out the sides b, c at an angle of 60°. These sides, however, were made 4, 6 inches too long, respectively, and the angle 10′ too large. Find the consequent error in the side a. [14 inches.]

18. In a tangent galvanometer the current passing through the instrument is proportional to the tangent of the deflection of the needle. Shew that the instrument is most sensitive when the deflection is 45°.

[The sensitiveness here considered is with respect to fractional increase of current.]

19. The position of a point C is determined by chaining from two base points A, B in the same horizontal plane. Employ the method of infinitesimals to shew that if the measured lengths are subject to errors p, q, the greatest possible distance of the point located by the survey measurements from the actual point is $\operatorname{cosec} C \sqrt{(p^2 + q^2 + 2pq \cos C)}$, provided the angle C is acute.

[Calculate the greatest uncertainties of location on the assumption that AC, BC are in turn correct.]

20. If in the preceding Example the position of C is determined by sighting from A, B and the measured angles are subject to errors of $m°$, $n°$, shew that the greatest distance of the point located by the survey measurements from the actual point is

$$\frac{\pi c}{180} \operatorname{cosec}^2 C \sqrt{(n^2 \sin^2 A + m^2 \sin^2 B + 2mn \sin A \sin B \cos C)},$$

provided that C is acute.

[Calculate the greatest uncertainties of location on the assumption that the angles A, B are in turn correct.]

EXAMPLES XXVIII

EXTREME VALUES

1. Investigate the local extreme values of the following functions:
(i) $4\sin x+\sin 2x$; (ii) $\tan x+\cot \frac{1}{2}x$; (iii) $\tan x+1/x$;
(iv) $a\cos^2 x+b\sin^2 x$; (v) $(\sin mx)/\sin x$; (vi) $(\sin^2 mx)/\sin^2 x$;
(vii) $\sin^2 x \operatorname{cosec}(x+a)\operatorname{cosec}(x+b)$, $(0<a<b<\pi)$;
(viii) $(\sin^2 \frac{1}{2}x)/\sin(x-a)$; (ix) $a-b\sin x+c\tan x$;
(x) $x\sin\sqrt{x}$; (xi) $\sin x-\frac{1}{2}\sin^2 x$; (xii) $\sec^4 x-6\sec^2 x+8$.

2. A line AB, of length l, moves so as to be always a tangent to a circle, of radius a, which has its centre on Oy at distance h from O, $(h<l)$. The end A lies on Ox and AB points in the positive direction of the x-axis, touching the circle above the level of its centre. Find the equation for the inclination of AB to Ox when B is as far to the right as possible and shew that, if $l=8h+4a\sqrt{3}$, the required inclination is $\frac{1}{6}\pi$.

3. Two fixed points A, B are taken in order along Oy, where $OA=a$, $AB=ka$. A line OP, of fixed length l and variable slope ψ to Ox, is drawn through O and the angle APB is denoted by θ. Investigate an expression for $\tan\theta$ in terms of ψ, and prove that if $l>a(1+k)$, the maximum value of θ occurs when $\sin\psi=\dfrac{la(2+k)}{l^2+a^2(1+k)}$.

4. If in the preceding Example the length of the line OP is variable and its slope ψ is fixed, shew that the maximum value of θ is given by
$$\tan\theta=\frac{k\cos\psi}{2\sqrt{(1+k)}-(2+k)\sin\psi}.$$
Verify that for $\tan\psi=\frac{1}{2}$, $k=2$, the maximum angle is $46^\circ\ 53'$, nearly.

5. From the point A, $(0, \frac{1}{2}\pi)$, a line is drawn to a point P on the curve $y=\cos x$. Find the positions of P for which the gradient of AP is stationary and discriminate between the maxima and minima. Shew that for large values of x the positions of the stationary values are given approximately by $x=n\pi+\{(-1)^n\pi-2\}/(2n\pi)$.

6. If OP, PQ are equal (variable) distances along the positive part of the y-axis, shew that the maximum angle subtended by PQ at a fixed point on Ox is about $19^\circ\ 28'$.

7. Two points A, $(h, 0)$, and B, $(0, k)$, are fixed, $(h, k>0)$, and variable distances BP, PQ are taken along Oy, in the positive sense, in the ratio $1:m$. Shew that the maximum value of the angle subtended by PQ at A is
$$\operatorname{artan}\frac{mh}{(2+m)k+2\sqrt{(1+m)}\sqrt{(h^2+k^2)}}.$$

8. From a circular paper, of radius a, a sector is removed and the remainder formed into a right circular cone. Find the angle of the sector so that the volume of the cone may be a maximum. [About 66°.]

9. A ball hits another ball which is at rest. Find the angle at which it must impinge in order that the change in its direction of motion may be a maximum, assuming that the tangents of the inclination of the motion to the line of centres after and before impact are in the ratio $k:1$.

[Maximize the tangent of the deviation. The required angle is $\operatorname{artan}(1/\sqrt{k})$.]

10. A rod ABC rotates about B, where $AB = r_1$, $BC = r_2$ (where $r_2 > r_1$). Shew that the angle φ subtended by the rod at a point O at a distance l from B has an extreme value when $\cos\theta = -\dfrac{l(r_2 - r_1)}{l^2 - r_1 r_2}$, where θ is the angle OBA.

$$\left[\text{Shew that } \cot\varphi = \frac{l^2 - r_1 r_2 + l(r_2 - r_1)\cos\theta}{l(r_1 + r_2)\sin\theta}.\right]$$

11. Shew that the greatest distance from the centre to a normal of the ellipse $x = a\cos\theta$, $y = b\sin\theta$ is $a - b$.

12. A ship sails between two ports whose latitude θ is the same. Shew that the distance saved by sailing along the shortest great-circle course instead of the shortest parallel-of-latitude course is a maximum when the ports differ in longitude by 180°. Shew also that the greatest saving of distance by this procedure occurs when $\theta = \operatorname{arsin}(2/\pi)$ and that the distance saved is then $a\{2\operatorname{arsin}(2/\pi) + \sqrt{(\pi^2 - 4)} - \pi\}$, where a is the earth's radius.

13. A point P, (θ_0), lies on the helix $x = a\cos\theta$, $y = a\sin\theta$, $z = c\theta$. Find the condition for the existence of other points on the helix which are at stationary distances from P, and give a graphical method for determining such points.

14. A helix $x = a\cos\theta$, $y = a\sin\theta$, $z = c\theta$ has its axis at an inclination α to the vertical. Given that $\tan\alpha > c/a$, find the points on the helix which are of stationary height and distinguish between those of maximum and minimum height. [Take the axis of x horizontal.]

EXAMPLES XXIX

MISCELLANEOUS EXAMPLES OF TRIGONOMETRICAL FUNCTIONS

1. Shew that for the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, the subnormal is $|a\sin\theta|$.

2. Investigate the points of contraflexure of the curves whose equations are the following, and sketch the curves:

(i) $y = x + \sin x$; (ii) $y = \tan x - 2\sin x$; (iii) $y = 4\sin x + \sin 2x$;
(iv) $y = 4\cos^3 x + 3\sin x$; (v) $y = 3\sin x/(2 + \cos x)$.

3. Sketch the curve $x = a\cos\theta$, $y = a(1 + \sin\theta)\cos\theta$ and determine the points of contraflexure.

4. Sketch the curve $y = a\operatorname{cosec} x + b\sec x$ in the range $(0, 2\pi)$ of x. Find the number of roots of the equation $a\operatorname{cosec} x + b\sec x = 1$ in the above range, distinguishing the cases in which $a^{2/3} + b^{2/3} >, =, < 1$.

5. A circle of radius b rolls on the outside of a fixed circle of radius a and a point carried by the rolling circle at distance c (less than b) from its centre describes a curve $\mathcal{C}$. Prove that $\mathcal{C}$ will or will not have points of contraflexure according as $c > b^2/(a + b)$ or $c \leqslant b^2/(a + b)$.

[The curve $\mathcal{C}$ is an *epitrochoid*.]

6. Find the curvature at any point of the curve $y = a \sin \dfrac{2\pi x}{\lambda}$. Shew that if the wave-length is large compared with the amplitude, $\kappa \approx -(2\pi/\lambda)^2 y$.

7. Find the curvature at any point θ of the ellipse $x = a \cos\theta$, $y = b \sin\theta$.

Use the result of Ex. 26, Set XXIV, to shew that the equation of the circle of curvature at θ is

$$ab(x^2+y^2) - 2(a^2-b^2)(xb\cos^3\theta - ya\sin^3\theta) - \tfrac{1}{2}ab\{(a^2+b^2) - 3(a^2-b^2)\cos 2\theta\} = 0.$$

8. Shew that the curvature of the ellipse

$$x = a + b\cos\theta + c\sin\theta,\quad y = \alpha + \beta\cos\theta + \gamma\sin\theta$$

at the point θ is given by

$$\kappa = \frac{b\gamma - c\beta}{\{(b^2+\beta^2)\sin^2\theta + (c^2+\gamma^2)\cos^2\theta - 2(bc+\beta\gamma)\sin\theta\cos\theta\}^{3/2}}.$$

Hence, or otherwise, shew that at the ends of the axes of the ellipse,

$$\tan 2\theta = \frac{2(bc+\beta\gamma)}{b^2+\beta^2-c^2-\gamma^2}.$$

9. Shew that the curvature at any point θ of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$ is $\pm 2/(3a\sin 2\theta)$, where the upper (lower) sign is taken when x is positive (negative).

Shew also that the coordinates (ξ, η) of the centre of curvature are given by $\xi + \eta = a(\cos\theta + \sin\theta)^3$, $\xi - \eta = a(\cos\theta - \sin\theta)^3$.

10. Shew that the evolute of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ is the cycloid $x = a(\theta + \sin\theta)$, $y = -a(1-\cos\theta)$.

[The evolute is the locus of the centre of curvature.]

11. From a point P, (x, y) on a curve, a distance $PQ(=h)$ is taken along the normal in the sense of the curvature. Find the coordinates (x', y') of Q, employing the slope ψ at P. Shew that if P moves along the curve with variable speed V, (h remaining constant), the rates of increase of x', y' are given by

$$dx'/dt = V(1 - h\kappa)\cos\psi,\qquad dy'/dt = V(1-h\kappa)\sin\psi.$$

Explain the result for a point P at which $\kappa = 1/h$.

12. A circular disc of radius a rolls round the outside of a disc of the same size. Shew that, by suitable choice of axes, the equation of the tangent to the curve C described by a point on the rim is $x\sin 3\theta - y\cos 3\theta = 3a\sin\theta$, where 2θ is the angle turned through by the line of centres.

Shew also that the curvature at any point of the curve C is $\frac{3}{8}(\operatorname{cosec}\theta)/a$.

[The curve C is an *epicycloid*.]

13. A circular disc of radius a rolls along a straight line and P is a point of it at distance b from the centre. Shew that the curvatures at the highest and lowest points of the path traced out by P are $-b/(a+b)^2$, $b/(a-b)^2$, respectively.

Shew also that if A is the point of contact of the rolling circle with the line and C is the centre of the circle, the curvature vanishes when $\angle APC = 90°$.

[The path of the carried point P is called a *trochoid*.]

14. Find approximately the positive root of the equation $\sin x = \frac{7}{8}x$, replacing $\sin x$ by its Taylor polynomial of the third degree. Employ Newton's method to improve the approximation. [$\frac{1}{2}\sqrt{3}$, 0·88348.]

15. Shew that in approximating to the largest root of the equation $x + 6 = 2\sin x$, if $-4{\cdot}23$ is taken as the starting point in the application of

Newton's method, a closer approximation is assured. Find this approximation and shew that it is not in error by more than $1 \cdot 2 \times 10^{-5}$.

[Approximations to the three roots of the equation are given in Ex. 5, Art. 142.]

16. Use Newton's method to shew that the root of the equation $\left(\frac{\sin x}{x}\right)^4 = \cos x$ which lies in the range $(0, \frac{1}{2}\pi)$ is approximately $1 \cdot 1945$.

[Start with the approximation $1 \cdot 19$, as given in Ex. 2, Art. 149.]

17. Shew that a seven-figure table of tangents in which the common difference of argument is 1 minute becomes incomplete for angles greater than about 48°.

EXAMPLES XXX

INVERSE TRIGONOMETRICAL FUNCTIONS

1. Establish the following results:

(i) $\text{arsin } x = \frac{1}{2} \text{ arsin } \{2x\sqrt{(1 - x^2)}\}$ if $|x| \leqslant 1/\sqrt{2}$;

(ii) $\text{arcos } x = \frac{1}{2} \text{ arcos } (2x^2 - 1)$ if $0 \leqslant x < 1$;

(iii) $\text{artan } x = \frac{1}{2} \text{ artan } \{2x/(1 - x^2)\}$ if $|x| < 1$;

(iv) $\text{arcos } x = 2 \text{ artan } \sqrt{\left(\frac{1 - x}{1 + x}\right)}$ if $|x| \leqslant 1$;

(v) $\text{artan } x = \text{arsin } \frac{x}{\sqrt{(1 + x^2)}}$;

(vi) $\text{artan } x = \pm \frac{1}{2} \text{ arcos } \frac{1 - x^2}{1 + x^2}$ according as $x \gtrless 0$.

2. Express $\cos(\text{arsin } x)$, $\tan(\text{arsin } x)$, $\sin(2 \text{ arsin } x)$, $\cos(2 \text{ arsin } x)$, $\tan(2 \text{ artan } x)$ as algebraic functions of x.

3. Sketch the graphs of the functions $\sin(\text{artan } x)$, $\sin 2(\text{artan } x + \frac{1}{2}\pi)$.

4. Investigate the potential continuity of the functions $\frac{\text{arsin } x}{x}$, $\frac{\text{artan } x}{x}$ for the value 0 of x. [The completing value in each case is 1.]

5. Obtain the derivative of $\text{arsin } x$ by finding the completing value of the potentially continuous function $\frac{\text{arsin}(x + \Delta x) - \text{arsin } x}{\Delta x}$ at x.

6. Find the first derivative of each of the following functions:

(i) $\text{arsin}(1/x)$; (ii) $\text{artan}(1/x)$; (iii) $\text{arsec}(1/x)$; (iv) $x \text{ arsin } x$;
(v) $\text{arsin } mx$; (vi) $\text{arcos } (x^2)$; (vii) $\sin x + \text{arsin}\sqrt{x}$; (viii) $x \text{ artan } x$;
(ix) $\text{artan}(\tan \frac{1}{2}x)$; (x) $\frac{2}{\sqrt{3}} \text{ artan } \frac{2x + 1}{\sqrt{3}}$; (xi) $\text{artan } \frac{\sqrt{x} + \sqrt{a}}{1 - \sqrt{x}\sqrt{a}}$;
(xii) $\text{arsin } \sqrt{(1 - x^2)}$; (xiii) $\text{arcos } \frac{1 - x^2}{1 + x^2}$; (xiv) $\text{arcos } \sqrt{\left(\frac{\cos 3x}{\cos^3 x}\right)}$;
(xv) $\tan(m \text{ artan } x)$; (xvi) $\text{artan } \frac{a + b \cos x}{b + a \cos x}$;
(xvii) $\frac{3}{8} \text{ artan } x - \frac{x(3 + 5x^2)}{8(1 + x^2)^2}$; (xviii) $\text{artan}(1/\sqrt{x})$;
(xix) $\text{artan } \left\{\sqrt{\left(\frac{a - b}{a + b}\right)} \tan \frac{1}{2}x\right\}$; (xx) $\text{arsin}\{2ax\sqrt{(1 - a^2x^2)}\}$.

7. Shew that the completed function $y=\{x \operatorname{artan}(1/x)\}^*$ has one-sided derivatives when $x=+0$ and when $x=-0$.

8. If $u=x^2 \operatorname{artan}(y/x)-y^2 \operatorname{artan}(x/y)$, find the values of $\partial u/\partial x$, $\partial^2 u/\partial x^2$, $\partial^2 u/(\partial x \partial y)$.

9. Find the nth derivative of $x/(a^2+x^2)$.

[Apply Leibniz's Theorem, using the result

$$\left(\frac{d}{dx}\right)^n \frac{a}{a^2+x^2}=n!\frac{\sin\{(n+1)(\operatorname{artan}\frac{x}{a}+\frac{1}{2}\pi)\}}{(a^2+x^2)^{(n+1)/2}}.$$

The required derivative is $-n!\dfrac{\cos\{(n+1)(\operatorname{artan}\frac{x}{a}+\frac{1}{2}\pi)\}}{(a^2+x^2)^{(n+1)/2}}$.]

10. If $y_m=\sin(m \operatorname{arsin} x)$, shew that

$$x^{m+1}y_{m+1}=\frac{x^2\sqrt{(1-x^2)}}{m}\frac{d}{dx}(x^m y_m)=-\frac{1}{m}\frac{d}{d\lambda}(x^m y_m)=\frac{(-1)^m}{m!}\left(\frac{d}{d\lambda}\right)^m(xy_1),$$

where $x^2=1/(1+\lambda^2)$. Deduce the mth derivative of $1/(1+\lambda^2)$ with respect to λ. [Here x, λ are taken to have the same sign.]

11. Establish the result

$$\left(\frac{d}{dx}\right)^m(1-x^2)^{m+\frac{1}{2}}=\frac{(2m+1)(2m-1)\dots 1}{(m+1)!}(1+\lambda^2)^{(m+1)/2}\left(\frac{d}{d\lambda}\right)^{m+1}\operatorname{artan}\lambda,$$

$$\text{where } \lambda=\frac{x}{\sqrt{(1-x^2)}}.$$

12. The formula $\dfrac{1}{2^{n-1}}\cos(n \operatorname{arcos} x)$, ($n$ a positive integer), defines the general Tchebychef polynomial $T_n(x)$. Shew that the explicit expression for $2^{n-1}T_n(x)$ is

$$(-1)^{n/2}\{1-\frac{n^2}{2!}x^2+\frac{n^2(n^2-2^2)}{4!}x^4-\dots\},\quad (n \text{ even}),$$

and $$(-1)^{(n-1)/2}n\{x-\frac{n^2-1^2}{3!}x^3+\frac{(n^2-1^2)(n^2-3^2)}{5!}x^5-\dots\},\quad (n \text{ odd}).$$

Shew also that $T_n(x)$ satisfies the equation $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+n^2y=0$.

13. If $y=\operatorname{arsin}^2 x$, shew that

$$(1-x^2)y^{(n+2)}-(2n+1)xy^{(n+1)}-n^2y^{(n)}=0,\quad (n>0).$$

Shew that the Taylor polynomial of the sixth degree for $\operatorname{arsin}^2 x$ is

$$x^2+\tfrac{1}{3}x^4+\tfrac{8}{45}x^6.$$

[Either employ the values of the successive derivatives at $x=0$ or square the corresponding polynomial for $\operatorname{arsin} x$, neglecting powers of x above the sixth.]

14. Shew that $\operatorname{artan} x > x/(1+x^2)$ if $x>0$.

15. Employ Gregory's series to shew that $\operatorname{artan}\frac{1}{6}=0{\cdot}165148677$ with an error of 1 at most in the last figure written. Obtain the same result by using the formula $\operatorname{artan}\frac{1}{6}=\operatorname{artan}\frac{1}{5}-\operatorname{artan}\frac{1}{31}$ and calculating the terms on the right-hand side by means of Gregory's series.

CHAPTER VII

THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS. ASSOCIATED FUNCTIONS (HYPERBOLIC, EPICENE, EXPOCYCLIC)

SECTION I. THE EXPONENTIAL FUNCTION

154. Introductory Remarks.

In the present Chapter we consider the elementary functions which are described as of *exponential type*. These form a second class of transcendental functions. In analytical importance they rank with the trigonometrical functions.

The fundamental exponential function is a correlative of the elementary algebraic function which is a fixed power[1] of the independent variable. If in the power a^b we suppose a to take all values in a range of which the terminals are both positive and typify one such value by x, b being supposed constant, we arrive at the function x^b, x being a *variable base* and b the constant *exponent*. If, on the other hand, we suppose b to take all values in a range and typify one such value by x, a being a positive constant, we arrive at the function a^x, x being now a *variable exponent* and a the constant *base*. This function a^x is called the *elementary exponential function* of base a, and is symbolized (alternatively) by[2] $\exp_a x$.

We now proceed to the treatment of this function and others derived from it. The exposition assumes a knowledge of the theorems concerning potentiation and radication (index laws, etc.), given in books on elementary Algebra. These theorems are demonstrated in Chap. XXI in the manner now regarded as rigorous and the proofs can be read at any stage.

[1] In the language of the Theory of Functions, this function is algebraic only when it is a *rational* power of the variable.

[2] The symbol '$\exp_a$' is to be regarded as a particular case of the general functional symbol 'f'. The form $\exp_a x$ is sometimes much more convenient than the form a^x, especially from a typographical standpoint; compare, for example, the form $10^{2^{\frac{1}{30}}}$ with $\exp_{10}(2^{1/30})$.

The particular results which we suppose known[1] are as follows :

(i) the existence of the general power a^b, where a, b are numbers, rational or irrational, and a is positive ; if b is the rational fraction m/n, $(n>0)$, a^b is the *positive* nth root of the mth power of a ;

(ii) that if $a>1$, $a^b>1$ if $b>0$ and $0<a^b<1$ if $b<0$;

(iii) that if $a>1$, $a^{b'}>a^b$ if $b'>b$;

(iv) that $a^b a^{b'}=a^{b+b'}$, where b, b' are numbers different from zero ; the *first index law*. Under the convention $a^0=1$, explained in the next Article, the restriction on b, b' is removed in this and the following two results ;

(v) that $a^b c^b=(ac)^b$, $(c>0)$; the *second index law* ;

(vi) that $(a^b)^{b'}=a^{bb'}=(a^{b'})^b$; the *third index law*.

155. Continuity of a^x.

A discussion of the continuity of a^x is given in Chap. XXI, Art. 390, as a step in the exposition of the theory of general powers. It is, however, desirable to insert here a verification that the standard ε-condition of continuity is satisfied in virtue of the results stated in the preceding Article.

The condition for the continuity of a^x at x_1, $(x_1 \neq 0)$, is expressed by

$$|a^x-a^{x_1}|<\varepsilon,\ (\varepsilon \text{ arbitrary}),\ \text{if } |x-x_1|<k,\ (k \text{ appropriate}). \quad \ldots\ldots(1)$$

Writing $x=x_1+h$, so that $a^x-a^{x_1}=a^{x_1+h}-a^{x_1}=a^{x_1}(a^h-1)$, this takes the form

$$a^{x_1}|a^h-1|<\varepsilon \text{ if } |h|<k,$$

or

$$|a^h-1|<\varepsilon/a^{x_1} \text{ if } |h|<k. \quad \ldots\ldots(2)$$

We proceed to shew that an appropriate value of k can be found to satisfy this condition.

(i) Suppose, at first, that $a>1$ and write $a=1+\alpha$, $(\alpha>0)$. If n is a positive integer, we have, as in Art. 50, Ex. 2,

$$\left(1+\frac{\alpha}{n}\right)^n>1+\alpha, \quad \ldots\ldots(3)$$

and hence

$$1+\frac{\alpha}{n}>(1+\alpha)^{1/n}, \quad \ldots\ldots(4)$$

the nth root of the greater number being the greater. We thus have $(1+\alpha)^{1/n}-1<\alpha/n$, that is, $a^{1/n}-1<(a-1)/n$; whence it follows that

$$a^{1/n}-1<\varepsilon',\ (\varepsilon' \text{ arbitrary}),\ \text{if } n>(a-1)/\varepsilon'. \quad \ldots\ldots(5)$$

[1] If the student is in doubt as to any one of the results stated, he should test it by taking a simple numerical case.

We deduce at once that if $h > 0$ we can make $a^h - 1 < \varepsilon'$ by taking $h < 1/n < \varepsilon'/(a-1)$, for then $a^h < a^{1/n} < 1 + \varepsilon'$.

Again, we have from (4),

$$1 - \frac{1}{(1+\alpha)^{1/n}} < 1 - \frac{1}{1+\alpha/n} = \frac{\alpha/n}{1+\alpha/n} < \alpha/n. \qquad \text{.............(6)}$$

For positive values of h we can therefore make $1 - \dfrac{1}{a^h} < \varepsilon'$, and thus $|\,a^{-h} - 1\,| < \varepsilon'$, by again taking $h < 1/n < \varepsilon'/(a-1)$.

Thus, whether h is positive or negative,

$$|\,a^h - 1\,| < \varepsilon', \quad (\varepsilon' \text{ arbitrary}), \quad \text{if}^{1}\ |\,h\,| < 1/n < \varepsilon'/(a-1), \qquad \text{......(7)}$$

where n is the integer next greater than $(a-1)/\varepsilon'$.

(ii) If, next, $0 < a < 1$, we write $a^h = (1/a)^{-h}$, and, by the above,

$$\left|\left(\tfrac{1}{a}\right)^{-h} - 1\right| < \varepsilon', \quad (\varepsilon' \text{ arbitrary}), \text{ if } |\,h\,| < 1/n < \varepsilon'/\left(\tfrac{1}{a} - 1\right), \qquad \text{........(8)}$$

where n is the integer next greater than $(1/a - 1)/\varepsilon'$.

The case of $a = 1$ requires no discussion.

We conclude that when $a > 0$ we can always make $|\,a^h - 1\,| < \varepsilon'$, ($\varepsilon'$ arbitrary), by taking $|\,h\,|$ less than $1/n$, where n is the integer next greater than $(a-1)/\varepsilon'$ or $\left(\frac{1}{a} - 1\right)/\varepsilon'$ according as a is greater or less than 1. The condition (2) follows at once by taking $k = \varepsilon/(\alpha a^{x_1})$, where α is equal to $a - 1$ or $\frac{1}{a} - 1$ according as $a \gtrless 1$, and the continuity of a^x at any value x_1, different from zero, is established.

The above results shew, moreover, that the function is potentially continuous at $x = 0$ and is completed by giving it the value 1 there. In conformity with the practice of denoting completed functions by the addition of an asterisk to the functional symbol, we should denote the completed exponential function by $(a^x)^*$, but the asterisk is not here necessary, and we shall henceforth use the symbol a^x to denote the exponential function whose value at $x = 0$ is defined to be 1. The range of values of this function is $[0, \infty]$, corresponding to the range $[-\infty, \infty]$ of x.

Ex. 1. Prove that $|\,10^x - 1\,| < 0{\cdot}2501$ if $|\,x\,| < 1/2^3$, while

$$|\,10^x - 1\,| < 2{\cdot}1445 \times 10^{-9} \text{ if } |\,x\,| < 1/2^{30}.$$

[The successive square roots of 10, up to the sixtieth, are given in Callet, *Table de Logarithmes*. Quoting from this we have

$n = 1/2^3 = 0{\cdot}125, \qquad 10^n = 1{\cdot}3335,$

$n = 1/2^{30} = 0{\cdot}313 \times 10^{-10}, \qquad 10^n = 1{\cdot}000\ 000\ 002\ 144\ 449.$

The above inequalities follow at once from these results.]

[1] We here assume ε' small, so that there is an integer n such that the condition $|\,h\,| < \frac{1}{n}$ is satisfied if $|\,h\,| < \varepsilon'/(a-1)$. A corresponding remark applies also to (8).

156. The Property of Geometric Progression.

Before going on to the construction of the graph of the exponential function, it is convenient to call attention to a property which is an immediate consequence of the index laws. If a set of equidistant values of the independent variable is chosen, thus forming an arithmetic progression, the corresponding values of the exponential function form a geometric progression. Denoting the successive values of x by x_1, x_1+h, x_1+2h, ... , the corresponding values of a^x are a^{x_1}, a^{x_1+h}, a^{x_1+2h}, ... , that is,

$$a^{x_1},\ a^{x_1}a^h,\ a^{x_1}(a^h)^2,\ \dots \qquad \text{.......................(1)}$$

The ratio of any one of these values to the immediately preceding one is the constant number a^h, which expresses the property stated.

157. Graph of the Function.

We consider, at first, the special value 2 of a, so that the function is $y=2^x$. Taking the integral values -3, -2, -1, 0, 1, 2, 3 for x, the corresponding values for y are $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 4, 8, giving a set of points on the curve such that each ordinate is double that of the preceding one.

In order to fill in between these points we may make use of the above property. For if x_1, x_1+h are successive integers, the value of 2^x for the half-integer $x_1+\frac{1}{2}h$ satisfies the relation $2^{x_1+\frac{1}{2}h}/2^{x_1}=2^{x_1+h}/2^{x_1+\frac{1}{2}h}$ and is therefore given by

$$2^{x_1+\frac{1}{2}h}=\sqrt{(2^{x_1}2^{x_1+h})}. \qquad \text{.........(1)}$$

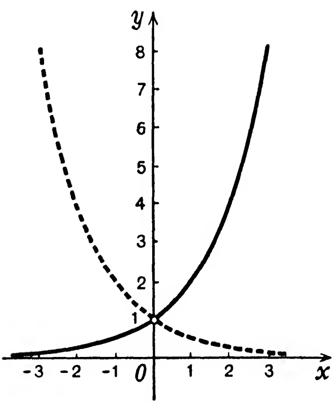

FIG. 129.

—— $y=2^x$,

- - - - $y=2^{-x}$.

By continued subdivision we can, in this way, find to any desired degree of accuracy the ordinates of as many points as we please on the curve corresponding to rational values of x. The general form[1] of the curve in the range $(-3, 3)$, obtained in this way, is shewn by the full line in Fig. 129. On account of the continuity of 2^x, plot-points for irrational values of x can be approximated to indefinitely by plot-points for rational values of x. This result can be expressed

[1] It is proved in Art. 159 that the curve is throughout concave upwards, as in the Figure.

by saying that the plot-points for irrational values of x also lie on the curve drawn.

If, next, we consider the function $(\frac{1}{2})^x$, the graph can be obtained from that of 2^x merely by reversing the direction of the x-axis, since $(\frac{1}{2})^x = 2^{-x}$. The graph of the function $(\frac{1}{2})^x$ relative to axes drawn in the conventional directions is shewn by the broken line in Fig. 129, the range of x being again $(-3, 3)$.

The form of the graph of a^x when $a > 1$ is evidently the same as that of 2^x, for the geometric law of increase of the function applies for any such value of a. Further, it is easily shewn that if a, b are any positive numbers, the functions a^x, b^x can be expressed in terms of each other. To demonstrate this we employ the elements of the theory of logarithmation, as given in elementary Algebra. According to this theory, we have

$$b = a^{\log_a b}, \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2)$$

from which it follows that for any chosen value of x,

$$b^x = (a^{\log_a b})^x = a^{x \log_a b} = a^{x'}, \text{ say}, \quad \dots\dots\dots\dots\dots\dots (3)$$

where $x' = x \log_a b$. This shews that the graph of a^x may be employed as a representation of b^x merely by changing the scale along the x-axis in the ratio $1 : \log_a b$.

The property of geometric progression may be employed to decide whether a given curve approximates to an exponential curve. If the function is decreasing, the lengths of intervals in which the function falls to *one-half* of its value at the beginning (left end) of each interval are compared. If these are approximately equal we conclude that the curve is approximately exponential. If the function is increasing, the intervals refer to the *doubling* of the function in each.

Ex. 1. Sketch the graph of the function $a^{|x|}$, $(a > 1)$.

158. Derivative of a^x.

Suppose, at first, that $a > 1$. We investigate the derivative at a point x_1, the corresponding value of the function being denoted by y_1. For increments Δx, Δy, we have

$$\frac{\Delta y}{\Delta x} = \frac{a^{x_1 + \Delta x} - a^{x_1}}{\Delta x} = a^{x_1} \cdot \frac{a^{\Delta x} - 1}{\Delta x}, \quad (\Delta x \neq 0). \quad \dots\dots\dots\dots (1)$$

The function on the right-hand side of (1) is the product of the constant term a^{x_1} and the function $(a^{\Delta x} - 1)/\Delta x$. We proceed to shew that the latter is potentially continuous at x_1, a property which, according to the argument of Art. 66, establishes the existence of

the derivative of a^x at x_1. The determination of the actual completing value is considered in Art. 160.

Writing h instead of Δx, for convenience, the function to be considered is $(a^h - 1)/h$. We shew that the variation of this function can be made as small as we please by confining $|h|$ to an appropriately chosen range.

(i) Let k be any chosen positive number and suppose, at first, that h is positive and a rational fraction m/n of k, so that $a^h = a^{\frac{m}{n}k}$, (m, n positive integers, $m \leqslant n$). Then

$$a^h - 1 = a^{\frac{m}{n}k} - 1 = \left(a^{\frac{1}{n}k}\right)^m - 1$$

$$= \left(a^{\frac{1}{n}k} - 1\right)\left(a^{\frac{m-1}{n}k} + a^{\frac{m-2}{n}k} + \dots + 1\right), \quad \dots\dots(2)$$

giving

$$\frac{a^{\frac{m}{n}k} - 1}{a^{\frac{1}{n}k} - 1} = \sum_{r=1}^{m} a^{\frac{r-1}{n}k}, \quad \dots\dots(3)$$

and, in particular, for $m = n$,

$$\frac{a^k - 1}{a^{\frac{1}{n}k} - 1} = \sum_{r=1}^{n} a^{\frac{r-1}{n}k}. \quad \dots\dots(4)$$

Consider now the expression

$$1 + a^{\frac{1}{n}k} + a^{\frac{2}{n}k} + \dots + a^{\frac{m-1}{n}k} + a^{\frac{m}{n}k} + \dots + a^{\frac{n-1}{n}k}, \quad \dots\dots(5)$$

in which the terms continually increase, so that the average value of the n terms is greater than that of the first m terms. Also, the average value of any number of terms is less than the greatest term, viz. $a^{\frac{n-1}{n}k}$, (and, *a fortiori*, less than a^k), and greater than the least term, viz. 1. We thus have the set of inequalities

$$a^k > \frac{1}{n}\sum_{r=1}^{n} a^{\frac{r-1}{n}k} > \frac{1}{m}\sum_{r=1}^{m} a^{\frac{r-1}{n}k} > 1, \quad \dots\dots(6)$$

which, with the aid of (3) and (4), becomes

$$a^k > \frac{a^k - 1}{n(a^{\frac{1}{n}k} - 1)} > \frac{a^{\frac{m}{n}k} - 1}{m(a^{\frac{1}{n}k} - 1)} > 1. \quad \dots\dots(7)$$

From the last two of these inequalities we have

$$\frac{a^k - 1}{n} > \frac{a^{\frac{m}{n}k} - 1}{m} > a^{\frac{1}{n}k} - 1,$$

and from the first one,

$$a^{\frac{1}{n}k} - 1 > \frac{a^k - 1}{na^k}.$$

Hence, on combining and dividing throughout by k/n, we get the fundamental double inequality

$$\frac{a^k - 1}{k} > \frac{a^{\frac{m}{n}k} - 1}{\frac{m}{n}k} > \frac{1 - a^{-k}}{k}, \quad \text{......................(8)}$$

or

$$\frac{a^k - 1}{k} > \frac{a^h - 1}{h} > \frac{1 - a^{-k}}{k}, \quad \text{......................(9)}$$

when h has the form $\frac{m}{n}k$.

We assume, for the present, that the same result is true when h/k is irrational and less than 1, so that (9) holds for any positive value of h in the range $(0, k)$. The mode of passage to irrational values of h/k will be seen in Chap. XXI.

(ii) In considering the case of h negative, we write $h = -\frac{m}{n}k$, and by an argument similar to the above, we reach the set of inequalities

$$1 > \frac{1 - a^{-\frac{m}{n}k}}{m(1 - a^{-\frac{1}{n}k})} > \frac{1 - a^{-k}}{n(1 - a^{-\frac{1}{n}k})} > a^{-k}, \quad \text{...............(10)}$$

which leads to the same result (9) when h is a negative rational fraction of k. Including again the case when h is an irrational fraction of k, the fundamental double inequality (9) holds when h is any number other than zero in the range $(-k, k)$.

If now h', h'' are any two such values of h, we have

$$\left| \frac{a^{h'} - 1}{h'} - \frac{a^{h''} - 1}{h''} \right| < \frac{a^k - 1}{k} - \frac{1 - a^{-k}}{k} = \frac{1 - a^{-k}}{k}(a^k - 1), \quad \text{.....(11)}$$

and the potential continuity of $(a^h - 1)/h$ at $h = 0$ is demonstrated by shewing that the expression on the left of (11) can be made less than an arbitrarily chosen number ε by suitably restricting the range of h. This readily follows, for, from (9), if k' is *any* positive number,

$$\frac{1 - a^{-k}}{k} < \frac{a^{k'} - 1}{k'} = M, \text{ say}, \quad \text{.....................(12)}$$

while, from Art. 155,

$$a^k - 1 < \varepsilon', \ (\varepsilon' \text{ small, arbitrary}), \text{ if } k < \varepsilon'/(a - 1). \quad \text{.......(13)}$$

The difference in question can therefore be made less than ε by taking k, and therefore $|h|$, less than $\varepsilon/\{M(a-1)\}$.

This establishes the potential continuity, and the completing value, which we denote temporarily by L_a, is given by

$$L_a=\left(\frac{a^h-1}{h}\right)^*_{h=0}=\lim_{h\to 0}\frac{a^h-1}{h}. \qquad\text{(14)}$$

The derivative of a^x at x_1 thus exists, its value, or the completing value of the function $\Delta y/\Delta x$, being $a^{x_1}L_a$. Thus, for any value of x,

$$\frac{da^x}{dx}=a^xL_a. \qquad\text{(15)}$$

This result shews that the gradient at any point of the graph of the function a^x is the product of the ordinate by L_a. It follows that the sub-tangent (Art. 83. 4) has the constant length $1/L_a$. It will appear later that the exponential curve is the only one with this property. Rough approximations to the value of L_a can be obtained by drawing the tangents to the curve at various points and taking the reciprocals of the corresponding sub-tangents.

The case in which $0<a<1$ can be dealt with separately by the above process, but it is most conveniently disposed of by considering the reciprocal, b say, of a. We have $a^h=b^{-h}$, and hence

$$\frac{a^h-1}{h}=\frac{b^{-h}-1}{h}=-b^{-h}\frac{b^h-1}{h}. \qquad\text{(16)}$$

Since $b^{-h}\to 1$ as $h\to 0$, the limit of $(a^h-1)/h$ is minus the limit of $(b^h-1)/h$, that is, $L_a=-L_b$. Hence

$$\frac{da^x}{dx}=a^xL_a=-a^xL_b. \qquad\text{(17)}$$

In all the general discussions it is sufficient to suppose the base greater than 1, and this we shall do.

159. Geometrical Interpretation of the Fundamental Double Inequality.
The double inequality

$$\frac{a^k-1}{k}>\frac{a^h-1}{h}>\frac{1-a^{-k}}{k} \qquad\text{(1)}$$

can be employed directly to establish the important geometrical property that the graph of the exponential function a^x, $(a>0)$, is throughout a curve of finite upward curvature. For if P_k, P_{-k}, P, A are the points on the curve of abscissae k, $-k$, h, 0, respectively, (Fig. 130), the successive members in (1) are the gradients of the chords AP_k, AP, AP_{-k}, and we therefore conclude that the chord

joining A to any point P_k (or P_{-k}) lies above all intermediate points P. The possession of this property is, according to Art. 111, the required condition for upward concavity.

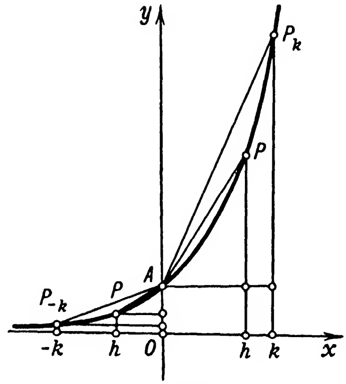

FIG. 130.

Conversely, the double inequality can be recovered geometrically (but not, of course, *proved*) in the above way when the general form of the curve is assumed.

It should be observed that from the above equation (15) we have

$$\frac{d^2a^x}{dx^2}=L_a\frac{da^x}{dx}=L_a{}^2a^x>0, \quad \text{....(2)}$$

so that the condition of Art. 112 for upward concavity is satisfied at all points of the graph.

160. Change of Base. Connection between the Values of L for Different Bases. The Special Bases e, 10.

For the function b^x, $(b>0)$, we have

$$\frac{db^x}{dx}=b^xL_b, \quad \text{..................................(1)}$$

where $L_b=\lim\limits_{h\to 0}\frac{b^h-1}{h}$. Now, recalling the result (2), Art. 157, we can write

$$b^h=a^{h\log_a b}=a^{h'}, \text{ say, } \quad \text{..........................(2)}$$

where $h'=h\log_a b$, so that $h'\to 0$ as $h\to 0$. Hence

$$\frac{b^h-1}{h}=\frac{a^{h'}-1}{h}=\frac{a^{h'}-1}{h'}\log_a b, \quad \text{......................(3)}$$

and therefore

$$\lim_{h\to 0}\frac{b^h-1}{h}=\lim_{h'\to 0}\frac{a^{h'}-1}{h'}\log_a b,$$

or

$$L_b=L_a\log_a b. \quad \text{................................(4)}$$

Interchanging a and b, we have the equivalent relation

$$L_a=L_b\log_b a. \quad \text{................................(5)}$$

It follows from (4) that if the value of L_a is known, that of L_b can be calculated provided that $\log_a b$ is known. This latter number can be found approximately by means of a table of logarithms. The theory of logarithms is discussed in Section III below, where it is

shewn how to calculate the logarithm of a positive number to a given base to any desired degree of accuracy.

From the analytical point of view there is one special base of great importance, namely that for which L has the value 1. The value b of the base must then satisfy the relation

$$1 = L_a \log_a b,$$

and is thus given by $b = a^{1/L_a}$. This special value of b is denoted by the letter e so that

$$e = a^{1/L_a}, \quad \text{.................................(6)}$$

and we have, from (4) and (5),

$$L_e = 1 = L_a \log_a e, \quad L_a = L_e \log_e a = \log_e a = \frac{1}{\log_a e}. \quad \text{..........(7)}$$

Thus L_a is the logarithm of a to the base e, (or the reciprocal of the logarithm of e to the base a); it can be calculated to any degree of accuracy by the methods of Section III below.

For the derivative of a^x we now have the formula

$$\frac{da^x}{dx} = a^x L_a = a^x \log_e a, \quad \text{.........................(8)}$$

while for the special function e^x we have

$$\frac{de^x}{dx} = e^x L_e = e^x. \quad \text{.................................(9)}$$

A second exponential base of practical importance is the number 10, and for this we have the results:

$$\frac{10^k - 1}{k} > L_{10} > \frac{1 - 10^{-k}}{k}, \quad (k > 0), \quad \text{...............(10)}$$

$$L_{10} = \log_e 10 = 1/\log_{10} e, \quad \text{.......................(11)}$$

$$\frac{d10^x}{dx} = 10^x L_{10} = 10^x \log_e 10. \quad \text{.....................(12)}$$

160. 1. *Approximate determination of $\log_e 10$.* We may employ (10) to find an approximation to the value of L_{10}, that is, of $\log_e 10$. We take for k the small value $1/2^{30}$, so that 10^k is the thirtieth of the successive square roots of 10. Quoting from Callet, *Table de Logarithmes*, we have

$$\frac{1}{k} = 2^{30} = 1{\cdot}073742 \times 10^9, \quad 10^k = 1{\cdot}000\,000\,002\,144\,449, \quad \text{.........(13)}$$

correct to the number of digits written, and hence

$$\frac{10^k - 1}{k} = \frac{2{\cdot}144449 \times 10^{-9}}{\dfrac{1}{1{\cdot}073742 \times 10^9}} = 2{\cdot}144449 \times 1{\cdot}073742 = 2{\cdot}302585, \quad \text{......(14)}$$

correct to six places of decimals. The effect of multiplying this number by 10^{-k} is insignificant so far as accuracy to six places is concerned, and hence we have the result

$$L_{10} = \log_e 10 = 2{\cdot}302585, \quad \text{..........................(15)}$$

correct to six places of decimals. This gives, correct to the same order,

$$\log_{10} e = 1/\log_e 10 = 1/2{\cdot}302585 = 0{\cdot}434294. \quad \text{.................(16)}$$

We also have the results

$$e = 10^{1/L_{10}} = 10^{0{\cdot}434294}, \quad \frac{d10^x}{dx} = 10^x \times 2{\cdot}302585 \ldots. \quad \text{...........(17)}$$

If we may employ a table of logarithms to base 10, the value of e can be found approximately by the process of 'reading backwards' from the tabulated value 0·434294 of the logarithm to the value of the argument. Correct to five decimal places, the value of e is 2·71828. We shew in Art. 162 below how to calculate e directly to any desired degree of accuracy without the aid of logarithm tables.

161. The Exponential Function e^x.

The simplicity of the result $de^x/dx = e^x$ is the justification for the introduction of e as the special base employed in Analysis. The student will recall that circular measure is introduced into Trigonometry for a similar reason. Since an exponential function to any (positive) base a is expressible as an exponential function to the base e, thus $a^x = e^{x \log_e a}$, all the properties of a^x become known as soon as those of e^x are known. For this and the above reasons, the function e^x is treated as the fundamental exponential function, and is usually referred to as *the exponential function of x*. In accordance with Art. 154, the alternative notation for this function is $\exp_e x$. As this symbolism is rarely employed except where the base is e, the indication of this base is usually omitted.

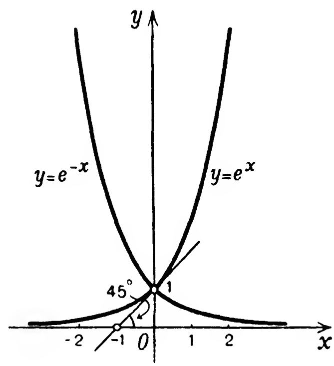

FIG. 131.

The graph of e^x for the range $(-3, 2)$ of x is shewn in Fig. 131, as well as that of e^{-x} in the range $(-2, 3)$. The ordinates of these curves have been taken from tabulated values of the functions. Methods for the tabulation of e^x, e^{-x} are given in Art. 162. Among the various tables in common use we may mention (i) Smithsonian Mathematical Tables, *Hyperbolic Functions*, and (ii) Hayashi, *Siebenstellige Tafeln der Kreis- u. Hyperbelfunktionen*. In the

former, the function e^x is tabulated correct to six, and e^{-x} to seven, places of decimals for values of x ranging up to 6. In the latter, the same functions are tabulated for values of x ranging up to 50, the number of decimal places in the tabulated values ranging from ten to twenty.[1]

161. 1. *Simple extensions.*

(i) *The function* $y=e^{-x}$. We write $y=1/e^x=1/u$, where $u=e^x$, and hence

$$\frac{dy}{dx}=-\frac{1}{u^2}\frac{du}{dx}=-\frac{1}{e^{2x}}e^x=-e^{-x}. \qquad (1)$$

(ii) $y=a^{-x}$. Here we find

$$\frac{dy}{dx}=-a^{-x}\log_e a. \qquad (2)$$

(iii) $y=e^{f(x)}$. Writing $f(x)=u$, $y=e^u$, the rule for the derivative of a function of a function gives

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=e^u f'(x)=f'(x)e^{f(x)}. \qquad (3)$$

In particular, putting $f(x)=-x$, $f'(x)=-1$, we recover the result (1). Again, putting $f(x)=kx$, (k constant), we find

$$\frac{de^{kx}}{dx}=ke^{kx}, \qquad (4)$$

a result of frequent application.

(iv) $y=f(e^x)$. Writing $e^x=u$, $y=f(u)$, we obtain the result

$$\frac{dy}{dx}=f'(u)\,e^x=f'(e^x)\,e^x, \qquad (5)$$

where, as in Art. 86, $f'(e^x)$ means the derivative of $f(e^x)$ with respect to e^x.

Ex. 1. Shew that $\quad da^{f(x)}/dx=\log_e a\,.f'(x)\,a^{f(x)}. \qquad (6)$

Ex. 2. Find the derivative of e^{x^2} and of e^{x^2/k^2}.

Ex. 3. Shew that $\quad \dfrac{d}{dx}(e^{x/a}+e^{-x/a})=\dfrac{1}{a}(e^{x/a}-e^{-x/a}). \qquad (7)$

Ex. 4. *The function* $y=e^{|x|}$. Here $y=e^x$ when $x>0$ and e^{-x} when $x<0$. We conclude that

$$\frac{de^{|x|}}{dx}=\pm e^{|x|}, \qquad (8)$$

the positive (negative) sign being taken if x is positive (negative).

[1] Tables of logarithms of logarithms (*lologs*) have been published by Chappell, *Five-Figure Mathematical Tables*, to facilitate the calculation of a^b. These are described in Art. 186.

Ex. 5. $y = e^{-1/x}$. This function is discontinuous when $x = 0$. Its derivative at any other point is given by

$$\frac{de^{-1/x}}{dx} = \frac{de^{-1/x}}{d(-1/x)} \frac{d(-1/x)}{dx} = \frac{1}{x^2} e^{-1/x}. \quad \text{.......................(9)}$$

See also Ex. 4, Art. 165, p. 392.

Ex. 6. $y = e^{mx} \sin nx$. We have

$$\frac{dy}{dx} = me^{mx} \sin nx + e^{mx} n \cos nx = e^{mx}(m \sin nx + n \cos nx). \quad \text{.........(10)}$$

The expression in parentheses can be written in the form

$$\sqrt{(m^2 + n^2)} \,.\, \sin(nx + c), \quad \text{.............................(11)}$$

where $\tan c = n/m$, and we may thus write

$$\frac{d}{dx} e^{mx} \sin nx = \sqrt{(m^2 + n^2)} \,.\, e^{mx} \sin(nx + c). \quad \text{..................(12)}$$

Functions of this type are called *expocyclic*. They are dealt with in Section VI of the present Chapter.

162. Higher Derivatives and Taylor's Theorem for the Exponential Function.

Since $de^x/dx = e^x$, the effect of each successive differentiation is to reproduce e^x, and so we have the general result

$$\frac{d^n e^x}{dx^n} = e^x. \quad \text{...............................(1)}$$

In the case of the general exponential function a^x, the effect of each differentiation is to introduce the multiplier L_a (i.e. $\log_e a$), and we now have

$$\frac{d^n a^x}{dx^n} = a^x L_a{}^n. \quad \text{...............................(2)}$$

The function e^x satisfies all the conditions for the $I.D_n.T.$ with starting point $x = 0$. We have

$$f^{(r)}(0) = 1, \quad f^{(n)}(\theta x) = e^{\theta x}, \quad \text{.......................(3)}$$

and the theorem becomes, in this case,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^{n-1}}{(n-1)!} + R_n(x), \quad \text{.............(4)}$$

where

$$R_n(x) = e^{\theta x} \frac{x^n}{n!}, \quad (0 < \theta < 1). \quad \text{....................(5)}$$

In investigating a remainder range it is convenient to consider separately the cases of x positive and negative.

(i) $x > 0$. We have $e^{\theta x} < e^x$, since e^x increases with x, and hence $R_n(x) < e^x x^n/n!$. Now, as in Art. 146. 2, for any given value of x, the value of $x^n/n!$ can be made as small as we please by increasing n sufficiently. Hence $R_n(x) \to 0$ as $n \to \infty$. The conditions for Taylor's Series are therefore satisfied and Taylor's Theorem for e^x, (x for the moment restricted to positive values), becomes

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots \quad \dots\dots\dots\dots(6)$$

Ex. 1. Shew that $e < 3$.

Taking the $I.D_3.T.$ for $x = 1$, we have

$$e = 1 + 1 + \tfrac{1}{2} + R_3(1), \quad \dots\dots\dots\dots(7)$$

where $R_3(1) < \frac{1}{6}e$. Hence $\frac{5}{6}e < \frac{5}{2}$, or $e < 3$.

It follows that for any positive value of x,

$$R_n(x) < 3^x x^n/n!. \quad \dots\dots\dots\dots(8)$$

Ex. 2. Shew that if P denotes the polynomial $1 + x + \frac{x^2}{2!} + \dots + \frac{x^{n-1}}{(n-1)!}$, and $x^n/n! < 1$, $(x > 0)$, then e^x lies between P and $P\Big/\left(1 - \frac{x^n}{n!}\right)$.

[This result shews that the error made in replacing e^x by P does not exceed $P\dfrac{x^n/n!}{1 - x^n/n!}$.]

(ii) $x < 0$. The exponent θx is now negative, and $e^{\theta x} < e^0$, that is, $e^{\theta x} < 1$, so that $|R_n(x)| < |x|^n/n!$ A remainder range may therefore be taken as $(0, x^n/n!)$. The magnitude of $R_n(x)$ decreases indefinitely with increase of n and the infinite series (6) again holds.

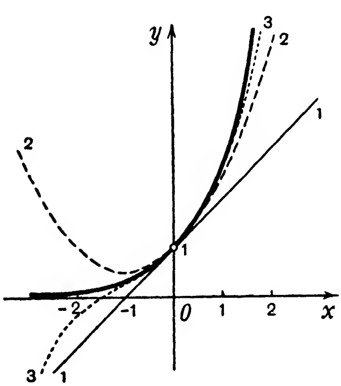

FIG. 132.

We thus have, as the general statement of Taylor's Theorem for e^x,

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots, \quad \dots(9)$$

which holds for all values of x. The series here is usually referred to as the *exponential series*.

The closeness of approximation of the earlier Taylor polynomials to e^x is illustrated in Fig. 132.

In the case of the function a^x, the $I.D_n.T.$ takes the form

$$a^x = 1 + xL_a + \frac{x^2 L_a{}^2}{2!} + \dots + \frac{x^{n-1}L_a{}^{n-1}}{(n-1)!} + a^{\theta x}\frac{x^n L_a{}^n}{n!}, \quad \dots\dots\dots(10)$$

where $L_a = \log_e a$, and Taylor's Theorem becomes

$$a^x = 1 + xL_a + \frac{x^2 L_a{}^2}{2!} + \dots + \frac{x^n L_a{}^n}{n!} + \dots, \quad \dots\dots\dots\dots\dots(11)$$

for all values of x. The series here is called the *general exponential series.*

162. 1. The series on the right-hand side of equation (9) is frequently taken as the *definition* of the exponential function e^x. The rigorous treatment of the function on this basis requires, however, the use of the theory of *Power Series*, for which we are not prepared at this stage. It may be mentioned that other ways of introducing the exponential function into Analysis have been adopted. One way that has been used is to introduce it after the logarithmic function, of which it is the inverse, making the logarithmic function the starting point. This mode of treatment is referred to in Art. 232. 1, Chap. IX.

Ex. 3. *Approximate evaluation of e.* Putting $x = 1$ in (9), we get

$$e = 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{n!} + \dots . \quad \dots\dots\dots\dots\dots\dots\dots\dots(12)$$

If we retain only 13 terms of the series, then, according to (8), the error committed will not exceed $3/13!$, or $4{\cdot}83 \times 10^{-10}$, and will not, therefore, affect the ninth place of decimals in the calculated value of e. We find

$$e \approx 1 + 1 + \frac{1}{2!} + \dots + \frac{1}{12!} = 2{\cdot}71828\ 18286. \quad \dots\dots\dots\dots\dots(13)$$

In this calculation there are 10 sources of error due to approximating in the tenth place in the reciprocals of the factorial terms from $1/3!$ to $1/12!$. Each such error will not exceed half a unit in the tenth place and the combined error will not, therefore, exceed 1 in the ninth place. The value of e correct to ten decimal places is actually $2{\cdot}71828\ 18285$.

Ex. 4. Shew that $1/e = 0{\cdot}36787\ 944$ correct to eight places.
[Put $x = -1$ in (9) and retain 13 terms, as in Ex. 3.]

Ex. 5. *Series for* $\dfrac{a^x - 1}{x}$. If $x \neq 0$, we have, from (10),

$$\frac{a^x - 1}{x} = L_a + \frac{x}{2!} L_a{}^2 + \frac{x^2}{3!} L_a{}^3 + \dots + \frac{x^{n-2}}{(n-1)!} L_a{}^{n-1} + a^{\theta x} \frac{x^{n-1}}{n!} L_a{}^n, \quad \dots\dots(14)$$

a relation which is also true when $x = 0$ provided we define the value of $(a^x - 1)/x$ there to be L_a, consistently with Art. 158. Since $a^{\theta x} \dfrac{x^{n-1}}{n!} L_a{}^n \to 0$ as $n \to \infty$, the conditions for Taylor's Series are satisfied,[1] and we write

$$\frac{a^x - 1}{x} = L_a + \frac{x}{2!} L_a{}^2 + \frac{x^2}{3!} L_a{}^3 + \dots + \frac{x^{n-1}}{n!} L_a{}^n + \dots . \quad \dots\dots\dots\dots(15)$$

[1] With respect to this statement, compare Ex. 3, Set LXX, p. 938.

Ex. 6. *Differentiation of* $(a^x-1)/x$. If $x \neq 0$, we have

$$\frac{d}{dx}\left(\frac{a^x-1}{x}\right)=\frac{1}{x}a^xL_a+(a^x-1)\left(-\frac{1}{x^2}\right)=\frac{a^x(xL_a-1)+1}{x^2}. \quad \text{.........(16)}$$

Employing the $I.D_3.T.$ for a^x, we can write this

$$\frac{d}{dx}\left(\frac{a^x-1}{x}\right)=\frac{\left(1+xL_a+\frac{x^2}{2!}L_a^2+a^{\theta x}\frac{x^3}{3!}L_a^3\right)(xL_a-1)+1}{x^2}$$

$$=\tfrac{1}{2}L_a^2+xL_a^3(\tfrac{1}{2}-\tfrac{1}{6}a^{\theta x})+\tfrac{1}{6}a^{\theta x}x^2L_a^4. \quad \text{.........(17)}$$

This shews that the derived function (16), which is undefined at $x=0$, is potentially continuous there, its completing value being $\frac{1}{2}L_a^2$.

It is easily demonstrated that this is the derivative of the function $\left(\frac{a^x-1}{x}\right)^*$ at $x=0$. The value of this function being L_a at $x=0$, the required derivative is the limiting value of the quotient $\left\{\left(\frac{a^x-1}{x}\right)^*-L_a\right\}\Big/x$ as $x \to 0$. Putting $n=3$ in (14), we find, on supposing $|x|<1$,

$$\left|\frac{\left(\frac{a^x-1}{x}\right)^*-L_a}{x}-\tfrac{1}{2}L_a^2\right|=|\tfrac{1}{6}a^{\theta x}xL_a^3|<|\tfrac{1}{6}axL_a^3|$$

$$<\varepsilon,\ (\varepsilon \text{ arbitrary}),\ \text{if } |x|<\frac{6\varepsilon}{a\,|L_a^3|}, \quad \text{.......(18)}$$

which establishes the desired result. The student should shew that the function $\left(\frac{a^x-1}{x}\right)^*$ has all its successive derivatives continuous at $x=0$.

Ex. 7. Find the Taylor polynomial for y of the sixth degree in x when y is defined by the equation $e^y+y=1+\frac{1}{2}x^2$.

We here illustrate the method of successive substitutions, (compare Art. 129 and Ex. 5, Art. 146). Thus we write

$$y=1+\tfrac{1}{2}x^2-e^y=\tfrac{1}{2}x^2-y-\frac{y^2}{2!}-\frac{y^3}{3!}-\dots, \quad \text{.........(19)}$$

giving

$$y=\tfrac{1}{4}x^2-\frac{1}{2}\cdot\frac{y^2}{2!}-\frac{1}{2}\cdot\frac{y^3}{3!}-\dots \quad \text{.........(20)}$$

The Taylor polynomial of the second degree is simply $\frac{1}{4}x^2$. That of the fourth degree is given by

$$y=\tfrac{1}{4}x^2-\frac{1}{2}\cdot\frac{(\frac{1}{4}x^2)^2}{2!}=\tfrac{1}{4}x^2-\tfrac{1}{64}x^4, \quad \text{.........(21)}$$

and that of the sixth by

$$y=\tfrac{1}{4}x^2-\tfrac{1}{4}(\tfrac{1}{4}x^2-\tfrac{1}{64}x^4)^2-\tfrac{1}{12}(\tfrac{1}{4}x^2-\tfrac{1}{64}x^4)^3 \quad \text{.........(22)}$$

after rejecting terms in the product involving powers of x higher than the sixth. This leads to the polynomial form

$$y=\tfrac{1}{4}x^2-\tfrac{1}{64}x^4+\tfrac{1}{1536}x^6. \quad \text{.........(23)}$$

The student should obtain the same result by the use of other methods, for example, the direct application of the $I.D_n.T.$ and the method of disposable coefficients.

163. Application of the Method of Proportional Parts to Tables of Exponential Functions.

(i) *Tables of* e^x. According to Art. 122, a formula for the uncertainty involved in the application of the *M.P.P.* to interpolation for a value of a function $f(x)$ lying between two successive tabulated values is $\frac{1}{8}h^2 \,|\, f''(x) \,|_{max}$, where h is the difference of argument. In the present case, $f''(x) = e^x$, and the uncertainty is therefore $\frac{1}{8}h^2(e^x)_{max}$.

Consider, for example, a table of values of e^x for the range (1, 3) of x, the interval of argument being 0·001. We have

$$\tfrac{1}{8}h^2(e^x)_{max} = \tfrac{1}{8} \times 10^{-6} \times e^3 \approx \tfrac{1}{8} \times 10^{-6} \times 20{\cdot}1 \approx 3 \times 10^{-6}, \quad \text{.......(1)}$$

whence it follows that accuracy to five places of decimals only is assured throughout the whole range. If the range extends to $x = 4$, the error will not exceed $\frac{1}{8} \times 10^{-6} \times e^4 \approx \frac{1}{8} \times 10^{-6} \times 54{\cdot}6 \approx 7 \times 10^{-6}$, and for the upper part of the range the formula does not exclude an error of a unit in the fifth place.

In Hayashi's tables the difference of argument for the range (1, 3) is 0·001 and the values of e^x are given correct to ten places of decimals. The *M.P.P.* is therefore plainly inadequate to give intermediate values to this accuracy in any portion of the range.

(ii) *Tables of* 10^x. Here we have

$$f(x) = 10^x, \quad f''(x) = 10^x(\log_e 10)^2 \approx 10^x(2{\cdot}303)^2, \quad \text{..........(2)}$$

and so

$$\tfrac{1}{8}h^2 \,|\, f''(x) \,|_{max} \approx \tfrac{1}{8}h^2 \times (2{\cdot}303)^2 \times (10^x)_{max} \approx 0{\cdot}66h^2(10^x)_{max}. \quad \text{......(3)}$$

Suppose that we consider a table of values of 10^x for the range [1] (0, 1) of x. The uncertainty of interpolation is then about $6{\cdot}6h^2$. If accuracy to seven significant figures is required, the permissible error is 5×10^{-7}, and for completeness of the table we must therefore have $6{\cdot}6h^2 < 5 \times 10^{-7}$, or

$$h < 2{\cdot}8 \times 10^{-4}. \quad \text{..........................(4)}$$

Tables of 10^x are generally described as *tables of antilogarithms.*[2] The most accessible comprehensive tables of this kind are the seven-place tables of Filipowski and of Shortrede, both of which have the range of argument (0, 1) and the interval of argument 0·00001. The *M.P.P.* can be applied correctly throughout the whole range of these tables.

[1] The value of 10^x for *any* value of x can be expressed in terms of 10^x for values of x in this range. For example, $10^{2\cdot314} = 10^2 \times 10^{0\cdot314}$.

[2] Antilogarithms are called *illogs* by Chappell in his *Five-Figure Tables.* This name may be treated as an abbreviation of *inverse logarithms.*

In the absence of tables of antilogarithms, the usual method of obtaining values of 10^x is by the process of reading backwards from a table of common logarithms. The question of the uncertainty attaching to the inverse use of such a table is considered in Art. 185. 3.

164. Special Limiting Values involving Exponential Functions.

Some special limits connected with the exponential functions are of frequent occurrence in the analytical theory. The fundamental result may be stated in the symbolic form

$$\lim_{x \to \infty} x^m a^{-nx} = 0, \quad \text{.............................(1)}$$

where $a > 1$, $n > 0$ and m is any number. On account of the relation $a^x = e^{x \log_e a}$, where $\log_e a > 0$, an equivalent result is expressed by

$$\lim_{x \to \infty} x^m e^{-nx} = 0, \quad (n > 0). \quad \text{........................(2)}$$

We consider separately the cases of m negative and positive.

(i) *m negative.* Writing $-m'$ for m, the function considered is $x^{-m'}e^{-nx}$, (m', n positive). Now each of the factors $x^{-m'}$, e^{-nx} converges to zero as x increases indefinitely, and hence the limiting value of the product is zero. Thus

$$\lim_{x \to \infty} x^{-m'} e^{-nx} = 0, \quad (m', n > 0), \quad \text{....................(3)}$$

or, reintroducing the letter m,

$$\lim_{x \to \infty} x^m e^{-nx} = 0, \quad (m < 0,\ n > 0). \quad \text{....................(4)}$$

(ii) *m positive.* Writing $y = x^m e^{-nx}$, we have

$$\frac{dy}{dx} = x^{m-1} e^{-nx}(m - nx) = -y(n - m/x). \quad \text{..............(5)}$$

Thus $dy/dx < 0$ if $x > m/n$ and y is then strictly down-way. Since y is here positive, it must have a lower bound which is either positive or zero. Suppose that this bound is positive and equal to B, so that

$$| dy/dx | > B(n - m/c) \text{ if } x > c > m/n. \quad \text{..............(6)}$$

Now consider the decrease of y when x increases from c to a greater value, $c + h$ say. According to the $I.D_1.T.$, this is equal to the product of the range and the magnitude of the derivative at some intermediate point, and is therefore greater than $B(n - m/c)h$, however large h may be. There is thus no positive lower bound to the

values of y and we conclude that the bound must be zero. Hence y converges to zero as x increases indefinitely, that is,

$$\lim_{x \to \infty} x^m e^{-nx} = 0, \quad (m>0, \ n>0). \quad \text{.....................(7)}$$

An alternative demonstration of this case, which depends only on the elementary properties of the exponential function, may be here outlined.

Consider a succession of positive values $r, r+1, \ldots, r+p, \ldots$ of x. Then, denoting the function by $f(x)$, it is easily shewn that if $r > r_1 > 1/(e^{n/m} - 1)$, we have

$$\frac{f(r+p+1)}{f(r+p)} < \left(\frac{r_1+1}{r_1 e^{n/m}}\right)^m < 1, \quad (p=0, 1, \ldots). \quad \text{.....................(8)}$$

Writing

$$(r_1+1)/(r_1 e^{n/m}) = k, \quad \text{...............................(9)}$$

we thus find

$$f(r+p+1) < k^{p+1} f(r). \quad \text{.............................(10)}$$

Suppose, now, that x is any point in the range $(r, r+1)$. We can replace r by x in (10), and thus, if X is the greatest value of $f(x)$ in the range $(r, r+1)$, we have

$$\begin{aligned} f(x+p+1) &< k^{p+1} X \\ &< \varepsilon \text{ (arbitrary) if } p+1 > \frac{\log_e (X/\varepsilon)}{\log_e (1/k)}. \quad \text{...........(11)} \end{aligned}$$

Therefore, for any value of x greater than $r+p+1$, where p is chosen as in (11), we have $f(x) < \varepsilon$, and hence $f(x) \to 0$ as $x \to \infty$.

The proof that $x^m e^{-nx}$, $(n>0)$, converges to zero when x diverges to $+\infty$ is now complete. The result is often quoted in the form that e^{nx}, $(n>0)$, is ultimately of a higher order of largeness than any power of x when x is increased indefinitely.

Cor. 1. Since $x^m e^{-nx} \to 0$ as $x \to \infty$, the reciprocal $x^{-m} e^{nx}$ increases indefinitely with x. We may express this symbolically by

$$\lim_{x \to \infty} x^{-m} e^{nx} = \infty, \quad (n > 0). \quad \text{......................(12)}$$

Cor. 2. Putting $m=1$, $n=1$ in (2), we have the special form

$$\lim_{x \to \infty} x e^{-x} = 0, \quad \text{..............................(13)}$$

of which an equivalent form, obtained by writing $-x$ for x, is

$$\lim_{x \to -\infty} x e^{x} = 0. \quad \text{...............................(14)}$$

Cor. 3. Writing $1/x$ for x in (2), we get the special form

$$\lim_{x \to +0} x^{-m} e^{-n/x} = 0, \quad (n > 0). \quad \text{.....................(15)}$$

165. Miscellaneous Examples.

Ex. 1. Prove that $(x-1)e^x + 1 > 0$ if $x > 0$.

[We have $f(0)=0$, $f'(x)>0$ for $x>0$, and the result follows from Cor. 1, Art. 98. 5.]

Ex. 2. Find the shortest distance of the point (3, 2) from the graph of the function $y=e^x$.

The distance r from (3, 2) to any point (x, y) on the graph is given by

$$r^2=(x-3)^2+(y-2)^2=(x-3)^2+(e^x-2)^2. \quad \text{.....................(1)}$$

The first derivative of r vanishes when

$$e^{2x}-2e^x+x-3=0, \quad \text{.................................(2)}$$

and then $r\dfrac{d^2r}{dx^2}=1+2e^x(e^x-1)$. It is clear that the value $x=1$ is a fair approximation to the root[1] of (2), since $e \approx 2 \cdot 72$, $e^2 \approx 7 \cdot 39$. The corresponding stationary value is therefore a minimum since d^2r/dx^2 is then positive. A closer approximation to the root is $1 \cdot 005$, a value found tentatively from the tables, and the shortest distance is then $\sqrt{(1 \cdot 995^2+0 \cdot 732^2)} \approx 2 \cdot 125$.

The application of Newton's method to the determination of the root of (2) to a high degree of accuracy is described in Ex. 8 below.

Ex. 3. Investigate the extreme values and the nature of the graph of the function $\frac{1}{x}e^{kx}$, $(k>0)$.

We have

$$\frac{dy}{dx}=\frac{kx-1}{x^2}e^{kx}, \quad \frac{d^2y}{dx^2}=\frac{(kx-1)^2+1}{x^3}e^{kx}. \quad \text{..(3)}$$

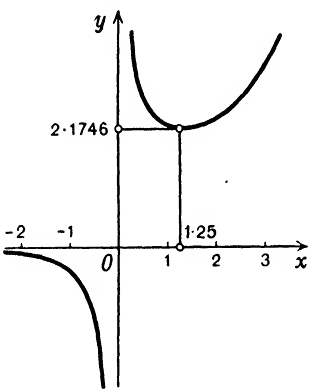

FIG. 133.

$y=\dfrac{1}{x}e^{0 \cdot 8x}$.

The first derivative vanishes when $x=1/k$ and the second is positive (negative) when x is positive (negative). The value $y=ke$ corresponding to $x=1/k$ is therefore a minimum. There are no points of contraflexure. The curve is concave upwards when $x>0$ and convex upwards when $x<0$.

The function is discontinuous (infinite) when $x=0$. Also $y \to \infty$ when $x \to \infty$, according to the result (12) of the preceding Article. Again, $e^{kx} \to 0$ when $x \to -\infty$ and then $y \to 0$. Portion of the graph of the function is shewn in Fig. 133, corresponding to the value $0 \cdot 8$ of k. The coordinates of the minimum point are $x=1 \cdot 25$, $y=2 \cdot 1746$.

Ex. 4. *The function* $y=e^{-1/x}$. As $|x| \to \infty$, $1/x \to 0$ and $e^{-1/x} \to 1$. The line $y=1$ is thus an asymptote. The function is discontinuous when $x=0$. If

[1] Since $f(\infty)>0$, $f(-\infty)<0$, there must be one real root. The equation $f'(x)=0$ is not here satisfied by any real value of e^x (or x), and hence it follows, from Art. 95. 1, that there cannot be more than one real root.

$x \to +0$, $1/x \to +\infty$ and $e^{-1/x} \to 0$. If, however, $x \to -0$, $1/x \to -\infty$ and therefore $e^{-1/x} \to \infty$. We have

$$\frac{dy}{dx} = \frac{1}{x^2} e^{-1/x}, \quad \frac{d^2y}{dx^2} = \frac{1-2x}{x^4} e^{-1/x}. \quad \text{..(4)}$$

The first derivative does not vanish and there are thus no stationary values. There is a point of contraflexure given by $x = 1/2$, $y = e^{-2} \approx 0{\cdot}1353$. Here the gradient is $4e^{-2}$, or $0{\cdot}5412$, the corresponding slope being about $28° \, 25'$.

Again, from the result (15) above, $\frac{1}{x^2} e^{-1/x} \to 0$ as $x \to +0$, and there is thus an upper zero gradient at the origin. Part of the graph is shewn in Fig. 134.

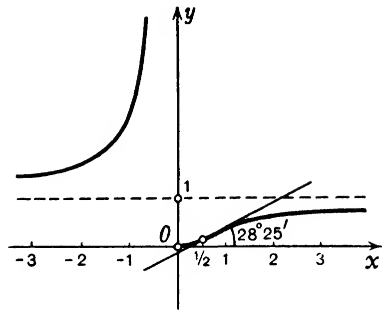

FIG. 134.
$y = e^{-1/x}$.

Ex. 5. *The function* e^{-x^2}. Here we have

$$f^{(1)}(x) = -2xe^{-x^2}, \quad f^{(2)}(x) = 2e^{-x^2}(2x^2-1), \quad f^{(3)}(x) = -4xe^{-x^2}(2x^2-3). \ldots(5)$$

Now $f^{(1)}(x) = 0$ when $x = 0$, $y = 1$, giving a maximum value of y, and $f^{(2)}(x) = 0$ when $x = \pm 1/\sqrt{2}$, $y = e^{-1/2} \approx 0{\cdot}6065$, giving the points of contraflexure. The gradients here are $\mp \sqrt{2} \, . \, e^{-1/2} \approx \mp 0{\cdot}8576$, corresponding to slopes of about $139° \, 23'$ and $40° \, 37'$.

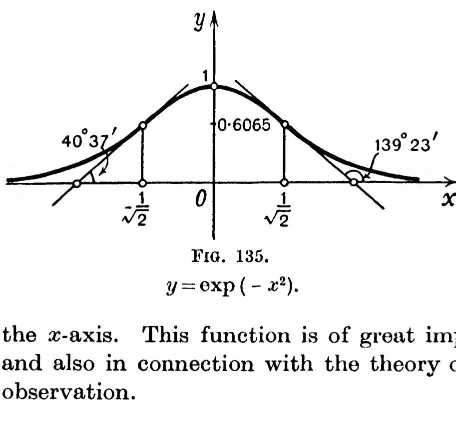

FIG. 135.
$y = \exp(-x^2)$.

We observe that $f^{(2)}(x) < 0$ when $x^2 < \frac{1}{2}$ and $f^{(2)}(x) > 0$ when $x^2 > \frac{1}{2}$. The curve is therefore convex upwards if $x^2 < \frac{1}{2}$ and concave upwards if $x^2 > \frac{1}{2}$. Also $f(x) \to 0$ as $|x| \to \infty$. Portion of the graph is shewn in Fig. 135.

The same graph obviously serves as a representation of the function $e^{-k^2x^2}$, (k a constant), merely by a change of scale along the x-axis. This function is of great importance in the theory of Statistics and also in connection with the theory of probability as applied to errors of observation.

Ex. 6. *The function* $y = 1/\{x^5(e^{a/x} - 1)\}$, $(a > 0)$. ....(6)

(i) Writing $x = a/z$, the function becomes $z^5/\{a^5(e^z - 1)\}$, or

$$z^5 e^{-z}/\{a^5(1 - e^{-z})\}. \quad \text{..............................(7)}$$

When $x \to +0$, $z \to +\infty$, so that $1 - e^{-z} \to 1$, $z^5 e^{-z} \to 0$ and hence $y \to 0$.

(ii) When $x \to -0$, $z \to -\infty$, so that $e^z - 1 \to -1$, $z^5 \to -\infty$ and hence $y \to +\infty$.

(iii) When $|x| \to \infty$, $z \to 0$, so that $z/(e^z - 1) \to 1$, $z^4 \to 0$ and hence $y \to 0$.

(iv) We have
$$\frac{dy}{dx} = \frac{\frac{a}{x^2} e^{a/x}}{x^5 (e^{a/x} - 1)^2} - \frac{5}{x^6} \frac{1}{e^{a/x} - 1}, \quad \text{........................(8)}$$

which vanishes when

$$\frac{a}{x}=5(1-e^{-a/x}). \quad\text{.................................(9)}$$

A solution of this equation in a/x may be obtained tentatively by the method of successive approximations, with the aid of a table of values of e^{-x}, such as those of Hayashi. Try as a first approximation $a/x=5$; then $e^{-a/x}=e^{-5}=0{\cdot}00674$, and a second approximation is given by $a/x=5(1-0{\cdot}00674)=4{\cdot}9663$. We now have $e^{-a/x}=e^{-4{\cdot}9663}=0{\cdot}00697$, and a third approximation is given by $a/x=5(1-0{\cdot}00697)=4{\cdot}9651$.

Taking $a/x=4{\cdot}965$, the corresponding value of y is given by

$$y_{max}=\frac{1}{(a/4{\cdot}965)^5(e^{4{\cdot}965}-1)}\approx\frac{3017{\cdot}1}{a^5\times 142{\cdot}3}\approx\frac{21{\cdot}2}{a^5}. \quad\text{...............(10)}$$

The function is of great importance in the theory of radiation, being the equivalent of an expression, known as *Planck's formula*, for the distribution of energy with respect to wave-length in the spectrum of the radiation from a black body.

Ex. 7. *The function* $e^{\sin x}$. This is defined for all values of x. We have

$$\frac{dy}{dx}=e^{\sin x}\cos x,\quad \frac{d^2y}{dx^2}=e^{\sin x}(\cos^2 x-\sin x). \quad\text{..................(11)}$$

The first derivative vanishes when $\cos x=0$, so that the extreme values correspond to those of the function $\sin x$. The second derivative vanishes when $\cos^2 x=\sin x$, that is, when $\sin x=0{\cdot}6180$, and then

$$x=n\pi+(-1)^n\times 0{\cdot}6662. \quad (12)$$

These give the points of contraflexure on the graph of the function. The range of values of the function is $(1/e, e)$ and it is periodic with period 2π. Portion of its graph is shewn in Fig. 136, together with that of $\sin x$.

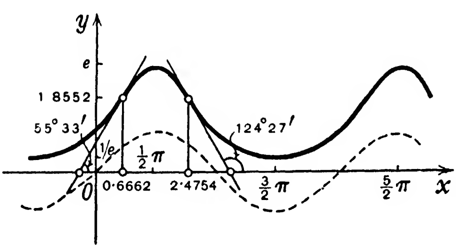

FIG. 136.
$y=e^{\sin x}$.

Ex. 8. Employ Newton's method to approximate to the root of the equation $e^{2x}-2e^x+x-3=0$.

We have

$$f'(x)=2e^{2x}-2e^x+1,\quad f''(x)=4e^{2x}-2e^x. \quad\text{....................(13)}$$

Taking the approximation $1{\cdot}005$ to the root, given in Ex. 2 above, we have $e^{1{\cdot}005}=2{\cdot}73190727$, $e^{2{\cdot}010}=7{\cdot}46331735$, giving

$$f(1{\cdot}005)=0{\cdot}00450281,\quad f'(1{\cdot}005)=10{\cdot}46282014,\quad f''(1{\cdot}005)>0. \quad\text{....(14)}$$

Since $f(x)$, $f''(x)$ are both positive for $x=1{\cdot}005$, a closer approximation to the root is $1{\cdot}005+h$, where, according to Newton,

$$h=-\frac{0{\cdot}00450281}{10{\cdot}46282014}=-0{\cdot}00043036. \quad\text{......................(15)}$$

This leads to the value $1{\cdot}00456964$.

We now investigate a barrier to the error involved in this approximation. According to the theory explained in Art. 119, the error is less than $\frac{1}{2}\{f(x_1)\}^2 |f''(x)|_u/\{|f'(x)|_l\}^3$, where x_1 is the starting point, here 1·005. Since $f'(x), f''(x)$ are both up-way and positive, we may take $f'(x)_l = f'(1) = 10{\cdot}34$, $f''(x)_u = f''(1{\cdot}005) = 24{\cdot}39$. The error is thus less than

$$\frac{1}{2} \times \frac{24{\cdot}39}{(10{\cdot}34)^3} \times (0{\cdot}0045)^2 \approx 2{\cdot}23 \times 10^{-7}. \quad \text{.....................(16)}$$

Ex. 9. *The function* $y = xe^{-k\left(x\sin\frac{1}{x}\right)^*}$. Here $\left(x\sin\frac{1}{x}\right)^*$ is a continuous function of x, its value at $x=0$ being 0, according to Art. 55. Therefore $e^{-k\left(x\sin\frac{1}{x}\right)^*}$ is continuous, having the value 1 at $x=0$, so that $y=0$ there. Thus y has the derivative 1 at $x=0$.

Now when $x \neq 0$, we have

$$\frac{dy}{dx} = e^{-kx\sin\frac{1}{x}}\left(1 - kx\sin\frac{1}{x} + k\cos\frac{1}{x}\right). \quad \text{.....................(17)}$$

If $k > 1$, we can find a value of x as near as we please to 0 for which dy/dx is negative, and therefore at which the function is decreasing with increase of x.

This Example illustrates a remark made in Art. 92, Chap. IV.

SECTION II. THE HYPERBOLIC FUNCTIONS

166. Introductory Remarks. The Primary Hyperbolic Functions.

In this Section we consider certain algebraic combinations of the exponential functions which are of such frequent occurrence that they are ranked as fundamental functions. They are called *hyperbolic functions* on account of a connection with the hyperbola which we explain below, but which is analytically of no importance. As tables of them are readily obtained they can be used freely for practical purposes.

The simplest and most useful of these functions are the odd and even functions obtained by the addition and subtraction of e^x, e^{-x}. It is convenient, for a reason explained later, to introduce a coefficient 1/2, so that the functions are defined to be $\frac{1}{2}(e^x + e^{-x})$, $\frac{1}{2}(e^x - e^{-x})$. The former is plainly unchanged in value if x is replaced by $-x$, so that it is an even function ; the sign of the latter is changed by the same replacement, so that it is an odd function. These functions correspond closely in some of their properties to the trigonometrical even and odd functions $\cos x$, $\sin x$, and the nomenclature and notation for them is suggested by this correspondence. The former is described as the *hyperbolic cosine of* x, the latter as the *hyperbolic sine of* x; the respective functional symbols are 'cosh', 'sinh',

obtained by modifying the corresponding symbols for the trigonometrical functions by the addition of the letter 'h', standing for 'hyperbolic'.[1] Thus we write

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}), \quad \sinh x = \tfrac{1}{2}(e^x - e^{-x}). \quad \ldots\ldots\ldots\ldots\ldots(1)$$

These are continuous functions for all values of x since e^x, e^{-x} are so. Plainly $\cosh x$ is always positive, having the minimum value 1 when[2] $x=0$. Thus we have $\cosh 0 = 1$. Also, as $x \to \infty$, $e^{-x} \to 0$ and $\cosh x/(\frac{1}{2}e^x) \to 1$, while as $x \to -\infty$, $e^x \to 0$ and $\cosh x/(\frac{1}{2}e^{-x}) \to 1$. The range of values of $\sinh x$ is unlimited. As $x \to \infty$, $\sinh x/(\frac{1}{2}e^x) \to 1$, and as $x \to -\infty$, $\sinh x/(-\frac{1}{2}e^{-x}) \to 1$. Also we have $\sinh 0 = 0$.

Unlike the trigonometrical functions $\cos x$, $\sin x$, the functions $\cosh x$, $\sinh x$ do not present any periodic properties.[3] The former is strictly down-way in the range $[-\infty, 0)$ and strictly up-way in the range $(0, \infty]$ of x. The latter is strictly up-way throughout the range $[-\infty, \infty]$.

The graphs of the functions are shewn in Figs. 137, 138, together

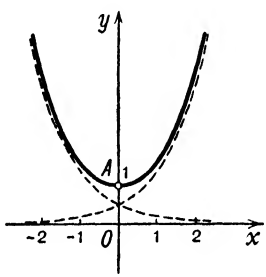

FIG. 137.
$y = \cosh x$.

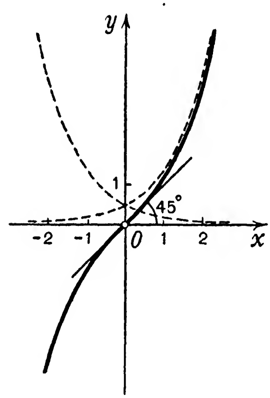

FIG. 138.
$y = \sinh x$.

with the curves $\frac{1}{2}e^x$, $\frac{1}{2}e^{-x}$ (shewn dotted). The point A on the graph of $\cosh x$ is usually referred to as the *vertex* of the curve.

The form taken by a uniform heavy chain hanging from two

[1] In print there will be no confusion between expressions of the forms $\cosh x$, $\cos hx$, the functional symbol being always in Roman characters, the argument in italics. In speaking of the new functions, it is customary to pronounce 'cosh' as written and 'sinh' as 'shine'.

[2] This follows from the fact that if z is positive, the minimum value of $\dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$ is 1, which occurs when $z = 1$.

[3] This statement does not apply in the theory of complex variables.

points is that of the graph of a hyperbolic cosine, the equation of which, on the introduction of a scale factor, may be written

$$y=c\cosh\frac{x}{c}. \quad\text{.............................(2)}$$

On account of this property, the graph of the function $c\cosh\frac{x}{c}$ is called the *catenary*, (common catenary); Ox is called the *directrix*. The height of the vertex A above O is now c. Further properties of the catenary are considered below.

167. The Secondary Hyperbolic Functions.

Secondary hyperbolic functions are defined from the primary ones, exactly as in the case of the trigonometrical functions.

167. 1. *The hyperbolic tangent.* This is defined as the quotient of the hyperbolic sine by the hyperbolic cosine, and is denoted by the symbol [1] $\tanh x$. Thus we have

$$\tanh x=\frac{\sinh x}{\cosh x} \quad\text{.....................................(1)}$$

$$=\frac{e^x-e^{-x}}{e^x+e^{-x}}=\frac{e^{2x}-1}{e^{2x}+1}. \quad\text{.......................(2)}$$

It is an odd function, vanishing when $x=0$. As $x\to\infty$, $\tanh x\to 1$, and as $x\to-\infty$, $\tanh x\to-1$. The function is throughout strictly up-way, its range of values being $[-1, 1]$. See Fig. 139.

167. 2. *The hyperbolic secant.* This is the reciprocal of the hyperbolic cosine and is denoted by [2] $\operatorname{sech} x$. Thus we have

$$\operatorname{sech} x=\frac{1}{\cosh x}=\frac{2}{e^x+e^{-x}}. \quad\text{.......................(3)}$$

It is an even function, with a range of values $[0, 1)$ being strictly

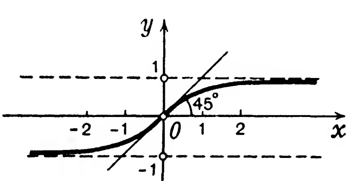

FIG. 139.
$y=\tanh x$.

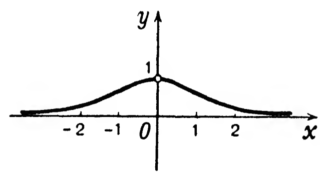

FIG. 140.
$y=\operatorname{sech} x$.

up-way in the range $[-\infty, 0)$ of x, and strictly down-way in the range $(0, \infty]$. See Fig. 140.

[1] Pronounced ' than x ', with the ' th ' sound as in ' thin '.
[2] ' shek x '.

167. 3. *The hyperbolic cosecant.* This is the reciprocal of the hyperbolic sine and is denoted by [1] cosech x. Thus

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}. \quad \text{......................(4)}$$

It is an odd function of unlimited range, defined for all values of x except 0, and down-way throughout. See Fig. 141.

167. 4. *The hyperbolic cotangent.* This is the reciprocal of the hyperbolic tangent and is denoted by [2] coth x. Thus

$$\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1}. \quad \text{............(5)}$$

It is an odd function, defined for all values of x except 0, but whose range of values lies outside the range $(-1, 1)$. It is down-way

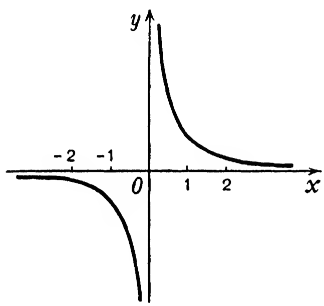

FIG. 141.
$y = \operatorname{cosech} x$.

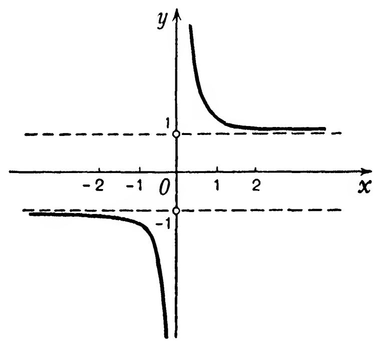

FIG. 142.
$y = \coth x$.

throughout and converges to 1, (-1), as x tends to ∞, $(-\infty)$. See Fig. 142.

Other notations (and rules for pronunciation) for the hyperbolic functions are employed. The common German practice is to employ gothic characters, writing 𝔈𝔬𝔰, 𝔖𝔦𝔫, 𝔗𝔫, etc. In some books the symbols 'cosh', 'sinh', 'tanh', are contracted to 'ch', 'sh', 'th', the symbols for the other functions being unabbreviated.

Although the values of the functions $\cosh x$, $\sinh x$ are immediately obtainable from tables of e^x, e^{-x}, special tables are available which give not only $\cosh x$, $\sinh x$, but also $\tanh x$, $\coth x$. Among these are the tables mentioned on p. 383. In the *Smithsonian Tables*, the functions $\cosh x$, $\sinh x$, $\tanh x$ are tabulated correct to five places of decimals, and $\coth x$ to five significant figures, for values of x ranging up to 6. In Hayashi, the first three functions are tabulated for values of x ranging up to 50, the number of decimal places ranging from 8 to 20. In addition, the Smithsonian collection gives the common logarithms of the four functions.

[1] 'coshek x'. [2] 'chot x', with the 'ch' sound as in 'chop'.

168. Fundamental Properties of the Hyperbolic Functions.

We now consider some properties which are of constant application. The reader will observe a close correspondence between the formulae obtained and well-known formulae of Trigonometry. This correspondence is capable of systematic treatment, and it can, in fact, be shewn how to pass directly from a formula relating to the trigonometrical functions to the corresponding one relating to the hyperbolic functions, and vice versa. The *Epicene Functions*, which are discussed in Section V of the present Chapter and which are ordinary functions of a real variable, can be used for this purpose, but the method commonly employed is to appeal to the properties of functions of a complex variable.

(i) From the definitions

$$\cosh x = \tfrac{1}{2}e^{x} + \tfrac{1}{2}e^{-x}, \quad \sinh x = \tfrac{1}{2}e^{x} - \tfrac{1}{2}e^{-x}, \quad \ldots\ldots\ldots\ldots\ldots(1)$$

we have, on addition and subtraction,

$$\cosh x + \sinh x = e^{x}, \quad \cosh x - \sinh x = e^{-x}. \quad \ldots\ldots\ldots\ldots(2)$$

By multiplication of these results, we have the identity [1]

$$\cosh^2 x - \sinh^2 x = 1. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

(ii) On division of (3) by $\cosh^2 x$, $\sinh^2 x$, in turn, we have

$$1 - \tanh^2 x = \operatorname{sech}^2 x, \quad \text{or} \quad \tanh^2 x = 1 - \operatorname{sech}^2 x, \quad \ldots\ldots\ldots(4)$$

and

$$\coth^2 x - 1 = \operatorname{cosech}^2 x, \quad \text{or} \quad \coth^2 x = 1 + \operatorname{cosech}^2 x. \quad \ldots\ldots(5)$$

(iii) *Addition Theorems.* We have

$$\begin{aligned}\cosh(x_1 + x_2) &= \tfrac{1}{2}\{e^{(x_1+x_2)} + e^{-(x_1+x_2)}\} \\ &= \tfrac{1}{2}(e^{x_1}e^{x_2} + e^{-x_1}e^{-x_2}) \\ &= \tfrac{1}{2}(\cosh x_1 + \sinh x_1)(\cosh x_2 + \sinh x_2) \\ &\qquad + \tfrac{1}{2}(\cosh x_1 - \sinh x_1)(\cosh x_2 - \sinh x_2) \\ &= \cosh x_1 \cosh x_2 + \sinh x_1 \sinh x_2. \quad \ldots\ldots\ldots\ldots\ldots(6)\end{aligned}$$

Also we find, in a similar manner,

$$\cosh(x_1 - x_2) = \cosh x_1 \cosh x_2 - \sinh x_1 \sinh x_2, \quad \ldots\ldots\ldots(7)$$

and (6), (7) together constitute the Addition Theorem for $\cosh x$.

The corresponding theorem for $\sinh x$ is expressed by

$$\sinh(x_1 \pm x_2) = \sinh x_1 \cosh x_2 \pm \cosh x_1 \sinh x_2, \quad \ldots\ldots\ldots(8)$$

[1] As in the case of the trigonometrical functions, $\cosh^2 x$ means the square of $\cosh x$, and so on.

and on combining this with (6), (7), we get that for $\tanh x$ in the form

$$\tanh(x_1 \pm x_2) = \frac{\tanh x_1 \pm \tanh x_2}{1 \pm \tanh x_1 \tanh x_2}. \quad \text{..................(9)}$$

The particular forms

$$\cosh 2x = \cosh^2 x + \sinh^2 x, \quad \text{........................(10)}$$

$$\sinh 2x = 2 \sinh x \cosh x, \quad \text{..........................(11)}$$

$$\tanh 2x = 2 \tanh x/(1 + \tanh^2 x), \quad \text{..................(12)}$$

obtained by putting $x_1 = x_2 = x$, are of frequent occurrence.

With the aid of (10) and the identity (3), we have

$$\cosh 2x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x; \quad \text{..............(13)}$$

whence we obtain the relations

$$\cosh^2 x = \tfrac{1}{2}(\cosh 2x + 1), \quad \text{.......................(14)}$$

$$\sinh^2 x = \tfrac{1}{2}(\cosh 2x - 1). \quad \text{.......................(15)}$$

Ex. 1. Prove the results

$$\sinh 3x = 3\sinh x + 4\sinh^3 x, \quad \cosh 3x = 4\cosh^3 x - 3\cosh x. \quad \text{.......(16)}$$

Ex. 2. Shew that $\tanh x$ differs from 1 by less than 1/2 per cent. of its value if x is greater than about 3.

We have

$$\frac{1 - \tanh x}{\tanh x} = \frac{2}{e^{2x} - 1}. \quad \text{..............................(17)}$$

If $x = 3$, $e^{2x} = e^6 \approx 403$, and the fraction written is less than 0·005.

169. Potential Continuity of the Function $\dfrac{\sinh x}{x}$ at $x = 0$.

This function is undefined at $x = 0$ but is elsewhere continuous. We have

$$\frac{\sinh x}{x} = \frac{1}{2}\frac{e^x - e^{-x}}{x} = e^{-x}\frac{e^{2x} - 1}{2x}. \quad \text{.....................(1)}$$

Now e^{-x} is continuous at $x = 0$, and $(e^{2x} - 1)/2x$ is potentially continuous there, by Art. 158, its completing value being 1. We conclude that the function in question is potentially continuous at $x = 0$ with a completing value 1, and we write

$$\lim_{x \to 0} \frac{\sinh x}{x} = 1. \quad \text{............................(2)}$$

The simplicity of this result and its correspondence with that for $\sin x$ form a sufficient justification for the insertion of the factor 1/2 in the definition of $\sinh x$ in terms of e^x, e^{-x}. By the introduction of the same factor in the definition of $\cosh x$, the correspondence,

remarked above, between the formulae for the trigonometrical and the hyperbolic functions is secured.

170. Parametric Specification of the Hyperbola.

Taking $$x^2/a^2 - y^2/b^2 = 1, \quad (a, b \text{ positive}), \quad \text{.................}(1)$$

as the equation of the hyperbola, we can evidently write

$$x = \pm a \cosh \varphi, \quad y = b \sinh \varphi, \quad \text{....................}(2)$$

introducing φ as parameter, for this gives

$$x^2/a^2 - y^2/b^2 = \cosh^2 \varphi - \sinh^2 \varphi = 1.$$

Referring to Fig. 143, we must take $x = a \cosh \varphi$ for the branch RAS, φ increasing from $-\infty$ to ∞ as the branch is described in the sense from R to S. For the branch $R'A'S'$ we must take $x = -a \cosh \varphi$, and φ now increases from $-\infty$ to ∞ as the branch is described in the sense R' to S'.

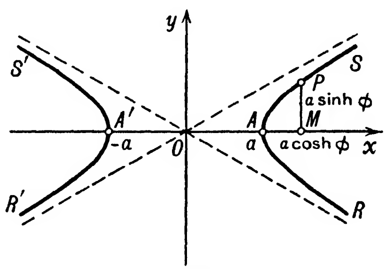

FIG. 143.

The student should compare the above with the parametric specification of the ellipse by means of the trigonometrical functions, Ex. 4, Art. 145.

170. 1. *The Gudermannian.* In the case of the *rectangular* hyperbola $x^2 - y^2 = a^2$ we take, as parametric equations, $x = \pm a \cosh \varphi$, $y = a \sinh \varphi$. The curve is shewn in Fig. 144, together with that of the circle $x^2 + y^2 = a^2$.

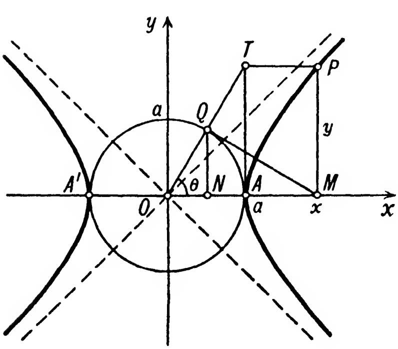

FIG. 144.

Let P, (x, y), be a point on the hyperbola for which x is positive, so that

$$\left.\begin{aligned} x &= OM = a \cosh \varphi, \\ y &= MP = a \sinh \varphi. \end{aligned}\right\} \quad \text{......}(3)$$

Complete the rectangle $MPTA$, join OT, and let this line cut the circle in Q. If we write $\angle MOQ = \theta$, we have $MP = AT = OA \tan \theta$, that is, $y = a \tan \theta$, and, comparing with (3), we have

$$\sinh \varphi = \tan \theta. \quad \text{..................................}(4)$$

This gives

$$\cosh \varphi = \sqrt{(1 + \sinh^2 \varphi)} = \sqrt{(1 + \tan^2 \theta)} = \sec \theta, \quad \text{...............}(5)$$

so that $x = a \sec \theta$. We conclude that QM is a tangent to the circle at Q.

Thus equivalent parametric equations for the hyperbola are

$$x = a\sec\theta, \quad y = a\tan\theta, \quad \text{.............................(6)}$$

while those of the circle itself are $x = a\cos\theta$, $y = a\sin\theta$.

Adding (4) and (5), we get the result

$$e^{\phi} = \tan\theta + \sec\theta = \tan(\tfrac{1}{2}\theta + \tfrac{1}{4}\pi), \quad \text{........................(7)}$$

so that $\theta = 2 \text{ artan } e^{\phi} - \frac{1}{2}\pi$. The angle θ, when associated with the parameter φ in this way, is called the *hyperbolic amplitude* or *Gudermannian*[1] of φ, and is denoted by $\text{amh}\,\varphi$ or $\text{gd}\,\varphi$. Tables of this function are given in the books mentioned on p. 383.

171. Derivatives of the Hyperbolic Functions.

The derivatives of the hyperbolic functions are easily determined. We have the following results.

(i) $$\frac{d\cosh x}{dx} = \frac{d}{dx}\{\tfrac{1}{2}(e^x + e^{-x})\} = \tfrac{1}{2}(e^x - e^{-x}) = \sinh x. \quad \text{..............(1)}$$

(ii) $$\frac{d\sinh x}{dx} = \frac{d}{dx}\{\tfrac{1}{2}(e^x - e^{-x})\} = \tfrac{1}{2}(e^x + e^{-x}) = \cosh x. \quad \text{..............(2)}$$

(iii) $$\frac{d\tanh x}{dx} = \frac{d}{dx}\cdot\frac{\sinh x}{\cosh x} = \frac{\cosh x\dfrac{d\sinh x}{dx} - \sinh x\dfrac{d\cosh x}{dx}}{\cosh^2 x}$$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \text{sech}^2 x. \quad \text{..............(3)}$$

(iv) $$\frac{d\,\text{sech}\, x}{dx} = \frac{d}{dx}\cdot\frac{1}{\cosh x} = -\frac{\sinh x}{\cosh^2 x} = -\text{sech}\, x \tanh x. \quad \text{.........(4)}$$

(v) $$\frac{d\,\text{cosech}\, x}{dx} = \frac{d}{dx}\cdot\frac{1}{\sinh x} = -\frac{\cosh x}{\sinh^2 x} = -\text{cosech}\, x \coth x. \quad \text{.......(5)}$$

(vi) $$\frac{d\coth x}{dx} = \frac{d}{dx}\cdot\frac{1}{\tanh x} = -\frac{\text{sech}^2 x}{\tanh^2 x} = -\frac{1}{\sinh^2 x} = -\text{cosech}^2 x. \quad (6)$$

Ex. 1. Shew that $$\frac{d}{dx}\sinh^2 x = \sinh 2x. \quad \text{...............................(7)}$$

Ex. 2. Shew that $$\frac{d}{dx}\text{artan}(\tanh\tfrac{1}{2}x) = \tfrac{1}{2}\,\text{sech}\, x. \quad \text{..........................(8)}$$

[Write $y = \text{artan}\, u$, $u = \tanh\frac{1}{2}x$, giving $\dfrac{dy}{du} = \dfrac{1}{1+u^2}$, $\dfrac{du}{dx} = \frac{1}{2}\,\text{sech}^2\,\frac{1}{2}x$.]

Ex. 3. Find the value of dy/dx when y is given implicitly by the equation

$$\sinh\frac{x}{a}\sinh\frac{y}{a} = 1. \quad \text{.................................(9)}$$

Differentiating throughout with respect to x, we get

$$\frac{1}{a}\cosh\frac{x}{a}\sinh\frac{y}{a} + \frac{1}{a}\sinh\frac{x}{a}\cosh\frac{y}{a}\cdot\frac{dy}{dx} = 0,$$

[1] After Gudermann, whose tables of this function were published in 1833.

giving
$$\frac{dy}{dx} = -\coth\frac{x}{a}\tanh\frac{y}{a}. \quad\text{.....(10)}$$

Since $\sinh\frac{y}{a} = \operatorname{cosech}\frac{x}{a}$, from (9), we have

$$\tanh\frac{y}{a} = \frac{\operatorname{cosech}\frac{x}{a}}{\sqrt{\left(1+\operatorname{cosech}^2\frac{x}{a}\right)}} = \frac{\operatorname{cosech}\frac{x}{a}}{\pm\coth\frac{x}{a}} = \pm\operatorname{sech}\frac{x}{a}, \quad\text{.....(11)}$$

the positive (negative) sign being taken if x/a is positive (negative). Hence

$$\frac{dy}{dx} = \mp\operatorname{cosech}\frac{x}{a}, \quad\text{.....(12)}$$

the negative (positive) sign being taken if x/a is positive (negative).

172. Taylor's Theorem for the Hyperbolic Functions.

172. 1. *Case of* $\cosh x$. The successive derivatives of $\cosh x$ are $\sinh x$, $\cosh x$, $\sinh x$, $\cosh x, \ldots$, so that, denoting the function by $f(x)$, we have

$$f^{(2r)}(x) = \cosh x, \quad f^{(2r+1)}(x) = \sinh x. \quad\text{.....(1)}$$

The odd derivatives all vanish when $x=0$, and the even ones are then equal to 1. Recalling the fact that $\cosh 0 = 1$, the $I.D_n.T.$ gives, in this case,[1]

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n-2}}{(2n-2)!} + R, \quad\text{.....(2)}$$

where R, the remainder term, is equal to

$$\frac{x^{2n-1}}{(2n-1)!}\sinh\theta x \quad\text{or to}\quad \frac{x^{2n}}{(2n)!}\cosh\theta' x, \quad\text{.....(3)}$$

according as the order of the intermediate derivative is the $(2n-1)$th or the $(2n)$th.

The polynomial expression, P say, obtained by omitting R from the right-hand side of (2), is the Taylor polynomial of degree $2n-2$ for $\cosh x$, and it contains n non-vanishing terms. If we take, as the expression for the remainder, $R_{2n-1}(x) = \frac{x^{2n-1}}{(2n-1)!}\sinh\theta x$, we have

$$|R_{2n-1}(x)| = \frac{|x|^{2n-1}}{(2n-1)!}|\sinh\theta x| < \frac{|x|^{2n-1}}{(2n-1)!}|\sinh x|$$
$$< \frac{|x|^{2n-1}}{(2n-1)!}\tfrac{1}{2}e^{|x|} < \tfrac{1}{2}\frac{|x|^{2n-1}}{(2n-1)!}3^{|x|}, \quad\text{.....(4)}$$

[1] The expansions for $\cosh x$, $\sinh x$ can, of course, be obtained from those for e^x, e^{-x} with remainders. The advantage of the direct method here given is that only one unknown fraction is introduced in the remainder.

which gives an upper barrier to the error committed in replacing $\cosh x$ by the Taylor polynomial P in question.

Further, since $|x|^{2n-1}/(2n-1)! \to 0$ as $n \to \infty$, the conditions for Taylor's Series are satisfied, and Taylor's Theorem for $\cosh x$ becomes

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!} + \ldots \qquad \text{.................(5)}$$

Ex. 1. Shew that $\cosh x$ lies between P and $P\Big/\left\{1 - \frac{x^{2n}}{(2n)!}\right\}$, provided that $x^{2n}/(2n)! < 1$.

[This follows from the fact that we can take

$$R = \frac{x^{2n}}{(2n)!}\cosh \theta x < \frac{x^{2n}}{(2n)!}\cosh x, \qquad \text{.......................(6)}$$

and hence $P < \cosh x < P + R < P + \frac{x^{2n}}{(2n)!}\cosh x$. The result quoted shews that the error committed in replacing $\cosh x$ by P does not exceed $P\frac{x^{2n}/(2n)!}{1 - x^{2n}/(2n)!}$.]

172. 2. *Case of* $\sinh x$. Here we have

$$f^{(2r)}(x) = \sinh x, \quad f^{(2r+1)}(x) = \cosh x, \qquad \text{.................(7)}$$

and the $I.D_n.T.$ gives

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n-1}}{(2n-1)!} + R, \qquad \text{..............(8)}$$

where R is equal to

$$\frac{x^{2n}}{(2n)!}\sinh \theta x \quad \text{or to} \quad \frac{x^{2n+1}}{(2n+1)!}\cosh \theta' x, \qquad \text{..............(9)}$$

according as the order of the intermediate derivative is the $(2n)$th or the $(2n+1)$th. The Taylor polynomial of degree $2n-1$ for $\sinh x$ is the polynomial obtained by omitting R from the right-hand side of (8), and an upper barrier to the error involved in replacing $\sinh x$ by this polynomial is obtained by observing that

$$|R_{2n}(x)| < \frac{x^{2n}}{(2n)!}|\sinh x| < \tfrac{1}{2}\frac{x^{2n}}{(2n)!}3^{|x|}. \qquad \text{.............(10)}$$

As in the case of $\cosh x$, the conditions for Taylor's Series are satisfied, and Taylor's Theorem for $\sinh x$ takes the form

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2n-1}}{(2n-1)!} + \ldots \qquad \text{.............(11)}$$

Ex. 2. Shew that if $x^{2n}/(2n)! < 1$, an upper barrier to the fractional error made in replacing $\sinh x$ by its Taylor polynomial of degree $2n-1$ is given by $\frac{x^{2n}/(2n)!}{-x^{2n}/(2n)!}$.

The student should illustrate the closeness of approximation of the earlier Taylor polynomials to the functions $\cosh x$, $\sinh x$ with the aid of diagrams, as in the corresponding cases of $\sin x$, e^x.

172. 3. *Case of* $\tanh x$. As in the case of $\tan x$, there is no simple formula for the nth derivative of $\tanh x$, even for the value 0 of x. We here shew how to determine the successive derivatives with the aid of a recurrence formula. Denoting the function by y, we have

$$y^{(1)} = \operatorname{sech}^2 x = 1 - \tanh^2 x = 1 - yy, \quad \text{.................}(12)$$

so that $y^{(1)} + yy = 1$, and Leibniz's Theorem now gives

$$y^{(n+1)} + y^{(n)}y + ny^{(n-1)}y^{(1)} + \frac{n(n-1)}{2!}y^{(n-2)}y^{(2)} + \dots$$
$$+ ny^{(1)}y^{(n-1)} + yy^{(n)} = 0, \quad \dots(13)$$

which is the formula sought. The even derivatives all vanish when $x=0$, and we then have the formula

$$y_0^{(n+1)} + ny_0^{(n-1)}y_0^{(1)} + \frac{n(n-1)(n-2)}{3!}y_0^{(n-3)}y_0^{(3)} + \dots + ny_0^{(1)}y_0^{(n-1)} = 0, \quad (14)$$

where n is an even integer. Giving to n the successive values 2, 4, 6, 8 in turn, we find

$$y_0^{(3)} = -2, \quad y_0^{(5)} = 16, \quad y_0^{(7)} = -272, \quad y_0^{(9)} = 7936, \quad \text{.........}(15)$$

and the ninth degree Taylor polynomial for $\tanh x$ is thus

$$x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \frac{62}{2835}x^9. \quad \text{..................}(16)$$

As in the case of $\tan x$, a more convenient process for the determination of the successive polynomials depends on the connection between the derivatives at $x=0$ and the Bernoulli numbers.

Ex. 3. *Calculation of* $\cosh 1$. Employing the polynomial approximation

$$\cosh 1 \approx 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \frac{1}{8!} + \frac{1}{10!}, \quad \text{.....................}(17)$$

we have the scheme

$$\begin{aligned} 1 &= 1, \\ 1/2! &= 0{\cdot}5, \\ 1/4! &= 0{\cdot}041\ 666\ 667, \\ 1/6! &= 0{\cdot}001\ 388\ 889, \\ 1/8! &= 0{\cdot}000\ 024\ 802, \\ 1/10! &= 0{\cdot}000\ 000\ 276, \\ \hline \cosh 1 &= 1{\cdot}543\ 080\ 634. \quad \text{...........................}(18) \end{aligned}$$

According to (4), the error due to omitting the remainder term will not exceed $\frac{1}{2} \times \frac{1}{11!} \times 3$, or about $3{\cdot}8 \times 10^{-8}$, and so, taking into account the errors

due to approximating in the last place of the decimals written, the result quoted is not in error by more than 4×10^{-8}.

A closer estimate of the error is obtained from the results of Ex. 1 above. The value of $\cosh 1$ will exceed $1{\cdot}543\,080\,632$, but will be less than $1{\cdot}543\,080\,636 \times 1\Big/\left(1 - \dfrac{1}{12!}\right) \approx 1{\cdot}543\,080\,639$. The actual value correct to nine places of decimals is $1{\cdot}543\,080\,635$.

Ex. 4. Shew that $\sinh 1$ lies between $1{\cdot}175\,201\,191$ and $1{\cdot}175\,201\,198$.

$\left[\text{Use the approximation } \sinh 1 \approx 1 + \dfrac{1}{3!} + \dfrac{1}{5!} + \dfrac{1}{7!} + \dfrac{1}{9!} + \dfrac{1}{11!}.\right]$

Ex. 5. Obtain the Taylor polynomial for $\tanh x$ of the seventh degree by division of the appropriate polynomials for $\sinh x$, $\cosh x$.

[According to Theorem III, Art. 129, the required polynomial is obtained by dividing the 7th degree polynomial for $\sinh x$ by the 6th degree polynomial for $\cosh x$ and rejecting terms in the quotient involving powers of x higher than the 7th.]

173. Miscellaneous Examples.

Ex. 1. Find the angle at which the curves $y = e^x$, $y = 1 + \cosh x$ intersect.

At the point of intersection we have

$$1 + \cosh x = e^x = \cosh x + \sinh x, \quad \text{.........................(1)}$$

so that $\sinh x = 1$ and thus $\cosh x = \sqrt{2}$, $e^x = 1 + \sqrt{2}$. The gradients of the respective curves are e^x, $\sinh x$, and if θ is the angle in question,

$$\tan\theta = \frac{e^x - \sinh x}{1 + e^x \sinh x} = \frac{1 + \sqrt{2} - 1}{1 + 1 + \sqrt{2}} = \sqrt{2} - 1. \quad \text{....................(2)}$$

The acute angle of intersection is therefore $\operatorname{artan}(\sqrt{2} - 1)$, or $\frac{1}{8}\pi$.

Ex. 2. *Curvature of the catenary.* Here $y = c\cosh\dfrac{x}{c}$, and therefore $\dfrac{dy}{dx} = \sinh\dfrac{x}{c}$, $\dfrac{d^2y}{dx^2} = \dfrac{1}{c}\cosh\dfrac{x}{c}$. The curvature is thus given by

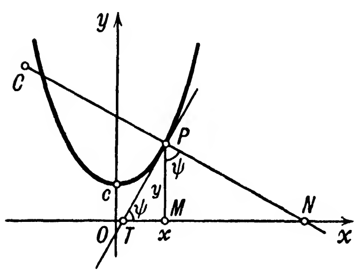

FIG. 145.
$y = c\cosh(x/c)$.

$$\kappa = \frac{\dfrac{1}{c}\cosh\dfrac{x}{c}}{\left(1 + \sinh^2\dfrac{x}{c}\right)^{3/2}} = \frac{1}{c}\cdot\frac{1}{\cosh^2\dfrac{x}{c}} = \frac{c}{y^2}, \quad (3)$$

and the radius of curvature by $\rho = y^2/c$.

If PT (Fig. 145) is the tangent and PN the normal at P, we have $\tan\psi = dy/dx = \sinh\dfrac{x}{c}$, and

$$PN = PM\sec\psi = y\sqrt{(1 + \tan^2\psi)} = y\cosh\frac{x}{c} = y^2/c. \quad \text{..............(4)}$$

It follows that if we measure a distance equal to PN along the inward drawn normal, the point reached is the centre of curvature C.

Ex. 3. *Curvature of the hyperbola.* Taking the parametric equations

$$x = \pm a \cosh \varphi, \quad y = b \sinh \varphi, \quad (a, b \text{ positive}),$$

we find

$$\frac{dy}{dx} = \frac{dy/d\varphi}{dx/d\varphi} = \pm \frac{b}{a} \coth \varphi, \quad \frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)\Big/ d\varphi}{dx/d\varphi} = -\frac{b}{a^2} \operatorname{cosech}^3 \varphi. \quad \text{.........(5)}$$

The standard formula for κ now leads to the result

$$\kappa = \frac{-\frac{b}{a^2} \operatorname{cosech}^3 \varphi}{\left(1 + \frac{b^2}{a^2} \coth^2 \varphi\right)^{3/2}} = \frac{\mp ab}{(a^2 \sinh^2 \varphi + b^2 \cosh^2 \varphi)^{3/2}}, \quad \text{..............(6)}$$

the negative (positive) sign being taken when φ is positive (negative), that is, when y is positive (negative). This is easily shewn to be in agreement with the result (33), Ex. 5, Art. 133, p. 302.

Ex. 4. Prove that $\dfrac{x}{\sinh x} - \dfrac{\tanh x}{x} > 0$ if $0 < x < 0{\cdot}881$.

[It is sufficient to prove that $x^2 - \cosh x + \operatorname{sech} x > 0$ in the range stated. Follow the procedure in Ex. 1, Art. 149, p. 345, shewing that the third derivative of this function is positive while the second vanishes at $x = 0$, etc.]

Ex. 5. With the aid of tables shew that the value $2{\cdot}68$ is a close approximation to the positive root of the equation $x^2 - \cosh x + \operatorname{sech} x = 0$, and employ Newton's method to obtain a closer approximation.

[Taking $x_1 = 2{\cdot}68$, we find $h = -0{\cdot}00391$, and so $x_2 = 2{\cdot}67609$. In estimating the numerical error involved in this approximation by means of the formula (1), Art. 119, p. 252, we may take

$$|f'(x)|_l = |f'(2{\cdot}67)| = 1{\cdot}981, \quad |f''(x)|_u = |f''(2{\cdot}68)| = 5{\cdot}195, \quad \text{.......(7)}$$

and we find $E < 0{\cdot}00002$.]

Ex. 6. Investigate the standard forms for the expression $ae^{\alpha x} + be^{\beta x}$ in terms of hyperbolic functions.

We suppose that α, β do not both vanish and that $\alpha > \beta$. Write $p = \frac{1}{2}(\alpha + \beta)$, $q = \frac{1}{2}(\alpha - \beta)$, so that

$$\alpha = p + q, \quad \beta = p - q. \quad \text{................................(8)}$$

Then

$$ae^{\alpha x} + be^{\beta x} = e^{px}(ae^{qx} + be^{-qx}) = e^{px}(A \cosh qx + B \sinh qx), \quad \text{.........(9)}$$

where $A = a + b$, $B = a - b$.

(i) If $A^2 > B^2$, we can write $A = C \cosh \gamma$, $B = C \sinh \gamma$, where $C^2 = A^2 - B^2$, $\tanh \gamma = B/A$, and then

$$ae^{\alpha x} + be^{\beta x} = Ce^{px} \cosh (qx + \gamma). \quad \text{.......................(10)}$$

(ii) If $A^2 < B^2$, we write $A = C' \sinh \gamma'$, $B = C' \cosh \gamma'$, where $C'^2 = B^2 - A^2$, $\tanh \gamma' = A/B$, and then

$$ae^{\alpha x} + be^{\beta x} = C'e^{px} \sinh (qx + \gamma'). \quad \text{.......................(11)}$$

(iii) If $A^2 = B^2$, either $a = 0$ or $b = 0$, and the expression is a monomial exponential function.

(iv) If $q = 0$, then $\alpha = \beta$, and we again have a monomial exponential function.

Ex. 7. *Solution of the cubic equation in terms of transcendental functions.* We have shewn in Ex. 4, Art. 95. 1, that the general cubic equation $x^3+3ax^2+3bx+c=0$ can be written in the form

$$f(\xi)=\xi^3+3H\xi+G=0, \quad \text{.................................}(12)$$

where $\xi=x+a$, $H=b-a^2$ and $G=c-3ab+2a^3$, and we have found the conditions for the existence of three real roots.

(i) If $H>0$ there is only one real root. Putting $\xi=2\sqrt{H}.z$, (12) becomes $4z^3+3z=-\frac{1}{2}G/H^{3/2}$, or, writing $\sinh\varphi=-\frac{1}{2}G/H^{3/2}$,

$$4z^3+3z=\sinh\varphi=4\sinh^3\tfrac{1}{3}\varphi+3\sinh\tfrac{1}{3}\varphi, \quad \text{..................}(13)$$

by (16), Art. 168. This equation is evidently satisfied by putting $z=\sinh\frac{1}{3}\varphi$, or $\xi=2\sqrt{H}.\sinh\frac{1}{3}\varphi$, which is therefore the root sought.

(ii) If $H<0$ and $4H^3+G^2>0$ there is again only one real root. Writing $\xi=2\sqrt{(-H)}.z$, (12) becomes $4z^3-3z=-\frac{1}{2}G/(-H)^{3/2}$.

(*a*) If $G<0$, then since $-\frac{1}{4}G^2/H^3>1$, we can write $\cosh\varphi=-\frac{1}{2}G/(-H)^{3/2}$ and the equation becomes

$$4z^3-3z=\cosh\varphi=4\cosh^3\tfrac{1}{3}\varphi-3\cosh\tfrac{1}{3}\varphi, \quad \text{..................}(14)$$

by (16), Art. 168. The equation is now plainly satisfied by putting $z=\cosh\frac{1}{3}\varphi$, or $\xi=2\sqrt{(-H)}.\cosh\frac{1}{3}\varphi$, which is therefore the root sought.

(*b*) If $G>0$ we can write $\cosh\varphi=\frac{1}{2}G/(-H)^{3/2}$ and the equation becomes

$$4z^3-3z=-\cosh\varphi=-4\cosh^3\tfrac{1}{3}\varphi+3\cosh\tfrac{1}{3}\varphi. \quad \text{..............}(15)$$

This is plainly satisfied by $z=-\cosh\frac{1}{3}\varphi$, or $\xi=-2\sqrt{(-H)}.\cosh\frac{1}{3}\varphi$, which is therefore the root sought.

(iii) If $H<0$ and $4H^3+G^2<0$ there are three real roots. We now have $-\frac{1}{4}G^2/H^3<1$ and we can write $\cos\theta=\frac{1}{2}G/(-H)^{3/2}$, the equation becoming

$$4z^3-3z=\cos\theta=4\cos^3\tfrac{1}{3}\theta-3\cos\tfrac{1}{3}\theta, \quad \text{..................}(16)$$

by (16), Art. 139. 7. There is therefore a root $z=\cos\frac{1}{3}\theta$, or $\xi=2\sqrt{(-H)}.\cos\frac{1}{3}\theta$ and since $\cos\theta=\cos(\theta+2\pi)=\cos(\theta+4\pi)$, the remaining roots of (12) in this case are $\xi=2\sqrt{(-H)}.\cos\frac{1}{3}(\theta+2\pi)$ and $\xi=2\sqrt{(-H)}.\cos\frac{1}{3}(\theta+4\pi)$.

SECTION III. THE LOGARITHMIC FUNCTION

174. Inversion of the Exponential Function.

The exponential function a^x, or $\exp_a x$, $(a>0)$, is positive, strictly one-way and continuous for all values of x, being up-way if $a>1$ and down-way if $a<1$. The relation $y=\exp_a x$ can therefore be inverted so as to define x as a single-valued, one-way, continuous function of y for all *positive* values of y. In the general notation of Art. 36, the inverse relation would be expressed by $x=\operatorname{arg}\exp_a y$, but, as remarked in that Article, a special notation is employed in this case, namely,

$$x=\log_a y, \quad \text{................................}(1)$$

the functional symbol here being an abbreviation of the phrase *logarithm to base a*. If, then, y is positive, the two relations

$$y=a^x, \quad x=\log_a y \qquad \text{.........(2)}$$

are equivalent.

Consider next the relation

$$y=-a^x, \qquad \text{.........(3)}$$

which defines y as a negative, strictly one-way, continuous function of x. Since a^x, or $\exp_a x$, is now equal to $-y$, the inverse relation expressing x as a continuous function of y for all *negative* values of y is, in the standard notation,

$$x=\log_a(-y). \qquad \text{.........(4)}$$

Hence if y is negative, the two relations

$$y=-a^x, \quad x=\log_a(-y) \qquad \text{.........(5)}$$

are equivalent.

By considering the two sets of relations (2), (5), it follows that the relations

$$x=\log_a|y| \quad \text{and} \quad \left.\begin{cases} y=a^x, & (y>0), \\ y=-a^x, & (y<0), \end{cases}\right\} \qquad \text{.........(6)}$$

are equivalent. Thus, starting from the two functional relations $y=a^x$, $y=-a^x$, we arrive, by simultaneous inversion, at a single-valued inverse function expressed by the relation $x=\log_a|y|$.

Leaving on one side the notion of inversion, we may regard '$\log_a$' as a single symbol for a function, and, resuming the practice of writing x as the independent variable, we have the function $\log_a|x|$, which is described as *the logarithm of the magnitude of x* (or *the logarithm of mod. x) to base a*, the ranges of values of function and argument being unlimited except for the exclusion of the value $x=0$. For any value of x except 0, $\log_a|x|$ is the index of the power of the number (base) a which is equal to the magnitude of x.

Two special values of the base are commonly employed, namely e and 10. Logarithms to the base e are called *natural* logarithms, since they are the ones which present themselves naturally in the analytical discussion. They are also called *Napierian* [1] logarithms and sometimes *hyperbolic* logarithms.

Various notations are employed for this special function, of which the commoner ones are $\log_e|x|$, $\log\text{nat}\,|x|$, $\ln|x|$, $l|x|$, $\log\text{hyp}\,|x|$, $\text{logh}\,|x|$. Throughout this Book we shall employ the first notation in which the base e is specified as a suffix. The indication of the base e may, of course, be omitted where no risk of confusion can arise, and this has, in fact, become the practice in many theoretical books, even where risk can hardly be said to be absent.

[1] After Napier (1550-1617), the discoverer of logarithms.

Logarithms to the base 10 are called *common* logarithms, or *Briggsian* logarithms, after their introducer. The student will know that their adoption in calculations depends on the fact that the decimal part (mantissa) alone of the logarithm need be tabulated, the integral part (characteristic) being determined by inspection. As in the case of natural logarithms, the base 10 will be indicated in all the theoretical discussions.

Since a logarithm to any one positive base is expressible in terms of that to another positive base, it is sufficient to consider one base only in the discussion of the properties of the general logarithmic function, and, as in the case of the exponential function, the special base chosen is e. This choice leads to the simplest form for the derivative of the function. On account of the analytical importance of this special function, it is referred to simply as the *logarithmic function* when there is no possibility of confusion as to the base to be employed.

As in the case of the 'inverse' trigonometrical functions, so here the term 'inverse' as applied to the logarithmic function loses significance from the analytical point of view. The logarithmic function may be treated as the primary function and the exponential function may then be derived from it by inversion. In other words, the two are mutually inverse functions. A method of introducing the logarithmic function as the primary function is indicated in Art. 232. 1, Chap. IX, when dealing with the subject of *Integration.*

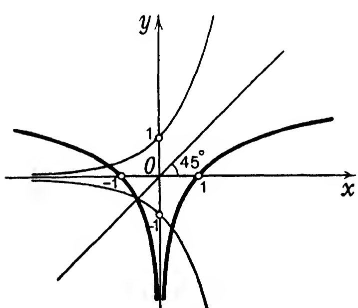

FIG. 146.
$y=\log_e|x|$.

175. Graph of $\log_e|x|$.

Since the relation $y=\log_e|x|$ is equivalent to the relations $x=e^y$, $(x>0)$, and $x=-e^y$, $(x<0)$, the required graph can be obtained by taking the geometrical image of the graphs of the functions $y=e^x$, $y=-e^x$ in the line bisecting the angle xOy, as explained in Art. 36. The graphs of the functions $y=e^x$, $y=-e^x$ are shewn in Fig. 146 by the light lines, and the graph of $y=\log_e|x|$, obtained in the manner described, by the heavy lines.

176. Fundamental Properties of the General Logarithmic Function.

The student is supposed to be familiar with the fundamental properties of logarithms from the arithmetic standpoint. We here

give a short list of formulae which are continually applied in this Section, the various arguments being supposed positive.[1]

(i) $\log_a (xy) = \log_a x + \log_a y$; ....(1)

(ii) $\log_a (x/y) = \log_a x - \log_a y$; ....(2)

(iii) $\log_a x^n = n \log_a x$, (n any number); ....(3)

(iv) $\log_a x = \log_b x \log_a b$. ....(4)

These correspond to fundamental formulae for the exponential function, given in Art. 154. If we put $x = a^w$, $y = a^z$, we have $xy = a^w a^z = a^{w+z}$, and hence

$$\log_a (xy) = \log_a a^{w+z} = w + z = \log_a x + \log_a y. \quad \text{..................(5)}$$

Similarly,

$$\log_a (x/y) = w - z = \log_a x - \log_a y. \quad \text{..................(6)}$$

If in (6) we put $x = 1$, we get the important case

$$\log_a y = -\log_a (1/y). \quad \text{..................(7)}$$

Again, $x^n = a^{nw}$ and hence

$$\log_a x^n = nw = n \log_a x. \quad \text{..................(8)}$$

Finally, since $x = a^w = (b^{\log_b a})^w = b^{w \log_b a}$, we have

$$\log_b x = w \log_b a = \log_a x \log_b a, \quad \text{..................(9)}$$

or, interchanging a and b,

$$\log_a x = \log_b x \log_a b. \quad \text{..................(10)}$$

The formula (4) gives the rule for the 'change of base' whereby the logarithm of a number to one base is expressed in terms of that to a second base. The coefficient $\log_a b$ is called the *modulus* of the transformation. On interchanging a and b, and comparing the formula so obtained with (4), it is seen that the corresponding moduli are connected by the relation $\log_a b = 1/\log_b a$.

In the case of the special bases e, 10, we have the results

$$\log_{10} x = \log_e x \log_{10} e = \log_e x \frac{1}{\log_e 10}. \quad \text{..................(11)}$$

The values of $\log_{10} e$, $\log_e 10$ have already been found, correct to six places of decimals, in Art. 160. 1, namely

$$\log_{10} e = 0{\cdot}434294, \quad \log_e 10 = 2{\cdot}302585.$$

We thus have the transformations

$$\log_{10} x = 0{\cdot}434294 \ldots \times \log_e x, \quad \log_e x = 2{\cdot}302585 \ldots \times \log_{10} x.$$

The modulus $\log_{10} e$, i.e. $0{\cdot}434294 \ldots$, is usually denoted by the letter M (or μ), and we thus write

$$\log_{10} x = M \log_e x, \quad \log_e x = \frac{1}{M} \log_{10} x. \quad \text{..................(12)}$$

[1] If x, y are negative, it is only necessary to replace x, y by $|x|$, $|y|$ in the formulae given.

These shew that the graph of the function $\log_e |x|$ can be taken as a representation of $\log_{10} |x|$ if the scale of ordinates is altered in the ratio $1 : M$.

Ex. 1. Shew that $$x = \tfrac{1}{2} \log_e \frac{1+\tanh x}{1-\tanh x}. \qquad (13)$$

[Express $\dfrac{1+\tanh x}{1-\tanh x}$ in terms of exponential functions, and invert.]

Ex. 2. Prove that $$\lim_{x\to 0} \frac{\log_e(1+x)}{x} = 1. \qquad (14)$$

This is a consequence of the result $\lim\limits_{h\to 0} \dfrac{e^h-1}{h} = 1$, (Art. 160), or its equivalent, $\lim\limits_{h\to 0} \dfrac{h}{e^h-1} = 1$. For, writing $e^h = 1+x$, $(x > -1)$, we have $h = \log_e(1+x)$, $e^h - 1 = x$, and the stated result follows since $h \to 0$ as $x \to 0$. In other words, the function $\dfrac{\log_e(1+x)}{x}$ is potentially continuous at $x=0$, its completing value there being 1.

The result (14) is employed in the following Article.

177. Special Limits involving Logarithmic Functions.

From the result $$\lim_{z\to\infty} z^m e^{-nz} = 0, \quad (n>0), \qquad (1)$$

established in Art. 164, we can deduce some important limits involving logarithmic functions. A fundamental form is obtained by writing $z = \log_e x$, so that $x = e^z$, $\frac{1}{x} = e^{-z}$. Since now x increases indefinitely with z, we have the result

$$\lim_{x\to\infty} \frac{(\log_e x)^m}{x^n} = 0, \quad (n > 0), \qquad (2)$$

of which special forms are

$$\lim_{x\to\infty} x^{-n} \log_e x = 0, \quad (n > 0), \qquad (3)$$

and

$$\lim_{x\to\infty} \tfrac{1}{x} \log_e x = 0. \qquad (4)$$

If we write $1/x$ for x in (4), so that now $x \to 0$ (through positive values), we have the result

$$\lim_{x\to +0} x \log_e x = 0. \qquad (5)$$

Another important limit is obtained from equation (14) of the preceding Article, which we can write in the form

$$\lim_{x\to 0} \log_e (1+x)^{1/x} = 1. \qquad (6)$$

Writing $\log_e (1+x)^{1/x} = 1+k$, we have $(1+x)^{1/x} = e^{1+k} = ee^k$, where, from equation (6), $k \to 0$ and therefore $e^k \to 1$ as $x \to 0$. Hence

$$\lim_{x\to 0} (1+x)^{1/x} = e. \qquad (7)$$

This result is proved independently of the manner in which x tends to zero. If, therefore, we give to x the sequence of values $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$, we have the result

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e. \quad \text{..............................(8)}$$

This proves that the number defined by the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$, Ex. 3, Art. 50, is the number e defined in Art. 160 of the present Chapter.

178. Differentiation of the Logarithmic Function.

The simplest procedure is to make use of the rule connecting the derivatives of direct and inverse functions, namely $\dfrac{dy}{dx}=\dfrac{1}{dx/dy}$.

178. 1. *Derivative of* $\log_e|x|$. Let $y=\log_e|x|$. If $x>0$ we have $x=e^y$, $dx/dy=e^y=x$, and so

$$\frac{dy}{dx}=\frac{1}{x}. \quad \text{..................................(1)}$$

If $x<0$, we have $x=-e^y$, $dx/dy=-e^y=x$, and again

$$\frac{dy}{dx}=\frac{1}{x}. \quad \text{..................................(2)}$$

We therefore have the general result

$$\frac{d\log_e|x|}{dx}=\frac{1}{x}, \quad (x\neq 0). \quad \text{........................(3)}$$

The derivative of the logarithmic function is thus an algebraic function of the independent variable, namely its reciprocal. It is on account of this property that the logarithmic function is of fundamental importance in Analysis.

Ex. 1. Obtain the derivative of $\log_e|x|$ directly from the definition of dy/dx. [Employ the result of Ex. 2, Art. 176.]

178. 2. *Derivative of* $\log_a|x|$. We have $\log_a|x|=\log_e|x|\log_a e$, and hence

$$\frac{d\log_a|x|}{dx}=\frac{d\log_e|x|}{dx}\log_a e=\frac{1}{x}\log_a e. \quad \text{................(4)}$$

For the special value 10 of a, we have

$$\frac{d\log_{10}|x|}{dx}=\frac{1}{x}\log_{10} e=\frac{M}{x}, \quad \text{......................(5)}$$

where M is the modulus $0{\cdot}434294\ldots$, (Art. 176).

178. 3. *The function* $y=\log_e|f(x)|$. Write $y=\log_e|u|$, where $u=f(x)$, $(u \neq 0)$. Then $dy/du=1/u$, $du/dx=f'(x)$, and the rule for the derivative of a function of a function gives

$$\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}=\frac{f'(x)}{f(x)},$$

or

$$\frac{d\log_e|f(x)|}{dx}=\frac{f'(x)}{f(x)}. \qquad \text{.........................(6)}$$

This rule is frequently applied in the sequel.

Ex. 2. Shew that $$\frac{d\log_e|1-x|}{dx}=-\frac{1}{1-x}. \qquad \text{.............................(7)}$$

[This follows from (6) on putting $f(x)=1-x$.]

Ex. 3. Shew that $$\frac{d}{dx}\log_e\left|\frac{1+x}{1-x}\right|=\frac{2}{1-x^2}. \qquad \text{..............................(8)}$$

$\left[\text{Write } \log_e\left|\frac{1+x}{1-x}\right|=\log_e|1+x|-\log_e|1-x|.\right]$

Ex. 4. Shew that $$\frac{d}{dx}\log_e|x+\sqrt{(x^2\pm 1)}|=1/\sqrt{(x^2\pm 1)}. \qquad \text{......................(9)}$$

[We observe here that $f'(x)=1+x/\sqrt{(x^2\pm 1)}$.]

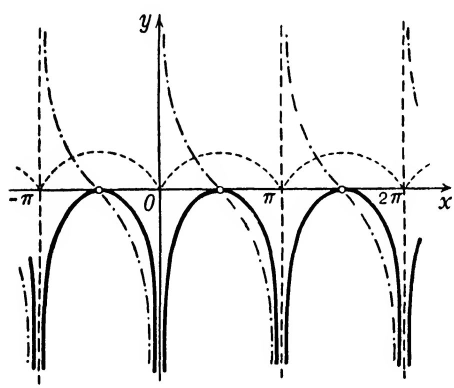

FIG. 147.
$y=\log_e|\sin x|$.

Ex. 5. *The function* $\log_e|\sin x|$, $(x \neq n\pi)$. We have, after equation (6),

$$\frac{d\log_e|\sin x|}{dx}=\frac{\cos x}{\sin x}=\cot x. \quad (10)$$

If the modulus signs are omitted, the ranges of x given by $(2n\pi-\pi, 2n\pi)$ must be excluded.

Portion of the graph of the function $\log_e|\sin x|$ is shewn in Fig. 147, the broken lines being the corresponding parts of the graphs of the functions $|\sin x|$, $\cot x$.

Ex. 6. *The function* $\log_e|\cos x|$, $\left(x \neq \frac{2n+1}{2}\pi\right)$. Here we have

$$\frac{d\log_e|\cos x|}{dx}=-\frac{\sin x}{\cos x}=-\tan x, \qquad \text{.......................(11)}$$

a result which we may write in the equivalent form

$$\frac{d\log_e|\sec x|}{dx}=\tan x. \qquad \text{..............................(12)}$$

Ex. 7. *The function* $\log_e|\tan\frac{1}{2}x|$, $(x \neq n\pi)$. We have

$$\frac{d\log_e|\tan\frac{1}{2}x|}{dx}=\frac{\frac{1}{2}\sec^2\frac{1}{2}x}{\tan\frac{1}{2}x}=\frac{1}{2\sin\frac{1}{2}x\cos\frac{1}{2}x}=\frac{1}{\sin x}=\operatorname{cosec} x. \qquad \text{.........(13)}$$

Ex. 8. *The function* $\log_e|\sec x+\tan x|$. Since the value of $f'(x)$ is here $\sec x(\sec x+\tan x)$, we have

$$\frac{d\log_e|\sec x+\tan x|}{dx}=\frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}=\sec x=\frac{1}{\cos x}. \quad\text{.........(14)}$$

Also we can write

$$\sec x+\tan x=\frac{1+\sin x}{\cos x}=\frac{(\cos\frac{1}{2}x+\sin\frac{1}{2}x)^2}{\cos^2\frac{1}{2}x-\sin^2\frac{1}{2}x}=\frac{\cos\frac{1}{2}x+\sin\frac{1}{2}x}{\cos\frac{1}{2}x-\sin\frac{1}{2}x}$$

$$=\frac{1+\tan\frac{1}{2}x}{1-\tan\frac{1}{2}x}=\tan(\tfrac{1}{2}x+\tfrac{1}{4}\pi), \quad\text{..........(15)}$$

and we therefore have the equivalent result

$$\frac{d}{dx}\log_e|\tan(\tfrac{1}{2}x+\tfrac{1}{4}\pi)|=\frac{1}{\cos x}. \quad\text{..........(16)}$$

The results (10)-(16) are of importance, especially in connection with integration, and should be remembered.

Ex. 9. Find the derivative of $\log_e|\log_e x|$, $(x>0)$.
[Put $\log_e x=u$. The result is $1/(x\log_e x)$.]

179. The 'Interest Law' of Change of a Quantity.

The 'interest law' of variation of a quantity is such that the rate of change of the quantity is proportional to its instantaneous value. In symbols, if y is the measure of the quantity corresponding to the value x of the independent variable, the statement of the law is

$$\frac{dy}{dx}=ky, \quad\text{..........(1)}$$

where k is a constant. This law is of frequent occurrence in Physics and Chemistry, illustrations from which are given below.

The problem to be solved here is to determine y as a function of x with the aid of the above relation and given initial conditions. It will be seen that this can be effected by employing the result for the derivative of the logarithmic function. It is sufficient to take y positive. We write (1) in the form

$$\frac{dx}{dy}=\frac{1}{ky}, \quad\text{..........(2)}$$

which, with the aid of the result referred to, can be written

$$\frac{dx}{dy}=\frac{1}{k}\cdot\frac{d\log_e y}{dy}=\frac{d}{dy}\left(\frac{1}{k}\log_e y\right),$$

or

$$\frac{d}{dy}\left(x-\frac{1}{k}\log_e y\right)=0. \quad\text{..........(3)}$$

Now, according to Art. 98. 1, if the derivative of a function is

zero throughout a range, the function is constant in the range. We conclude that

$$x - \tfrac{1}{k}\log_e y = C, \quad \text{.........................(4)}$$

where the value of the constant C is fixed by the initial conditions of the problem. For example, if the initial amount of the quantity is y_0, corresponding to the value 0 of x, the value of C is given by

$$0 - \tfrac{1}{k}\log_e y_0 = C, \quad \text{.........................(5)}$$

and the special function y which satisfies this condition and the equation (1) is given by

$$x - \tfrac{1}{k}\log_e y = -\tfrac{1}{k}\log_e y_0,$$

or

$$kx = \log_e (y/y_0). \quad \text{.........................(6)}$$

On inverting this relation, we obtain

$$y = y_0 e^{kx}, \quad \text{.........................(7)}$$

which is the expression for the function sought. If $k > 0$, this represents a function increasing exponentially with x, while if $k < 0$, the function decreases exponentially from the initial value y_0 to the final value 0. In the latter case it is usually more convenient to replace k by $-k$, so that the decreasing law is expressed by $y = y_0 e^{-kx}$, $(k > 0)$.

In practical applications of the law the independent variable is commonly the time, which is denoted by the letter t. The laws of increase and decrease are then expressed by $y = y_0 e^{kt}$, $y = y_0 e^{-kt}$, respectively, where $k > 0$.

In the case of a physical phenomenon, the value of k can be determined experimentally by observing the time in which the varying quantity increases or decreases in a certain ratio. In the latter case, if the time in which the quantity decreases to one-half of its initial value is τ, we have

$$\tfrac{1}{2}y_0 = y_0 e^{-k\tau}, \quad \text{.........................(8)}$$

whence $e^{k\tau} = 2$, or $k\tau = \log_e 2$. This gives

$$k = \frac{\log_e 2}{\tau} = \frac{0{\cdot}693147}{\tau}, \quad \text{.........................(9)}$$

on introducing the value of $\log_e 2$, (see Art. 182. 1). Hence the law of variation is given by

$$y = y_0 e^{-0{\cdot}693147 t/\tau}. \quad \text{.........................(10)}$$

The time τ is usually described as the *half-value period* of the quantity considered.

Ex. 1. Shew that if the varying quantity decreases in the ratio $1 : e$ in time τ, k is equal to $1/\tau$.

We now give some illustrations of the law.

(i) *Growth of capital at continuous compound interest.* Considering at first the case when the interest is added at the ends of finite intervals of time, let y be the capital at the beginning t of an interval and let Δy be the interest added at the end $t+\Delta t$ of the interval. If the rate of interest is p per cent. per annum, and if the unit for the measurement of the time is the year, we have

$$\Delta y = \frac{p}{100} y\, \Delta t, \qquad \text{.....(11)}$$

so that the average rate of increase of capital in this interval is given by $\Delta y/\Delta t = py/100$.

In the case of instantaneous compound interest, this average rate becomes the exact rate, corresponding to an indefinitely small time interval, and we thus have the law

$$\frac{dy}{dt} = \frac{py}{100}, \qquad \text{.....(12)}$$

so that, if y_0 is the initial capital, that at the end of t years is given by $y = y_0 e^{pt/100}$. The capital after one year $(t=1)$ is given by

$$y_1 = y_0 e^{p/100} = y_0 \left(1 + \frac{p}{10^2} + \frac{1}{2!}\cdot\frac{p^2}{10^4} + \frac{1}{3!}\cdot\frac{p^3}{10^6} + \dots\right). \qquad \text{.....(13)}$$

For example, if $p=5$, the successive terms in the parentheses are equal to 1, 0·05, 0·00125, 0·00002 ..., ..., and we have $y \approx 1{\cdot}05127 y_0$, which is sufficiently exact for most purposes.

(ii) *Radioactive transformations.* The mathematical theory of radioactive transformations is based on the law $y = y_0 e^{-kt}$. In the simplest case, in which a quantity Q of radioactive matter is being transformed at a rate proportional to the amount of the quantity at the instant in question, we have

$$\frac{dQ}{dt} = -\lambda Q, \qquad \text{.....(14)}$$

the constant λ being called the *radioactive constant* of the substance. This gives, for the amount transformed after time t,

$$Q = Q_0 e^{-\lambda t} = Q_0 e^{-0{\cdot}693147 t/\tau}, \qquad \text{.....(15)}$$

where τ is the half-value period.

(iii) *Newton's law of cooling.* Under certain conditions, the rate of cooling of a hot body is proportional to the difference between the temperature of the body and that of the surrounding medium, which is here supposed constant. The excess θ of the temperature of the body over that of the medium then varies according to the interest law, and we have $\theta = \theta_0 e^{-\lambda t}$, where λ is a constant and θ_0 is the excess when $t=0$.

180. Use of Logarithms in the Differentiation of Products and Powers.

The ordinary rule for the derivative of a product can be recovered with the aid of the derivative of the logarithmic function as follows. Let $y = u_1 . u_2 . u_3 \dots u_n$, where $u_r \equiv u_r(x)$. Then we have

$$|y| = |u_1| . |u_2| \dots |u_n|, \quad \log_e |y| = \log_e |u_1| + \log_e |u_2| + \dots + \log_e |u_n|,$$

and, on differentiating throughout with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u_1}\frac{du_1}{dx} + \frac{1}{u_2}\frac{du_2}{dx} + \dots + \frac{1}{u_n}\frac{du_n}{dx}. \quad \text{.................(1)}$$

Multiplication by y now leads to the ordinary formula.

The same method applies to a quotient. Thus, if $y = u/v$, so that $\log_e |y| = \log_e |u| - \log_e |v|$, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} - \frac{1}{v}\frac{dv}{dx},$$

which leads to the ordinary formula.

The above method of differentiation can also be used to find the derivative for functions of the type $y = \{f(x)\}^{\phi(x)}$. This function is defined by the equation

$$\{f(x)\}^{\phi(x)} = e^{\phi(x) \log_e f(x)}, \quad \text{..........................(2)}$$

where $f(x)$ is supposed positive. Direct differentiation gives

$$\frac{dy}{dx} = e^{\phi(x)\log_e f(x)} \left\{\varphi'(x) \log_e f(x) + \varphi(x) \frac{f'(x)}{f(x)}\right\}$$

$$= \{f(x)\}^{\phi(x)} \left\{\varphi'(x) \log_e f(x) + \varphi(x) \frac{f'(x)}{f(x)}\right\}, \quad \text{.............(3)}$$

which is also obtained by first writing

$$\log_e y = \varphi(x) \log_e f(x), \quad \text{..........................(4)}$$

and then differentiating throughout with respect to x.

In the discussion of small corrections, where the fractional error is frequently required, it is sometimes an advantage to take logarithms before differentiating. Thus, let $y = f(x)$ be the relation between a measured quantity x and a calculated quantity y, and suppose that there is a small error δx in x. The corresponding fractional error in y is given approximately by

$$\frac{\delta y}{y} = \frac{1}{y}\frac{dy}{dx}\delta x, \quad \text{.....................................(5)}$$

and the value of $\frac{1}{y}\frac{dy}{dx}$ is obtained immediately by the process suggested.

Ex. 1. $y=\sqrt{\{(1-x)(2-x)\}}$. Here $x \leqslant 1$ or $x \geqslant 2$. We have

$$\log_e y = \tfrac{1}{2}\log_e |1-x| + \tfrac{1}{2}\log_e |2-x|, \qquad (6)$$

which gives, on differentiating,

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\frac{-1}{1-x}+\frac{1}{2}\frac{-1}{2-x}=-\frac{1}{2}\frac{3-2x}{(1-x)(2-x)}, \qquad (7)$$

so that

$$\frac{dy}{dx}=-\frac{1}{2}\frac{3-2x}{\sqrt{\{(1-x)(2-x)\}}}. \qquad (8)$$

Ex. 2. The area S of a triangle ABC is computed from the measurements of the sides a, b and the included angle C. If there is an error δC in C, find the percentage error in the calculated value of S.

We have $S=\frac{1}{2}ab\sin C$, whence

$$\log_e S = \log_e(\tfrac{1}{2}ab)+\log_e \sin C. \qquad (9)$$

Therefore

$$\frac{1}{S}\frac{dS}{dC}=\cot C \qquad (10)$$

and so

$$\frac{\delta S}{S}\approx\frac{1}{S}\frac{dS}{dC}\delta C=\cot C\,\delta C. \qquad (11)$$

181. Successive Derivatives and Taylor's Theorem for $\log_e |x|$.

Since $\dfrac{d\log_e|x|}{dx}=\dfrac{1}{x}$, the nth derivative of $\log_e|x|$ is equal to the $(n-1)$th derivative of $1/x$. Referring to equation (15), Ex. 2, Art. 86, p. 181, we thus have

$$\frac{d^n \log_e|x|}{dx^n}=\frac{d^{n-1}(1/x)}{dx^{n-1}}=(-1)^{n-1}\frac{(n-1)!}{x^n}. \qquad (1)$$

Now the function $\log_e|x|$, together with all its derivatives, is undefined (infinite) at $x=0$. This point cannot, therefore, be used as a starting point for the application of the $I.D_n.T.$ to $\log_e|x|$. Any other starting point may be chosen and the theorem then applied to a range of which the end points have the same sign, and which does not accordingly include the value $x=0$. It is sufficient to consider the case in which the end points are positive. Let a be the starting point and x the end point, so that the increment is $x-a$. We have

$$\log_e x=\log_e\{a+(x-a)\}=\log_e\left\{a\left(1+\frac{x-a}{a}\right)\right\}$$

$$=\log_e a+\log_e\left(1+\frac{x-a}{a}\right), \qquad (2)$$

and this shews that it is sufficient to consider the special starting value 1, taking $\dfrac{x-a}{a}$ as increment. With a change in the meaning of x, this increment can be taken as $x-1$, and we write

$$\log_e x=\log_e\{1+(x-1)\}. \qquad (3)$$

It is plain that a simplification is effected by a shift of origin to the point $x=1$, the function $\log_e x$ transforming to $\log_e(1+x')$, say, where $x'=x-1$, and the starting point being $x'=0$. Using now x instead of x', we are thus led to consider the application of the theorem to the function $\log_e(1+x)$, $(x>-1)$, with starting point $x=0$ and end point any value of x greater than -1.

Writing, then, $f(x)=\log_e(1+x)$, $(x>-1)$, we have, from (1),

$$f^{(n)}(x)=(-1)^{n-1}\frac{(n-1)!}{(1+x)^n}, \quad\text{.......................(4)}$$

so that

$$f(0)=0,\quad f^{(r)}(0)=(-1)^{r-1}(r-1)!,\quad f^{(n)}(\theta x)=(-1)^{n-1}\frac{(n-1)!}{(1+\theta x)^n}. \quad\text{...(5)}$$

The $I.D_n.T.$ for $\log_e(1+x)$ is thus expressed by

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots+(-1)^{n-2}\frac{x^{n-1}}{n-1}+R_n(x), \quad\text{.......(6)}$$

where $$R_n(x)=(-1)^{n-1}\frac{1}{n}\left(\frac{x}{1+\theta x}\right)^n,\quad (x>-1). \quad\text{..............(7)}$$

We now investigate conditions under which the remainder converges to zero with increase of n. It will be shewn that if $-1<x\leqslant 1$, the convergence is assured.

Case (i). $0<x\leqslant 1$. In this case $\theta x>0$, and hence

$$|R_n(x)|<\frac{x^n}{n}. \quad\text{..............................(8)}$$

It thus follows that $|R_n(x)|$ can be made as small as we please by taking sufficiently large values of n. The conditions for Taylor's infinite series are therefore satisfied in the range (0, 1) of x, and Taylor's Theorem for this range is expressed by

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots+(-1)^{n-1}\frac{x^n}{n}+\dots. \quad\text{..........(9)}$$

Case (ii). $-1<x<0$. Write, for convenience, $x=-x'$, so that $0<x'<1$. We now have, from (7),

$$|R_n(x)|=\frac{1}{n}\left(\frac{x'}{1-\theta x'}\right)^n<\frac{1}{n}\left(\frac{x'}{1-x'}\right)^n, \quad\text{...............(10)}$$

since $1-x'<1-\theta x'$. We conclude, as in Case (i), that $|R_n(x)|\to 0$ as $n\to\infty$ provided that $x'/(1-x')\leqslant 1$, that is, $x'\leqslant\frac{1}{2}$. It can be shewn, however, that the convergence of $|R_n(x)|$ to zero is assured for values of x' greater than this, the range of convergence being, in fact, given by $0<x'<1$, that is, by $-1<x<0$.

One step in this direction is made by using the identity

$$\log_e (1 - x') = \log_e \frac{1 - x'^2}{1 + x'} = \log_e (1 - z) - \log_e (1 + x'), \quad \text{...........(11)}$$

where $z = x'^2$, so that $0 < z < 1$. If, now, $0 < z \leqslant \frac{1}{2}$, the remainder in the expansion of $\log_e (1 - z)$ converges to zero as $n \to \infty$, and this corresponds to the range $0 < x' \leqslant 1/\surd 2$. Also, the remainder in the expansion of $\log_e (1 + x')$ has been proved to converge to zero as $n \to \infty$ if $0 < x' \leqslant 1$, and by combining these results we establish the convergence to zero of $| R_n(x) |$ for $0 < x' \leqslant 1/\surd 2$. This process can be repeated.

A direct proof is afforded by using Cauchy's formula for the remainder term, Art. 128, equation (19), which, in the present case, becomes

$$R_n(x) = (-1)^{n-1} \frac{(1 - \theta)^{n-1} x^n}{(1 + \theta x)^n}, \quad (0 < \theta < 1). \text{............(12)}$$

Therefore

$$| R_n(x) | = \frac{1}{1 - \theta} \left| \frac{x - \theta x}{1 + \theta x} \right|^n = \frac{1}{1 - \theta} \left(\frac{x' - \theta x'}{1 - \theta x'} \right)^n. \text{............(13)}$$

If, now, $x' < 1$, we have $x' - \theta x' < 1 - \theta x'$, and hence $\dfrac{x' - \theta x'}{1 - \theta x'} < 1$. It follows that if $x' < 1$, $| R_n(x) | \to 0$ as $n \to \infty$. The value $x' = 1$, that is, $x = -1$, is excluded from the discussion since the function $\log_e (1 + x)$ is not defined for that value. Actually the Taylor polynomials do not here converge to a finite value.

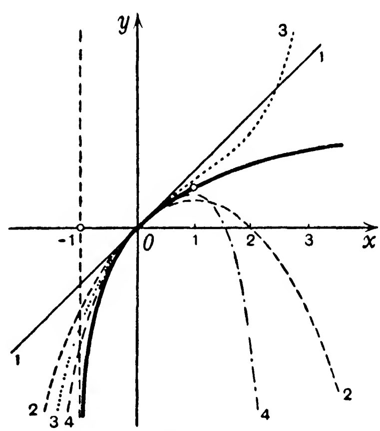

FIG. 148.

We may therefore assert that Taylor's Theorem for $\log_e (1 + x)$, as expressed by equation (9), holds if $-1 < x < 0$, and we have thus proved the validity of the theorem for the range $[-1, 1)$ of x. The infinite series on the right-hand side of (9) may thus be taken as the definition of the function $\log_e (1 + x)$ in this range. It is called the *logarithmic series* for $\log_e (1 + x)$.

Case (iii). $| x | > 1$. The Taylor polynomials do not now satisfy the necessary condition for the validity of Taylor's Theorem that the general term $\pm x^n/n$ should converge to zero when n increases indefinitely. For if $u_n = | x |^n/n$, we have

$$\frac{u_{n+1}}{u_n} = \frac{n}{n+1} | x |, > 1 \quad \text{if} \quad n > \frac{1}{| x | - 1} \text{..............(14)}$$

Thus after a certain value of n, the term u_n increases with n and so cannot converge to zero when $n \to \infty$. The Taylor Theorem expressed by (9) therefore does not hold if $|x| > 1$.

The closeness of approximation of the earlier Taylor polynomials to $\log_e(1+x)$ is illustrated geometrically in Fig. 148, which clearly indicates the range of convergence.

Ex. 1. Obtain the earlier Taylor polynomials for $\log_e(1+x)$ by the process of successive substitutions, employing the corresponding polynomials for the exponential function.

[Write $y = \log_e(1+x)$, so that $1 + x = e^y = 1 + y + \frac{y^2}{2!} + \dots$, and hence

$$y = x - \frac{y^2}{2!} - \frac{y^3}{3!} - \dots .]$$

Ex. 2. Shew that $\log_e(1+x) = x$, correct to seven places of decimals, if $0 < x < 0{\cdot}000316$.

We have $\log_e(1+x) = x + R_2(x)$, where $|R_2(x)| = \frac{1}{2}x^2 \frac{1}{(1+\theta x)^2} < \frac{1}{2}x^2$. Thus $|R_2(x)| < 5 \times 10^{-(n+1)}$ if $x^2 < 10^{-n}$, or $x < 10^{-n/2}$. If $n = 7$, this condition becomes $x < \sqrt{10} \times 10^{-4} \approx 0{\cdot}000316$.

Ex. 3. Shew that for small rates of interest, the time taken for a sum of money to double itself at compound interest at the rate of p per cent. per annum (added annually) is about $69/p$ years. A rule employed for high rates of interest is $73/p$ years. Examine the validity of this for interest rates of 5 and 10 per cent.

If y_0 is the initial sum, that at the end of t years is given by

$$y = y_0\left(1 + \frac{p}{100}\right)^t . \qquad \text{(15)}$$

If τ is the time for the sum to double, this gives $2 = \left(1 + \frac{p}{100}\right)^\tau$, and hence

$$\log_e 2 = \tau \log_e\left(1 + \frac{p}{100}\right)$$

$$= \tau\left(\frac{p}{100} - \frac{p^2}{2 \times 100^2} + \dots\right), \qquad \text{(16)}$$

so that

$$\tau = \frac{100 \log_e 2}{p\left(1 - \frac{p}{200} + \dots\right)} . \qquad \text{(17)}$$

If p is small, we may neglect the terms $p/200$ and those which follow it in the denominator, as being small compared with 1, and we have

$$\tau \approx \frac{100 \log_e 2}{p} \approx 69{\cdot}315/p \text{ years}, \qquad \text{(18)}$$

introducing the approximate value $0{\cdot}69315$ for $\log_e 2$, (Art. 182. 1).

A second approximation is given by

$$\tau \approx \frac{100 \log_e 2}{p\left(1 - \frac{p}{200}\right)} \approx \frac{100 \log_e 2}{p}\left(1 + \frac{p}{200}\right), \dots \qquad \text{(19)}$$

which gives, for $p=5$,

$$\tau \approx \frac{100 \log_e 2 \times 1{\cdot}025}{p} = \frac{71{\cdot}05}{p}, \quad \text{.............................(20)}$$

and for $p=10$,

$$\tau \approx \frac{100 \log_e 2 \times 1{\cdot}05}{p} = \frac{72{\cdot}78}{p}. \quad \text{.............................(21)}$$

182. Calculation of Natural Logarithms.

It is beyond the scope of this Book to discuss the various methods [1] that have been employed to calculate tables of logarithms. We confine ourselves, for the most part, to the illustration of the application of the logarithmic series for $\log_e(1+x)$. As we have seen, this series is theoretically applicable for values of x in the range $[-1, 1)$, and hence for the calculation of logarithms of numbers in the range $[0, 2)$, but for rapid convergence of the series x must be a small fraction. For example, corresponding to the end $x=1$ of the range we have

$$\log_e 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \ldots + (-1)^{n-1}\tfrac{1}{n} + \ldots, \quad \text{.............(1)}$$

and some 20,000 terms of the series are required to secure accuracy to four places of decimals.

For numbers outside the stated range, as well as those in the range which are not nearly equal to 1, the practical method is to employ linear combinations of the logarithmic series for different small values of x. We proceed to illustrate the different cases.

Let us first suppose that $x=0{\cdot}1$. Then we get

$$\log_e 1{\cdot}1 = \frac{1}{10} - \frac{1}{2\times 10^2} + \frac{1}{3\times 10^3} - \frac{1}{4\times 10^4} + \frac{1}{5\times 10^5} - \ldots, \quad \text{......(2)}$$

and if we retain the five terms written, the absolute error involved will, according to (8), Art. 181, not exceed $\dfrac{1}{6\times 10^6} = 1{\cdot}67\times 10^{-7}$. We have

$$\begin{aligned}\log_e 1{\cdot}1 &= 0{\cdot}1 - 0{\cdot}005 + 0{\cdot}0003333 - 0{\cdot}0000250 + 0{\cdot}0000020\\ &= 0{\cdot}0953103, \quad \text{...(3)}\end{aligned}$$

and here there is only one source of error due to approximating in the seventh place, and it is less than 4 in the eighth place. The total error will not therefore exceed about 2 in the seventh place.

If, now, we take $x=-0{\cdot}1$, we get

$$\log_e\left(1-\frac{1}{10}\right) = -\frac{1}{10} - \frac{1}{2\times 10^2} - \frac{1}{3\times 10^3} - \ldots, \quad \text{............(4)}$$

[1] The student may refer to *Encyclopaedia Britannica*, Art. *Logarithms*, and to Shortrede, *Logarithmic Tables*.

and, including only five terms of the series, this gives

$$\log_e \tfrac{9}{10} = -0{\cdot}1053603. \quad\ldots\ldots(5)$$

The error here introduced due to omitting the terms after the fifth is, by (10), Art. 181, less than $\frac{1}{6}\left(\frac{\frac{1}{10}}{1-\frac{1}{10}}\right)^6 = \frac{1}{6\times 9^6} \approx 3{\cdot}1\times 10^{-7}$, and the total error is therefore less than 4×10^{-7}. The value of $\log_e 9/10$ correct to ten places is found to be

$$\log_e \tfrac{9}{10} = -0{\cdot}105\,360\,515\,7,$$

so that

$$\log_e \tfrac{10}{9} = 0{\cdot}105\,360\,515\,7. \quad\ldots\ldots(6)$$

Ex. 1. Evaluate $\log_e 2$ correct to five places of decimals.

[Write $\log_e 2 = \log_e \frac{1+\frac{1}{3}}{1-\frac{1}{3}} = \log_e(1+\frac{1}{3}) - \log_e(1-\frac{1}{3})$, and employ the logarithmic series.]

Ex. 2. Find the value of $\log_e \frac{25}{24}$ correct to ten places of decimals.

[The given logarithm is equal to $-\log_e(1-\frac{4}{100})$. The value is found to be 0·040 821 994 5.]

Ex. 3. Find the value of $\log_e \frac{81}{80}$ correct to ten places of decimals.

[Write $\log_e \frac{81}{80} = -\log_e\left(1-\frac{1}{9^2}\right)$. The required value is 0·0124225200.]

Ex. 4. *Adams's formulae.* If $a = \log_e \frac{10}{9}$, $b = \log_e \frac{25}{24}$, $c = \log_e \frac{81}{80}$, shew that

$$\left.\begin{aligned} \log_e 2 &= 7a - 2b + 3c, & \log_e 3 &= 11a - 3b + 5c, \\ \log_e 5 &= 16a - 4b + 7c, & \log_e 10 &= 23a - 6b + 10c. \end{aligned}\right\} \quad\ldots\ldots(7)$$

Hence evaluate $\log_e 2$, $\log_e 3$, $\log_e 5$, $\log_e 10$ correct to seven places of decimals.

We now shew how to calculate the logarithms of the natural numbers by the application of the logarithmic series for values of x so small that the series is rapidly convergent. For this purpose we use the two following equations, in which $0<x<1$,

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-2}\frac{x^{n-1}}{n-1} + \ldots + \frac{x^{2n-1}}{2n-1} + R_{2n}(x), \ldots(8)$$

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots - \frac{x^{n-1}}{n-1} - \ldots - \frac{x^{2n-1}}{2n-1} + R^*_{2n}(x). \quad\ldots\ldots(9)$$

The former of these is equivalent to equation (6) of the preceding Article, and the latter is obtained from it by replacing x by $-x$. The remainder in each case converges to zero as $n\to\infty$. On subtracting, we obtain

$$\log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \ldots + \frac{x^{2n-1}}{2n-1}\right) + R, \quad\ldots\ldots(10)$$

where R is equal to $R_{2n}(x) - R^*_{2n}(x)$, and so converges to zero when n increases indefinitely. The conditions for the expansion

of $\log_e \dfrac{1+x}{1-x}$ in an infinite series are thus satisfied, and Taylor's Theorem for the function is expressed by

$$\log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \dots + \frac{x^{2n-1}}{2n-1} + \dots\right). \quad\dots\dots\dots\dots(11)$$

An explicit form for R can be obtained from equation (7) of the preceding Article, but it is not very convenient for the determination of an upper barrier to the error committed in replacing $\log_e \dfrac{1+x}{1-x}$ by its Taylor polynomial of any stated degree. We shall subsequently obtain an expression for R as a definite integral, Ex. 2, Art. 239, which leads directly to an upper barrier. We here indicate an indirect method of estimating the error. Write (10) in the form $\log_e \dfrac{1+x}{1-x} = S_n + R$, and let $S_m = 2\left(x + \dfrac{x^3}{3} + \dots + \dfrac{x^{2m-1}}{2m-1}\right)$, $(m < n)$. Then

$$\log_e \frac{1+x}{1-x} - S_m = S_n - S_m + R, \quad\dots\dots\dots\dots(12)$$

where we have

$$S_n - S_m = 2\left(\frac{x^{2m+1}}{2m+1} + \frac{x^{2m+3}}{2m+3} + \dots + \frac{x^{2n-1}}{2n-1}\right)$$
$$< 2\frac{x^{2m+1}}{2m+1}(1 + x^2 + \dots + x^{2n-2m-2}) < 2\frac{x^{2m+1}}{2m+1}\frac{1}{1-x^2}, \quad\dots(13)$$

which is independent of n. Since $R \to 0$ as $n \to \infty$, we can therefore assert that

$$\log_e \frac{1+x}{1-x} - S_m < 2\frac{x^{2m+1}}{2m+1}\frac{1}{1-x^2}, \quad\dots\dots\dots\dots(14)$$

for any value of m. The error committed is therefore less than the fraction $1/(1-x^2)$ of the first term of the series which is omitted.

We now introduce special forms for x.

(i) Put
$$\frac{1+x}{1-x} = \frac{1+p}{p}, \quad (p > 0), \quad\dots\dots\dots\dots(15)$$

giving $x = 1/(2p+1)$, so that if p is an integer, $x \leqslant \frac{1}{3}$. The series (11) now becomes

$$\log_e \frac{1+p}{p} = 2\left\{\frac{1}{2p+1} + \frac{1}{3(2p+1)^3} + \dots + \frac{1}{(2n-1)(2p+1)^{2n-1}} + \dots\right\}, \quad (16)$$

from which we can find $\log_e(1+p)$ as soon as $\log_e p$ is known, since

$$\log_e \frac{1+p}{p} = \log_e(1+p) - \log_e p. \quad\dots\dots\dots\dots(17)$$

Also, since $x \leqslant \frac{1}{3}$, we have $\frac{1}{1-x^2} \leqslant \frac{1}{1-\frac{1}{9}} = \frac{9}{8}$, and the error committed by neglecting the terms after a certain one is less than 9/8ths of the first term neglected.

(ii) More flexibility in application is obtained by writing

$$\frac{1+x}{1-x} = \frac{p+q}{p}, \quad (p, q > 0), \quad \text{......................(18)}$$

which gives $x = q/(2p+q)$, the series (11) then leading to

$$\log_e(p+q) = \log_e p$$
$$+2\left\{\frac{q}{2p+q} + \frac{1}{3}\left(\frac{q}{2p+q}\right)^3 + \dots + \frac{1}{2n-1}\left(\frac{q}{2p+q}\right)^{2n-1} + \dots\right\}. \quad \text{....(19)}$$

(iii) A second modification of (15) is obtained by writing $1+p=k^2$, so that

$$\frac{1+x}{1-x} = \frac{k^2}{k^2-1}, \quad (k>1), \quad \text{......................(20)}$$

which gives $x = 1/(2k^2-1)$, and then, from (11),

$$\log_e k = \tfrac{1}{2}\log_e(k-1) + \tfrac{1}{2}\log_e(k+1)$$
$$+\left\{\frac{1}{2k^2-1} + \frac{1}{3}\cdot\frac{1}{(2k^2-1)^3} + \dots + \frac{1}{2n-1}\cdot\frac{1}{(2k^2-1)^{2n-1}} + \dots\right\}. \quad \text{...(21)}$$

The use of the series (16), (19), (21) in calculation will now be illustrated.

182. 1. *Use of series* (16). *Calculation of* $\log_e 2$. Put $p=1$ in (16) and we have

$$\log_e 2 = 2\left(\frac{1}{3} + \frac{1}{3\times 3^3} + \frac{1}{5\times 3^5} + \dots\right) = \frac{2}{3}\left(1 + \frac{1}{3\times 9} + \frac{1}{5\times 9^2} + \dots\right). \quad \text{........(22)}$$

Working to eight decimal places, we have the following scheme :

$1 = 1$	$1 = 1$
$1/9 = 0{\cdot}111\ 111\ 11$	$1/(3\times 9) = 0{\cdot}037\ 037\ 04$
$1/9^2 = 0{\cdot}012\ 345\ 68$	$1/(5\times 9^2) = 0{\cdot}002\ 469\ 14$
$1/9^3 = 0{\cdot}001\ 371\ 74$	$1/(7\times 9^3) = 0{\cdot}000\ 195\ 96$
$1/9^4 = 0{\cdot}000\ 152\ 42$	$1/(9\times 9^4) = 0{\cdot}000\ 016\ 94$
$1/9^5 = 0{\cdot}000\ 016\ 94$	$1/(11\times 9^5) = 0{\cdot}000\ 001\ 54$
$1/9^6 = 0{\cdot}000\ 001\ 88$	$1/(13\times 9^6) = 0{\cdot}000\ 000\ 14$
$1/9^7 = 0{\cdot}000\ 000\ 21$	$1/(15\times 9^7) = 0{\cdot}000\ 000\ 01$
	$1{\cdot}039\ 720\ 77$

Hence, retaining eight terms of the series, we get

$$\log_e 2 = \tfrac{2}{3}\times 1{\cdot}039\ 720\ 77 = 0{\cdot}693\ 147\ 18. \quad \text{....................(23)}$$

The first term omitted in the series is $\frac{2}{3}\times\frac{1}{17\times 9^8}$, and the error committed in neglecting the terms after the eighth is less than 9/8ths of this, i.e. less than about 10^{-9}. In the actual calculations there are seven errors due to approximating in the eighth place, and these together will not exceed $7\times 5\times 10^{-9}$. Hence the total error will not exceed $\frac{2}{3}\times 35\times 10^{-9}+10^{-9}\approx 2\cdot 4\times 10^{-8}$. Actually, the result obtained is correct to the eight places given.

Calculation of $\log_e 3$. Put $p=2$, $2p+1=5$, and we obtain

$$\log_e 3=\log_e 2+2\left(\frac{1}{5}+\frac{1}{3\times 5^3}+\frac{1}{5\times 5^5}+\dots\right), \qquad (24)$$

from which, with the aid of (23), we find, correct to seven places,

$$\log_e 3=1\cdot 098\ 612\ 3. \qquad (25)$$

The evaluation of the logarithms of successive integers can be carried out by this step-by-step process, care being taken to ensure that the accumulated errors do not become significant by carrying out the calculations in the early stages to more than the number of places finally desired. The student should verify the following results:

$$\log_e 4=1\cdot 38629436,\quad \log_e 5=1\cdot 60943791,\quad \log_e 6=1\cdot 79175947,$$
$$\log_e 7=1\cdot 94591015,\quad \log_e 8=2\cdot 07944154,\quad \log_e 9=2\cdot 19722458,$$
$$\log_e 10=2\cdot 30258509.$$

Since the logarithm of a product is the sum of the logarithms of its factors, the logarithms of prime numbers only are required in the construction of a table of logarithms. In the *Thesaurus* of Vega, a table prepared by Wolfram gives the natural logarithms to forty-eight places of decimals of all the integers from 1 to 2200, and of all the primes from 2201 to 100 009. This allows the logarithm of any number of four figures to be obtained at once. For example,

$$\begin{aligned}\log_e 3\cdot 133&=\log_e 3133-\log_e 1000=\log_e 241+\log_e 13-\log_e 1000\\&=5\cdot 48479693+2\cdot 56494936-6\cdot 90775528=1\cdot 14199101. \qquad (26)\end{aligned}$$

182. 2. *Use of series* (19). *Calculation of* $\log_e 4\cdot 3$. We have

$$\log_e 4\cdot 3=\log_e 43-\log_e 10. \qquad (27)$$

Putting $p=40$, $q=3$, so that $q/(2p+q)=3/83$, (19) gives

$$\log_e 43=\log_e 40+2\{\tfrac{3}{83}+\tfrac{1}{3}(\tfrac{3}{83})^3+\dots\}, \qquad (28)$$

and therefore, subtracting $\log_e 10$,

$$\log_e 4\cdot 3=2\log_e 2+2\{\tfrac{3}{83}+\tfrac{1}{3}(\tfrac{3}{83})^3+\dots\}. \qquad (29)$$

With the aid of Barlow's tables, we find:

$3/83=0\cdot 0361446$	$3/83=0\cdot 0361446$
$(3/83)^3=0\cdot 0000472$	$\frac{1}{3}(3/83)^3=0\cdot 0000157$
$(3/83)^5=0\cdot 0000001$	$\frac{1}{5}(3/83)^5=0\cdot 0000000$
	$0\cdot 0361603$

Hence

$$\log_e 4{\cdot}3 = 1{\cdot}3862944 + 2 \times 0{\cdot}0361603 = 1{\cdot}4586150, \quad \ldots\ldots\ldots\ldots(30)$$

a result which is actually correct to the seven places of decimals given.

182. 3. *Use of series* (21). *Calculation of* $\log_e 4{\cdot}3$. Put $k=43$, so that $k-1=42$, $k+1=44$, $2k^2-1=3697$. Then (21) gives

$$\log_e 43 = \tfrac{1}{2}\log_e 42 + \tfrac{1}{2}\log_e 44 + \frac{1}{3697} + \frac{1}{3\times 3697^3} + \ldots$$

$$= \tfrac{1}{2}\log_e 2 + \tfrac{1}{2}\log_e 3 + \tfrac{1}{2}\log_e 7 + \log_e 2 + \tfrac{1}{2}\log_e 11 + \tfrac{1}{3697} + \ldots$$

$$= \tfrac{3}{2}\log_e 2 + \tfrac{1}{2}\log_e 3 + \tfrac{1}{2}\log_e 7 + \tfrac{1}{2}\log_e 10 + \tfrac{1}{2}\log_e(1+\tfrac{1}{10}) + \tfrac{1}{3697} + \ldots . \quad (31)$$

Now by equation (3) above, $\log_e 1{\cdot}1 = 0{\cdot}0953103$, and also $1/3697 = 0{\cdot}0002705$. Working to six decimal places we thus obtain

$$\log_e 43 = \tfrac{3}{2}\times 0{\cdot}693147 + \tfrac{1}{2}\times 1{\cdot}098612 + \tfrac{1}{2}\times 1{\cdot}945910$$
$$+ \tfrac{1}{2}\times 2{\cdot}302585 + \tfrac{1}{2}\times 0{\cdot}095310 + 0{\cdot}000270$$
$$= 3{\cdot}761199. \quad \ldots\ldots\ldots\ldots(32)$$

Therefore

$$\log_e 4{\cdot}3 = 3{\cdot}761199 - 2{\cdot}302585 = 1{\cdot}458614, \quad \ldots\ldots\ldots\ldots(33)$$

which is in error actually by only 1 in the sixth place of decimals.

Ex. 5. *Glaisher's formulae.* If $a = \log_e \frac{16}{15}$, $b = \log_e \frac{25}{24}$, $c = \log_e \frac{81}{80}$, shew that

$$\log_e 2 = 7a + 5b + 3c, \quad \log_e 3 = 11a + 8b + 5c, \quad \log_e 5 = 16a + 12b + 7c. \ \ldots(34)$$

Evaluate a, b, c correct to eight decimal places from (16) with $p = 15$, 24, 80, in turn, and hence evaluate $\log_e 2$, $\log_e 3$, $\log_e 5$ correct to six places.

183. Calculation of Common Logarithms.

The values of the common logarithms of numbers may be found at once from the natural logarithms by multiplying by the modulus $M = 0{\cdot}4342945 \ldots$. Conversion tables to obviate the multiplication are usually included in logarithm tables, as in Chambers's *Mathematical Tables* or Shortrede's *Logarithmic Tables*.

184. Weddle's Method for finding Logarithms.

We here describe Weddle's method for finding the logarithm of any given number to a large number of decimal places by means of a small table carried to that number of places.

For ordinary purposes the existing seven-figure tables give the common logarithm of any number to the necessary accuracy by means of simple interpolation. Occasionally it is required to find the logarithm of a number to a larger number of places than that of any accessible table of a complete kind,[1] and much study has been devoted to the discovery of rapid methods

[1] A table, *Logarithmetica Britannica*, of common logarithms to twenty decimal places of numbers up to 100,000, by A. J. Thompson, is in course of publication by the Cambridge University Press.

of effecting this. Those most frequently employed are based on the possibility of expressing any number with sufficient accuracy as the product of a small number of factors of some simple kind.[1] The required logarithm is then the sum of the logarithms of the factors.

Any such method is made available by the calculation of a table for the logarithms of all the factors of the chosen kind which it is necessary to employ. Weddle considered factors of the form $(1 - r \times 10^{-n})^{-1}$, where n is a positive integer and r is one of the nine digits 1, 2, ..., 9. Indicating a product of such factors by prefixing Π to the one written, any number can be readily expressed in the form $p \times 10^m \times \Pi(1 - r \times 10^{-n})^{-1} \times (1 + z)$, where p is the first digit in the number, $m + 1$ is the number of digits in the number and z is so small that the logarithm of the factor $1 + z$ is negligible. A table of common logarithms of Weddle's factors is given, for example, in Shortrede's *Logarithms*, Table IV, to sixteen places (with a supplement for twenty-five places).

To illustrate the process, take the number 4973, working to sixteen places. We have $4973 = 4 \times 10^3 \times 1{\cdot}24325$, and we have to express $1{\cdot}24325$ in the form $\Pi(1 - r \times 10^{-n})^{-1} \times (1 + z)$. The Weddle factors for $1{\cdot}24325$ are found by multiplying it in succession by factors of the type $1 - r \times 10^{-n}$ for increasing values of n, the digits r being chosen by inspection so as to reduce that number to one having sixteen noughts after the decimal point. Thus the first factor is $1 - \frac{1}{10}$, and we get the following scheme :

N	1·2432 5
$N \times 10^{-1}$	·1243 25
$N_1 = N(1 - 1 \times 10^{-1})$	1·1189 25
$N_1 \times 10^{-1}$	·1118 925
$N_2 = N_1(1 - 1 \times 10^{-1})$	1·0070 325
$N_2 \times 6 \times 10^{-3}$	·0060 4219 5
$N_3 = N_2(1 - 6 \times 10^{-3})$	1·0009 9030 5
$N_3 \times 9 \times 10^{-4}$	·0009 0089 1274 5
$N_4 = N_3(1 - 9 \times 10^{-4})$	1·0000 8941 3725 5
$N_4 \times 8 \times 10^{-5}$	·0000 8000 7153 0980 4
$N_5 = N_4(1 - 8 \times 10^{-5})$	1·0000 0940 6572 4019 6
$N_5 \times 9 \times 10^{-6}$	·0000 0900 0084 6591 5
$N_6 = N_5(1 - 9 \times 10^{-6})$	1·0000 0040 6487 7428 1
$N_6 \times 4 \times 10^{-7}$	·0000 0040 0000 1626 2
$N_7 = N_6(1 - 4 \times 10^{-7})$	1·0000 0000 6487 5801 9
$N_7 \times 6 \times 10^{-9}$	·0000 0000 6000 0000 4

From the last step onwards, the multipliers $1 - 6 \times 10^{-9}$, $1 - 4 \times 10^{-10}$, ... replace the significant digits 6, 4, ... by 0, 0, ... without affecting the remaining ones. Thus the Weddle transform of the number 4973 is

$$4 \times 10^3 \times \genfrac{\{}{\}}{0pt}{}{1}{1}^2 \genfrac{\{}{\}}{0pt}{}{6}{3} \genfrac{\{}{\}}{0pt}{}{9}{4} \genfrac{\{}{\}}{0pt}{}{8}{5} \genfrac{\{}{\}}{0pt}{}{9}{6} \genfrac{\{}{\}}{0pt}{}{4}{7} \genfrac{\{}{\}}{0pt}{}{6}{9} \genfrac{\{}{\}}{0pt}{}{4}{10} \genfrac{\{}{\}}{0pt}{}{8}{11} \genfrac{\{}{\}}{0pt}{}{7}{12} \genfrac{\{}{\}}{0pt}{}{5}{13} \genfrac{\{}{\}}{0pt}{}{8}{14} \genfrac{\{}{\}}{0pt}{}{2}{16},$$

where $\genfrac{\{}{\}}{0pt}{}{1}{1}$ is written for $(1 - 1 \times 10^{-1})^{-1}$, $\genfrac{\{}{\}}{0pt}{}{6}{3}$ for $(1 - 6 \times 10^{-3})^{-1}$, and so on.

[1] See Henderson, *Bibliotheca Tabularum*, Lüroth, *Numerisches Rechnen*, Shortrede, *Logarithms*, (*Preface*).

Taking the common logarithm of these factors from Shortrede, Table IV, we find, for $\log_{10}4973$,

$$\begin{array}{r}
3{\cdot}6020\ 5999\ 1327\ 9624\\
457\ 5749\ 0560\ 6751\\
457\ 5749\ 0560\ 6751\\
26\ 1361\ 5602\ 6867\\
3\ 9104\ 1028\ 5829\\
3474\ 4948\ 3687\\
390\ 8667\ 9262\\
17\ 3717\ 8275\\
2605\ 7669\\
173\ 7178\\
34\ 7435\\
3\ 0400\\
2171\\
347\\
1\\
\hline
3{\cdot}6966\ 1845\ 9232\ 2247
\end{array}$$

If for the sake of a check we convert this to base e by means of the conversion table in Shortrede, Table VI, we find

$$\log_e 4973 = 8{\cdot}5117\ 7855\ 8714\ 7380,$$

which differs from Wolfram's result, in the *Thesaurus* of Vega, only in the last digit.

Ex. 1. Use the logarithmic series to shew that

$$\log_{10}(1 - 6 \times 10^{-8})^{-1} = 0{\cdot}0000\ 0002\ 6057\ 6697.$$

185. Application of the Method of Proportional Parts to Tables of Logarithms of Functions.

To test the validity of the method, we require to determine the value of $\frac{1}{8}h^2|f''(x)|_{max}$, where $f(x)$ is the function tabulated and h is the common difference of argument.

185. 1. *Tables of* $\log_{10}x$. Here we have $f''(x) = -M/x^2$, and so

$$\tfrac{1}{8}h^2|f''(x)|_{max} = \tfrac{1}{8}Mh^2(1/x^2)_{max}, \qquad \text{(1)}$$

where M is, as usual, the modulus $0{\cdot}434294 \ldots$.

Consider a seven-figure table extending from[1] $x = 10{,}000$ to $x = 100{,}000$ with constant difference of argument 1. Since the greatest value of $1/x^2$ is 10^{-8}, the uncertainty of interpolation throughout the table will not exceed $\frac{1}{8}M \times 1 \times 10^{-8} \approx 0{\cdot}0543 \times 10^{-8}$, which is negligible to the order of accuracy of tabulation. We may thus interpolate correctly by the *M.P.P.* throughout the table.

[1] The tabulated values of $\log_{10} x$ in such a table are always the decimal parts of the logarithms, and the range of x is accordingly more correctly described as extending from $1{\cdot}0000$ to $10{\cdot}0000$ with common difference of argument $0{\cdot}0001$.

In the *Thesaurus* of Vega, the common logarithms are given correct to ten places of decimals, the range and difference of argument being the same as in the seven-figure tables. The above investigation shews that accuracy to ten places cannot be assured by the *M.P.P.* in the portion of the table extending up to about 30,000. In this statement the permissible uncertainty is taken to be 5×10^{-11}.

In those five-figure tables where the range is from 1,000 to 10,000 and the difference of argument is 1, the uncertainty will not exceed $0{\cdot}0543 \times 10^{-6}$.

185. 2. *Tables of* $\log_e x$. The uncertainty of interpolation is now $\frac{1}{8}h^2(1/x^2)_{max}$. Let us investigate the necessary subdivision of the range (1, 3) of x in order to get accuracy to ten places. The permissible error is 5×10^{-11}, and the desired accuracy will be secured throughout the range if we take $\frac{1}{8}h^2 < 5 \times 10^{-11}$, or $h < 2 \times 10^{-5}$, corresponding to the maximum value 1 of $1/x^2$.

In the tables of Hayashi, the natural logarithms are tabulated for numbers in the range (1, 3) to ten decimal places, the difference of argument being 0·001. The *M.P.P.* is plainly inadequate to give intermediate values to this accuracy in any portion of the range.

185. 3. *Inverse reading of tables of* $\log_{10} x$. As explained in Art. 163, if tables of 10^x are not available, the antilogarithm of a number is to be obtained by reading backwards in a table of $\log_{10} x$. The tabulated values range from 0 to 1 and the corresponding values of x from 1 to 10.

Consider a seven-figure table of $\log_{10} x$ in which the common difference of x is 0·0001. Write $y = \log_{10} x$, so that $x = 10^y$. The uncertainty U of a value x_1 taken as the value of 10^y corresponding to a tabulated value y_1 is, according to Art. 125. 1, given by

$$U = 5 \times 10^{-8} \Big/ \left(\frac{dy}{dx}\right)_1 = 5 \times 10^{-8} \times x_1/M. \quad\text{..................}(2)$$

The uncertainty U^* of an interpolated value x^* lying between two consecutive tabulated values x_1, x_2, and determined by means of the *M.P.P.*, is, according to Art. 125. 2, given by

$$U^* = \tfrac{1}{8}h^2 \left|\frac{d^2y}{dx^2}\right|_1 \Big/ \left|\frac{dy}{dx}\right|_1 = \tfrac{1}{8} \times 10^{-8} \left|\frac{M/x_1^2}{M/x_1}\right| = \tfrac{1}{8} \times 10^{-8}/x_1. \quad\text{......}(3)$$

For completeness of the table for inverse reading, we must have

$$\tfrac{1}{8} \times 10^{-8}/x_1 < 5 \times 10^{-8} \times x_1/M, \quad\text{or}\quad x_1^2 > 0{\cdot}01, \quad\text{..........}(4)$$

a condition which is satisfied throughout the table.

From (2), accuracy to seven significant figures is assured only if $5 \times 10^{-8} \times x_1/M < 5 \times 10^{-7}$, which gives $x_1 < 4{\cdot}343$. For values of x

greater than this, the possibility of an error of a unit in the seventh significant digit is not excluded.

185. 4. *Tables of* $\log_{10}\sin x$. Here we have $f(x) = M\log_e \sin x$, so that $f''(x) = -M\operatorname{cosec}^2 x$. The uncertainty of an interpolated value found from the *M.P.P.* is thus given by

$$\tfrac{1}{8}Mh^2 \operatorname{cosec}^2 x, \quad \dots\dots\dots\dots(5)$$

h being the difference of argument. Since $\operatorname{cosec} x$ decreases from ∞ to 1 as x increases from 0 to $\frac{1}{2}\pi$, it is clear that for small values of x the *M.P.P.* cannot be applied correctly.

In tables of common logarithms of trigonometrical functions, the arguments are usually expressed in degree measure, and in order to avoid the introduction of negative characteristics into the tables, the logarithms are generally given in excess of their true values by 10. The student will be familiar with this procedure.

Consider a seven-figure table of logarithmic sines in which the argument is in degrees, the common difference being 1 minute. The equivalent difference in a table of $\log_{10}\sin x$ is now equal to $\pi/(180\times 60)$, and from (5) the uncertainty of interpolation is $\frac{1}{8}M\dfrac{\pi^2}{(180\times 60)^2}\operatorname{cosec}^2 x$, or about $5\times 10^{-9}\times\operatorname{cosec}^2 x$. The permissible error being 5×10^{-8}, the table ceases to be complete when

$$\operatorname{cosec}^2 x \geqslant 10,$$

that is, when $\sin^2 x \leqslant 1/10$, or[1] when $\sin^2 x < \pi^2/100$, which is satisfied when $x < \pi/10$, the degree measure of which is 18°. The *M.P.P.* is thus not to be relied on for values of the argument less than this.

If the common difference of argument is 10″, as in the tables of Callet, Gardiner and Vega (*Thesaurus*), the *M.P.P.* gives accuracy to seven places for arguments down to 3°, about. If the difference of argument is 1″, as in Shortrede, the angle may be as small as 18′. The verification of these results is left to the student.

186. The Functions $\operatorname{lolog} x$, $\operatorname{illolog} x$.

Chappell in his *Five-Figure Mathematical Tables* gives tables for $\log_{10}|\log_{10}x|$ and the inverse function. These functions he denotes by '$\operatorname{lolog} x$' and '$\operatorname{illolog} x$'. The object of these tables is to shorten the calculation of powers where the index is not a simple integer or fraction. Let it be required to find a^b, $(a>0)$. If we write $c = a^b$, we have

$$|\log_{10}c| = |b|\,.\,|\log_{10}a| \quad\text{and}\quad \log_{10}|\log_{10}c| = \log_{10}|b| + \log_{10}|\log_{10}a|. \quad \dots\dots\dots(1)$$

[1] We here make use of the approximation $\pi^2 \approx 10$.

The direct table has a as argument and $\log_{10}|\log_{10}a|$ as tabular value, so that $\log_{10}|b|+\log_{10}|\log_{10}a|$ is found by one entry in a common-logarithm table and one in the lolog table, followed by addition. We have then to find c from a knowledge of $\log_{10}|\log_{10}c|$, where, if b is positive, the sign of $\log_{10}c$ is positive or negative according as a is greater or less than 1, and if b is negative, is positive or negative according as a is less or greater than 1. The inverse, or illolog, table consists of two parts, one printed in black and the other in red. If $\log_{10}c$ is positive, the black table is used; if negative, the red table. In either case c is the number tabulated against the argument $\log_{10}|\log_{10}c|$, so that a single entry gives the required value of a^b. The method of proportional parts can, of course, be employed where necessary, as in the case of ordinary logarithms.

187. Miscellaneous Examples.

Ex. 1. Shew that $\log_e(1+x)>x-\frac{1}{2}x^2$ if $x>0$.

[The function $\log_e(1+x)-x+\frac{1}{2}x^2$ vanishes when $x=0$ and has a positive derivative when $x>0$. The result stated follows from Cor. 1, Art. 98. 5.]

Ex. 2. Shew that $\dfrac{x}{1+x}<\log_e(1+x)<x$ if $x>-1$.

[Apply the same corollary to the functions $\dfrac{x}{1+x}-\log_e(1+x)$, $\log_e(1+x)-x$.]

Ex. 3. Investigate the sense of the curvature of the graph of the function $x\log_e x$, $(x>0)$.

We have $d^2y/dx^2=1/x$, which is positive throughout, so that the curve is everywhere concave upwards, (Fig. 149). By (5), Art. 177, $y\to 0$ as $x\to 0$, so that the origin is a limiting point on the curve, to the left of which the curve does not extend. There is a negative infinite upper derivative at this point, since $dy/dx=1+\log_e x\to -\infty$ as $x\to +0$.

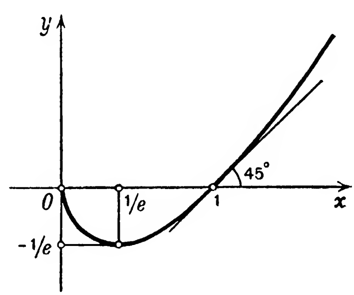

FIG. 149.
$y=x\log_e x$.

The curve crosses Ox where $x=1$ and the gradient there is 1, or the slope $\frac{1}{4}\pi$. The point of minimum ordinate has coordinates $(1/e, -1/e)$.

Ex. 4. Find the general Taylor polynomial for $\{\log_e(1+x)\}^2$.

Denoting this function by y, we find

$$(1+x)y^{(1)}-2\log_e(1+x)=0. \quad\text{..........................(1)}$$

On employing Leibniz's Theorem, this leads to the recurrence formula

$$(1+x)y^{(n+1)}+ny^{(n)}+(-1)^n\frac{2(n-1)!}{(1+x)^n}=0, \quad\text{..................(2)}$$

which, on putting $x=0$, becomes, for $n\geqslant 1$,

$$y_0^{(n+1)}=-ny_0^{(n)}-(-1)^n 2(n-1)!, \quad\text{.........................(3)}$$

while from (1) we have $y_0^{(1)}=0$.

Now (3) can be put into the form

$$(-1)^n \frac{1}{n!} y_0^{(n+1)} - (-1)^{n-1} \frac{1}{(n-1)!} y_0^{(n)} = -\frac{2}{n}, \quad \text{.................}(4)$$

and hence, adding the equations obtained by giving n the values 1, 2, 3, ..., n in (4), we get

$$(-1)^n \frac{1}{n!} y_0^{(n+1)} = -2\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \dots + \frac{1}{n}\right). \quad \text{.................}(5)$$

The required Taylor polynomial is found to be

$$2\left\{\tfrac{1}{2}x^2 - \tfrac{1}{3}(1+\tfrac{1}{2})x^3 + \tfrac{1}{4}(1+\tfrac{1}{2}+\tfrac{1}{3})x^4 + \dots + (-1)^n \frac{1}{n}\left(1 + \tfrac{1}{2} + \dots + \frac{1}{n-1}\right)x^n\right\}. \quad \dots(6)$$

The same result is established by squaring the Taylor polynomial for $\log_e(1+x)$ of the $(n-1)$th degree and rejecting terms involving powers of x higher than the nth from the product. The earlier terms of the series are often found informally in the following manner. Writing

$$(1+x)^n = e^{n \log_e(1+x)}, \quad \text{...............................}(7)$$

where we suppose $|x| < 1$, we have, by the Binomial Theorem and the exponential series,

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + \frac{n(n-1)\dots(n-r-1)}{r!}x^r + \dots$$

$$= 1 + n\log_e(1+x) + \frac{n^2}{2!}\{\log_e(1+x)\}^2 + \dots + \frac{n^r}{r!}\{\log_e(1+x)\}^r + \dots, \quad \dots(8)$$

which is to hold for all values of n. Hence, comparing coefficients of n^2 in the two series, we get

$$\frac{1}{2!}\{\log_e(1+x)\}^2 = \frac{x^2}{2!} - \frac{3x^3}{3!} + \frac{11x^4}{4!} - \dots, \quad \text{.......................}(9)$$

which leads to the required form.

It may be pointed out that the logarithmic series for $\log_e(1+x)$ follows in the same way by equating the coefficients of n.

Ex. 5. Investigate the Taylor polynomials of the sixth degree for

(i) $\log_e \dfrac{\sin x}{x}$, (ii) $\log_e \cos x$, (iii) $\log_e \dfrac{\tan x}{x}$,

(iv) $\log_e \dfrac{\sinh x}{x}$, (v) $\log_e \cosh x$, (vi) $\log_e \dfrac{\tanh x}{x}$,

[Results : (i) $-\frac{1}{6}x^2 - \frac{1}{180}x^4 - \frac{1}{2835}x^6$ (ii) $-\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6$,

(iii) $\frac{1}{3}x^2 + \frac{7}{90}x^4 + \frac{62}{2835}x^6$, (iv) $\frac{1}{6}x^2 - \frac{1}{180}x^4 + \frac{1}{2835}x^6$,

(v) $\frac{1}{2}x^2 - \frac{1}{12}x^4 + \frac{1}{45}x^6$, (vi) $-\frac{1}{3}x^2 + \frac{7}{90}x^4 - \frac{62}{2835}x^6$.]

SECTION IV. THE INVERSE HYPERBOLIC FUNCTIONS

188. Definitions of the Inverse Hyperbolic Functions.

We next consider the inversion of the hyperbolic functions. From the forms of the expressions for the direct functions in terms of

exponential functions, it follows, as we shall see below, that the inverse functions are expressible as logarithms of algebraic functions of the independent variable. The introduction of these functions is therefore not so urgent as in the case of the inverse trigonometrical functions for which no corresponding alternative expressions exist.[1] Nevertheless, from the analytical point of view there is a considerable advantage in introducing these inverse functions as fundamental functions, in order to maintain the parallelism between the trigonometrical and the hyperbolic functions. Moreover, from the point of view of application, the increasing availability of tables of the hyperbolic functions has led to a rapid increase in the use of the inverse functions. In the tables of Hayashi the direct and inverse functions are tabulated in parallel columns, while in the Smithsonian collection the inverse functions have to be obtained by the process of 'reading backwards' from the values of the direct functions.

188. 1. *Inversion of* $y=\cosh x$, $(y \geqslant 1)$. The function $\cosh x$ is strictly down-way in the range $[-\infty, 0)$ and strictly up-way in the range $(0, \infty]$. Either of these is thus a suitable range for inversion. According to the usual convention, the inverse function[2] $\operatorname{arcosh} y$ is defined only for the range $(0, \infty]$ of x, so that the inverse relation $x=\operatorname{arcosh} y$ gives x as an up-way continuous function of y, increasing indefinitely from the value 0 as y increases from the value 1.

Associated with the function $\operatorname{arcosh} y$ there is an adjoined function $-\operatorname{arcosh} y$, which corresponds to the inversion of the relation $y=\cosh x$ in the range $[-\infty, 0)$ of x.

The existence of a definite, single-valued inverse hyperbolic cosine being thus established, there is no longer any reason for retaining y to denote the independent variable, and we now replace it by x, y being available to denote the function $\operatorname{arcosh} x$, which is therefore a positive, up-way function, defined only in the range $(1, \infty]$ of x.

The graph of the function $y=\operatorname{arcosh} x$ is the same as that of the function $y=\cosh x$ in the range $(0, \infty]$ of x when the rôles of the axes Ox, Oy are interchanged. It is most simply obtained by taking the geometrical image of the latter graph in the line bisecting

[1] The statement ceases to be true when complex numbers are introduced.

[2] Other notations employed are $\cosh^{-1} y$, $\mathfrak{Ar}\mathfrak{Cos}\, y$, the latter mainly in German texts. The functional signs 'arcosh,' etc., are pronounced as in the case of the direct functions with the syllable 'ar' prefixed.

the angle xOy. These graphs are shewn as continuous lines in Fig. 150.

188. 2. *Inversion of* $y = \sinh x$. Here y is a strictly up-way function of x throughout. We can therefore invert without any restrictions of range, the inverse relation being written $x = \operatorname{arsinh} y$.

Treating 'arsinh' now as a symbol for a new function and taking x as independent variable, we have the function arsinh x, which is up-way

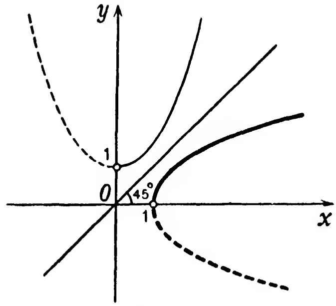

FIG. 150.
——— $y = \operatorname{arcosh} x$.

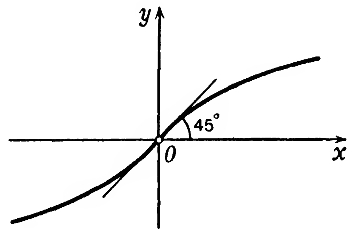

FIG. 151.
$y = \operatorname{arsinh} x$.

throughout, having an unlimited range of values and being defined for all values of x. The graph of the function $y = \operatorname{arsinh} x$ is shewn in Fig. 151.

188. 3. *Inversion of* $y = \tanh x$, $(|y| < 1)$. Since y is strictly up-way throughout, we can again invert without restrictions of range. The inverse relation is expressed by $x = \operatorname{artanh} y$, which defines x as an up-way function of y throughout the range $[-1, 1]$ of y.

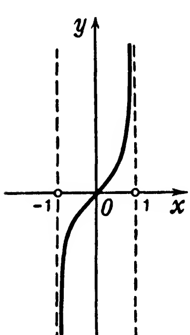

FIG. 152.
$y = \operatorname{artanh} x$.

Treating 'artanh' as a symbol for a new function and taking x as independent variable, we arrive at the function artanh x, which is up-way and defined only for values of x in the range $[-1, 1]$. The graph of the function $y = \operatorname{artanh} x$ is shewn in Fig. 152.

188. 4. *The inverse hyperbolic secant.* According to the usual convention, the inverse hyperbolic secant, which is denoted by $y = \operatorname{arsech} x$ when the independent variable is x, corresponds to the inversion of the hyperbolic secant, $x = \operatorname{sech} y$, for positive values only of its argument, here y. The function arsech x is thus a positive, down-way function, defined only in the range $[0, 1)$ and having the range of values $(0, \infty]$.

As in the case of the inverse hyperbolic cosine, there is the adjoined function $y = -\operatorname{arsech} x$, which corresponds to the inversion of $x = \operatorname{sech} y$ for negative values of y. The graphs of the functions are shewn in Fig. 153, together with those of $\operatorname{arcosh} x$ and $-\operatorname{arcosh} x$.

188. 5. *The inverse hyperbolic cosecant.* Here we have the inverse relations $y = \operatorname{arcosech} x$, $x = \operatorname{cosech} y$, the ranges of x and y being

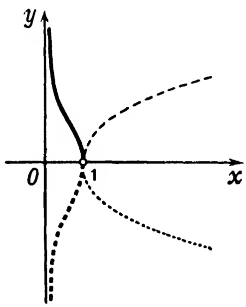

FIG. 153.
—— $y = \operatorname{arsech} x$.

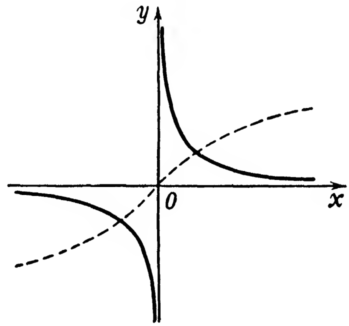

FIG. 154.
—— $y = \operatorname{arcosech} x$.

unrestricted except for the exclusion of the value 0 in each case. The graph is shewn in Fig. 154, together with that of $\operatorname{arsinh} x$.

188. 6. *The inverse hyperbolic cotangent.* Here we have the relations $y = \operatorname{arcoth} x$, $x = \coth y$, the value 0 only being excluded from the range of y and the ranges of x being $[-\infty, -1]$, $[1, \infty]$. The graph is shewn in Fig. 155, together with that of $\operatorname{artanh} x$.

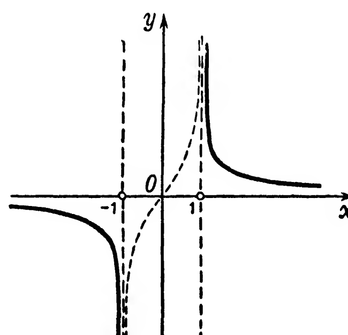

FIG. 155.
—— $y = \operatorname{arcoth} x$.

189. Relations between the Inverse Functions.[1] Their Expression in terms of Logarithms.

We begin with some relations between the inverse hyperbolic functions. These are inverse forms of relations already established for the direct functions. Proofs are supplied in selected cases and the student should treat the other cases on similar lines. Complete proofs are given for the logarithmic expressions which follow.

[1] The student should observe that there are restrictions on the various arguments introduced, corresponding to the restricted ranges of definition of the inverse functions.

(i) $\operatorname{arsech} x = \operatorname{arcosh}\frac{1}{x}$, $\operatorname{arcosech} x = \operatorname{arsinh}\frac{1}{x}$, $\operatorname{arcoth} x = \operatorname{artanh}\frac{1}{x}$. (1)

These are inverse forms of the relations $\operatorname{sech} x = 1/\cosh x$, etc.

If $1/x \geqslant 1$ we can put $1/x = \cosh\varphi$, $(\varphi \geqslant 0)$, and hence $x = \operatorname{sech}\varphi$. We therefore have

$$\operatorname{arsech} x = \varphi = \operatorname{arcosh}(1/x). \quad \text{.....(2)}$$

Again, we can always put $1/x = \sinh\varphi$ and hence $x = \operatorname{cosech}\varphi$. Therefore

$$\operatorname{arcosech} x = \varphi = \operatorname{arsinh}(1/x). \quad \text{.....(3)}$$

Finally, if $|x| > 1$ we can put $1/x = \tanh\varphi$ and hence $x = \coth\varphi$. Therefore

$$\operatorname{arcoth} x = \varphi = \operatorname{artanh}(1/x). \quad \text{.....(4)}$$

These relations are also evident from the graphs by comparing the abscissae of the pairs of curves $y = \operatorname{arsech} x$ and $y = \operatorname{arcosh} x$, etc.; see Figs. 153-155.

(ii)
$$\operatorname{arcosh}|x| = \operatorname{arsinh}\sqrt{(x^2-1)}, \quad \text{.....(5)}$$
$$\operatorname{arsinh}|x| = \operatorname{arcosh}\sqrt{(x^2+1)}. \quad \text{.....(6)}$$

These are inverse forms of the relations $\sinh|x| = \sqrt{(\cosh^2 x - 1)}$, $\cosh x = \sqrt{(\sinh^2 x + 1)}$.

If $|x| \geqslant 1$ we can put $|x| = \cosh\varphi$, $\sqrt{(x^2-1)} = \sinh\varphi$, $(\varphi \geqslant 0)$. Therefore

$$\operatorname{arcosh}|x| = \varphi = \operatorname{arsinh}\sqrt{(x^2-1)}. \quad \text{.....(7)}$$

Again, since $\sqrt{(x^2+1)} \geqslant 1$ we can always put $|x| = \sinh\varphi$, $\sqrt{(x^2+1)} = \cosh\varphi$, $(\varphi \geqslant 0)$. Therefore

$$\operatorname{arsinh}|x| = \varphi = \operatorname{arcosh}\sqrt{(x^2+1)}. \quad \text{.....(8)}$$

(iii)
$$\operatorname{arcosh}|x| = \operatorname{artanh}\frac{\sqrt{(x^2-1)}}{|x|}, \quad \text{.....(9)}$$

$$\operatorname{arsinh} x = \operatorname{artanh}\frac{x}{\sqrt{(x^2+1)}}. \quad \text{.....(10)}$$

These are inverse forms of the relation $\tanh x = \sinh x/\cosh x$.

(iv) *Inverse forms of the Addition Theorems.*

$$|\operatorname{arcosh} x_1 \pm \operatorname{arcosh} x_2| = \operatorname{arcosh}\{x_1x_2 \pm \sqrt{(x_1{}^2-1)}\sqrt{(x_2{}^2-1)}\}, \quad \text{.(11)}$$

$$\operatorname{arsinh} x_1 \pm \operatorname{arsinh} x_2 = \operatorname{arsinh}\{x_1\sqrt{(x_2{}^2+1)} \pm x_2\sqrt{(x_1{}^2+1)}\}, \quad \text{..(12)}$$

$$\operatorname{artanh} x_1 \pm \operatorname{artanh} x_2 = \operatorname{artanh}\frac{x_1 \pm x_2}{1 \pm x_1x_2}. \quad \text{.....(13)}$$

These are inverse forms of the relations

$$\cosh(x_1 \pm x_2) = \cosh x_1 \cosh x_2 \pm \sinh x_1 \sinh x_2, \quad \text{.....(14)}$$

$$\sinh(x_1 \pm x_2) = \sinh x_1 \cosh x_2 \pm \cosh x_1 \sinh x_2, \quad \text{.....(15)}$$

$$\tanh(x_1 \pm x_2) = (\tanh x_1 \pm \tanh x_2)/(1 \pm \tanh x_1 \tanh x_2). \quad \text{...(16)}$$

To prove (11) observe that since[1] $x_1 \geqslant 1$, $x_2 \geqslant 1$, we can put $x_1 = \cosh\varphi_1$, $\sqrt{(x_1^2-1)} = \sinh\varphi_1$, $(\varphi_1 \geqslant 0)$, and $x_2 = \cosh\varphi_2$, $\sqrt{(x_2^2-1)} = \sinh\varphi_2$, $(\varphi_2 \geqslant 0)$. Then

$$\begin{aligned} x_1x_2 &+ \sqrt{(x_1^2-1)}\sqrt{(x_2^2-1)} \\ &= \cosh\varphi_1\cosh\varphi_2 + \sinh\varphi_1\sinh\varphi_2 = \cosh(\varphi_1+\varphi_2), \quad \text{..........(17)} \end{aligned}$$

where $\varphi_1 + \varphi_2 \geqslant 0$, and hence

$$\operatorname{arcosh} x_1 + \operatorname{arcosh} x_2 = \varphi_1 + \varphi_2 = \operatorname{arcosh}\{x_1x_2 + \sqrt{(x_1^2-1)}\sqrt{(x_2^2-1)}\}. \quad \text{......(18)}$$

Also we have

$$\begin{aligned} x_1x_2 - \sqrt{(x_1^2-1)}\sqrt{(x_2^2-1)} &= \cosh\varphi_1\cosh\varphi_2 - \sinh\varphi_1\sinh\varphi_2 \\ &= \cosh(\varphi_1-\varphi_2) = \cosh|\varphi_1-\varphi_2|, \quad \text{.............(19)} \end{aligned}$$

and hence

$$|\operatorname{arcosh} x_1 - \operatorname{arcosh} x_2| = |\varphi_1 - \varphi_2| = \operatorname{arcosh}\{x_1x_2 - \sqrt{(x_1^2-1)}\sqrt{(x_2^2-1)}\}. \text{ (20)}$$

(v) *Expression in terms of logarithms.*

$$\begin{aligned} &(a)\ \operatorname{arcosh} x = \log_e\{x + \sqrt{(x^2-1)}\} && \text{..................(21)} \\ &\qquad\qquad = -\log_e\{x - \sqrt{(x^2-1)}\}; && \text{...............(22)} \\ &(b)\ \operatorname{arsinh} x = \log_e\{x + \sqrt{(x^2+1)}\} && \text{..................(23)} \\ &\qquad\qquad = -\log_e\{\sqrt{(x^2+1)} - x\}; && \text{...............(24)} \\ &(c)\ \operatorname{artanh} x = \tfrac{1}{2}\log_e\{(1+x)/(1-x)\}; && \text{...............(25)} \\ &(d)\ \operatorname{arcoth} x = \tfrac{1}{2}\log_e\{(x+1)/(x-1)\}. && \text{...............(26)} \end{aligned}$$

The relations (*a*), (*b*) correspond to the relation $\cosh x + \sinh x = e^x$, and (*c*), (*d*) to the relation $\tanh x = (e^{2x}-1)/(e^{2x}+1)$.

(*a*) If $x \geqslant 1$ we can put $x = \cosh\varphi$, $\sqrt{(x^2-1)} = \sinh\varphi$, $(\varphi \geqslant 0)$. Then we have $x + \sqrt{(x^2-1)} = \cosh\varphi + \sinh\varphi = e^\varphi$, and hence

$$\operatorname{arcosh} x = \varphi = \log_e\{x + \sqrt{(x^2-1)}\}. \quad \text{.....................(27)}$$

Since $\{x + \sqrt{(x^2-1)}\}\{x - \sqrt{(x^2-1)}\} = 1$, we can write

$$\operatorname{arcosh} x = -\log_e\{x - \sqrt{(x^2-1)}\}. \quad \text{.......................(28)}$$

(*b*) Without any restriction on the value of x, we can put $x = \sinh\varphi$, $\sqrt{(x^2+1)} = \cosh\varphi$. Then $x + \sqrt{(x^2+1)} = \sinh\varphi + \cosh\varphi = e^\varphi$, and hence

$$\operatorname{arsinh} x = \varphi = \log_e\{x + \sqrt{(x^2+1)}\}. \quad \text{.....................(29)}$$

Since $\{\sqrt{(x^2+1)} + x\}\{\sqrt{(x^2+1)} - x\} = 1$, we can write

$$\operatorname{arsinh} x = -\log_e\{\sqrt{(x^2+1)} - x\}. \quad \text{.......................(30)}$$

(*c*) If $|x| < 1$ we can put $x = \tanh\varphi$. Then

$$\frac{1+x}{1-x} = \frac{1+\tanh\varphi}{1-\tanh\varphi} = \frac{\cosh\varphi + \sinh\varphi}{\cosh\varphi - \sinh\varphi} = \frac{e^\varphi}{e^{-\varphi}} = e^{2\varphi}, \quad \text{..............(31)}$$

and hence

$$\begin{aligned} \operatorname{artanh} x = \varphi &= \tfrac{1}{2}\log_e\frac{1+x}{1-x} && \text{..........................(32)} \\ &= \tfrac{1}{2}\log_e(1+x) - \tfrac{1}{2}\log_e(1-x). && \text{..........(33)} \end{aligned}$$

[1] The student will recall that the function $\operatorname{arcosh} x$ is defined only when $x \geqslant 1$.

(*d*) Similarly, if $|x| > 1$ we can put $x = \coth\varphi$. Then $(x+1)/(x-1) = e^{2\phi}$ and hence

$$\operatorname{arcoth} x = \varphi = \tfrac{1}{2}\log_e \frac{x+1}{x-1} \quad \text{.........................(34)}$$

$$= \tfrac{1}{2}\log_e |x+1| - \tfrac{1}{2}\log_e |x-1|. \quad \text{........(35)}$$

Ex. 1. Evaluate arsinh 1·694, given $\log_{10} 3{\cdot}661 = 0{\cdot}5636$. [1·298.]

Ex. 2. Obtain the following results, employing a table of common logarithms:

$$\operatorname{arcosh} 2{\cdot}461 = 1{\cdot}5496, \ \operatorname{arsinh} 0{\cdot}1 = 0{\cdot}0998, \ \operatorname{arsinh} 1 = 0{\cdot}8814,$$
$$\operatorname{arsinh} 2{\cdot}461 = 1{\cdot}6326, \ \operatorname{artanh} 0{\cdot}1 = 0{\cdot}1003, \ \operatorname{artanh} 0{\cdot}536 = 0{\cdot}5985,$$
$$\operatorname{arcoth} 1{\cdot}626 = 0{\cdot}7169.$$

Ex. 3. Establish the following results:

$$\operatorname{arcosh}(-x) = \log_e\{-x + \sqrt{(x^2-1)}\}, \quad (x \leqslant -1), \quad \text{...............(36)}$$
$$\operatorname{arcosh}|x| = \log_e\{|x| + \sqrt{(x^2-1)}\}, \quad (|x| \geqslant 1). \quad \text{.............(37)}$$

Ex. 4. Prove that $\operatorname{arsinh} x - \log_e x$ converges to $\log_e 2$ as $x \to \infty$.

We have

$$\operatorname{arsinh} x - \log_e x = \log_e\{x + \sqrt{(x^2+1)}\} - \log_e x$$
$$= \log_e \frac{x + \sqrt{(x^2+1)}}{x} = \log_e\left\{1 + \sqrt{\left(1 + \frac{1}{x^2}\right)}\right\},$$

and as $x \to \infty$, $1/x^2 \to 0$.

190. Derivatives of the Inverse Hyperbolic Functions.

We shew that for each function the derivative is an algebraic function of the independent variable, as in the case of the inverse trigonometrical functions and the logarithmic function. As usual, we make use of the rule connecting the derivatives of the direct and inverse functions, namely, $dy/dx = \dfrac{1}{dx/dy}$.

190. 1. *Derivative of* $\operatorname{arcosh} x$. Write $y = \operatorname{arcosh} x$, $(y \geqslant 0, x \geqslant 1)$. Then $x = \cosh y$, $dx/dy = \sinh y$, and therefore, $\sinh y$ being positive,

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\sinh y} = \frac{1}{\sqrt{(\cosh^2 y - 1)}} = \frac{1}{\sqrt{(x^2-1)}}.$$

We thus have the result

$$\frac{d \operatorname{arcosh} x}{dx} = \frac{1}{\sqrt{(x^2-1)}}, \quad \text{..........................(1)}$$

the derivative being positive throughout.

Ex. 1. *The function* $\operatorname{arcosh}\dfrac{x}{a}$. We must here have $x/a \geqslant 1$, and so x, a must have the same sign and $|x| \geqslant |a|$. To find the derivative, write

$y = \operatorname{arcosh} u$, where $u = x/a$, and employ the rule for the derivative of a function of a function. We find

$$\frac{d \operatorname{arcosh} \frac{x}{a}}{dx} = \pm \frac{1}{\sqrt{(x^2 - a^2)}}, \quad \text{.........................(2)}$$

where the positive (negative) sign is to be taken if a (and therefore x) is positive (negative).

190. 2. *Derivative of* arsinh x. Write $y = \operatorname{arsinh} x$, so that $x = \sinh y$, $dx/dy = \cosh y$. Then

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{(\sinh^2 y + 1)}} = \frac{1}{\sqrt{(x^2 + 1)}}.$$

We thus have the result

$$\frac{d \operatorname{arsinh} x}{dx} = \frac{1}{\sqrt{(x^2 + 1)}}. \quad \text{.........................(3)}$$

Ex. 2. Shew that $\dfrac{d}{dx}\left(\operatorname{arsinh} \dfrac{x}{a}\right) = \pm 1/\sqrt{(x^2 + a^2)}$ according as $a \gtrless 0$. ....(4)

190. 3. *Derivative of* artanh x. Writing $y = \operatorname{artanh} x$, $(|x| < 1)$, we have $x = \tanh y$, $dx/dy = \operatorname{sech}^2 y$, and hence

$$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y} = \frac{1}{1 - \tanh^2 y} = \frac{1}{1 - x^2},$$

or

$$\frac{d \operatorname{artanh} x}{dx} = \frac{1}{1 - x^2}. \quad \text{.............................(5)}$$

Ex. 3. Shew that $\dfrac{d}{dx}\left(\operatorname{artanh} \dfrac{x}{a}\right) = \dfrac{a}{a^2 - x^2}$. ....(6)

190. 4. *Derivatives of* arsech x, arcosech x, arcoth x. These are most simply obtained by making use of the relations (1), Art. 189, putting $1/x = u$, say, and employing the rule for the derivative of a function of a function. Thus, writing

$$y = \operatorname{arsech} x = \operatorname{arcosh} \frac{1}{x} = \operatorname{arcosh} u, \quad \text{where} \quad u = \frac{1}{x}, \quad (0 < x \leqslant 1),$$

we find

$$\frac{d \operatorname{arsech} x}{dx} = -\frac{1}{x\sqrt{(1 - x^2)}}. \quad \text{.........................(7)}$$

For the remaining functions we find

$$\frac{d \operatorname{arcosech} x}{dx} = \mp \frac{1}{x\sqrt{(x^2 + 1)}}, \quad \text{..................................(8)}$$

$$\frac{d \operatorname{arcoth} x}{dx} = -\frac{1}{x^2 - 1}, \quad (|x| > 1), \quad \text{......................(9)}$$

where in (8), the negative (positive) sign is to be taken if x is positive (negative).

Ex. 4. Shew that $\dfrac{d}{dx}(\operatorname{arsinh}^2 x)=\dfrac{2\operatorname{arsinh} x}{\sqrt{(x^2+1)}}$. [Write $y=u^2$, $u=\operatorname{arsinh} x$.]

Ex. 5. Shew that $\dfrac{d\operatorname{artanh}(\tan\frac{1}{2}x)}{dx}=\frac{1}{2}\sec x$. [Put $\tan\frac{1}{2}x=u$.]

191. Successive Derivatives of the Inverse Hyperbolic Functions.

As in the case of the inverse trigonometrical functions, there are no simple formulae for the nth derivatives of the functions $\operatorname{arcosh} x$, $\operatorname{arsinh} x$. The successive derivatives are thus to be found either by repeated differentiation, or by use of a recurrence formula.

For the function $y=\operatorname{arcosh} x$, $(x\geqslant 1)$, we have

$$y^{(1)}=1/(x^2-1)^{1/2},\quad y^{(2)}=-x/(x^2-1)^{3/2}, \quad\text{.............(1)}$$

which give

$$(x^2-1)y^{(2)}+xy^{(1)}=0. \quad\text{...........................(2)}$$

Applying Leibniz's Theorem to this equation, we obtain

$$(x^2-1)y^{(n+2)}+x(2n+1)y^{(n+1)}+n^2y^{(n)}=0, \quad\text{.............(3)}$$

which gives the derivative of order $n+2$ linearly in terms of those of orders $n+1$ and n. The successive derivatives can now be found by a step-by-step process, employing the results in (1).

For the function $y=\operatorname{arsinh} x$, we find, similarly,

$$y^{(1)}=1/(x^2+1)^{1/2},\quad y^{(2)}=-x/(x^2+1)^{3/2}, \quad\text{..............(4)}$$

$$(1+x^2)y^{(2)}+xy^{(1)}=0, \quad\text{.........................(5)}$$

and the recurrence formula is

$$(1+x^2)y^{(n+2)}+x(2n+1)y^{(n+1)}+n^2y^{(n)}=0. \quad\text{..............(6)}$$

For the function $y=\operatorname{artanh} x$, we have

$$y^{(1)}=\frac{1}{1-x^2}=\frac{1}{2}\left(\frac{1}{1+x}+\frac{1}{1-x}\right), \quad\text{.....................(7)}$$

and hence, referring to Ex. 2, Art. 86, we can write

$$y^{(n)}=\tfrac{1}{2}(n-1)!\left\{\frac{(-1)^{n-1}}{(1+x)^n}+\frac{1}{(1-x)^n}\right\}. \quad\text{................(8)}$$

Ex. 1. Shew that the Taylor polynomial of the $(2n+1)$th degree for $\operatorname{arsinh} x$ is

$$x-\frac{x^3}{3!}+3^2\frac{x^5}{5!}-3^25^2\frac{x^7}{7!}+\ldots+(-1)^n3^25^2\ldots(2n-1)^2\frac{x^{2n+1}}{(2n+1)!}. \quad\text{........(9)}$$

The conditions for the application of the $I.D_n.T.$ to arsinh x with starting point $x=0$ are satisfied. Putting $x=0$ in (6), we get the formula

$$y_0^{(n+2)}+n^2y_0^{(n)}=0,$$

which, together with the results $y_0^{(1)}=1$, $y_0^{(2)}=0$, shews that all the even derivatives vanish when $x=0$, and that

$$y_0^{(2n+1)}=(-1)^n 3^2 5^2 \dots (2n-1)^2. \quad \text{.......................(10)}$$

This leads to the polynomial form given. It can be proved, moreover, that when x lies in the range $[-1, 1]$, arsinh x can be expanded in a convergent Taylor infinite series, of which the first $n+1$ terms constitute the above polynomial.

Ex. 2. Shew that the Taylor Series for artanh x, $(|x|<1)$, is

$$x+\frac{x^3}{3}+\frac{x^5}{5}+\dots+\frac{x^{2n-1}}{2^{n-1}}+\dots . \quad \text{..........................(11)}$$

[This follows from the result (10) of Art. 182, observing that artanh $x=\frac{1}{2}\log_e\{(1+x)/(1-x)\}$.]

SECTION V. THE EPICENE FUNCTIONS

192. Definition and Fundamental Properties of the Epicene Functions.

These functions have been referred to in Section II above when considering the fundamental properties of the hyperbolic functions. The results established in that section exhibit a close correspondence with well-known formulae of Trigonometry. With the aid of the *Epicene Functions*, which are functions of a real variable, we can discuss this correspondence systematically and in a very simple, direct manner, thereby avoiding the introduction of complex numbers in the treatment of the elementary theory of functions of real variables.

It is convenient, at the present stage, to define the epicene functions and to investigate their fundamental properties. Some of these properties are utilized in Chap. XX when dealing with the subject of *Linear Differential Equations of the Second Order.* There are two primary epicene functions, which are denoted by $\mathfrak{C}(x)$, $\mathfrak{S}(x)$ when the independent variable is x, and which are defined by the equations

$$\mathfrak{C}(x)=\begin{cases}\cosh\sqrt{x}, & (x>0),\\ \cos\sqrt{(-x)}, & (x<0),\end{cases}\qquad \mathfrak{S}(x)=\begin{cases}\dfrac{1}{\sqrt{x}}\sinh\sqrt{x}, & (x>0),\\[2ex] \dfrac{1}{\sqrt{(-x)}}\sin\sqrt{(-x)}, & (x<0).\end{cases}$$

A more natural definition of the functions is in terms of convergent

infinite series. If we introduce the Taylor infinite series for the above hyperbolic and trigonometrical functions, we readily obtain, as the infinite series for $\mathfrak{C}(x)$, $\mathfrak{S}(x)$,

$$\left.\begin{aligned}\mathfrak{C}(x)&=1+\frac{1}{2!}x+\frac{1}{4!}x^2+\dots+\frac{1}{(2n)!}x^n+\dots,\\ \mathfrak{S}(x)&=1+\frac{1}{3!}x+\frac{1}{5!}x^2+\dots+\frac{1}{(2n+1)!}x^n+\dots.\end{aligned}\right\}$$

We may take these equations as *defining* the functions in question, but as the theory of Power Series has not been considered, the investigation of the properties of the functions from these definitions must be postponed.

The graphs of the functions[1] $5\mathfrak{C}(x)$, $5\mathfrak{S}(x)$ are shewn in Figs. 156, 157, the change of the functions from the trigonometrical to the hyperbolic type when x crosses the origin from the negative to the positive side being clearly seen.

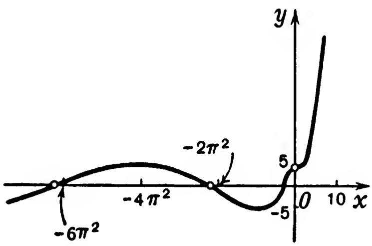

FIG. 156.
$y=5\mathfrak{C}(x)$.

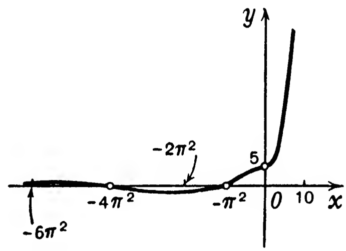

FIG. 157.
$y=5\mathfrak{S}(x)$.

The following properties follow immediately from the definitions of the functions :

(i) $\mathfrak{C}(0)=1$; $\mathfrak{S}(0)=1$. ....(1)

(ii) $\mathfrak{C}(x)\to\infty$ as $x\to\infty$; $\mathfrak{S}(x)\to\infty$ as $x\to\infty$. ....(2)

$$\left.\begin{aligned}&\text{(iii) } \mathfrak{C}(x)=0 \text{ when } \sqrt{(-x)}=\frac{2n+1}{2}\pi,\\ &\qquad\qquad\text{that is, when } x=-\left(\frac{2n+1}{2}\right)^2\pi^2;\\ &\mathfrak{S}(x)=0 \text{ when } \sqrt{(-x)}=n\pi, \text{ that is, when } x=-n^2\pi^2.\end{aligned}\right\}\ \dots\dots(3)$$

(iv) $|\mathfrak{C}(x)|=1$ when $x=-n^2\pi^2$; $\mathfrak{S}(x)\to 0$ when $x\to-\infty$. .....(4)

Some of these results are illustrated in the above Figures.

[1] The factor 5 is introduced in each case to give a sufficiently open scale along Oy.

(v) The fundamental relation between the functions is expressed by[1]

$$\mathfrak{C}^2(x) - x\mathfrak{S}^2(x) = 1. \quad \text{..........................(5)}$$

(vi) For the derivatives we have

$$\frac{d\mathfrak{C}(x)}{dx} = \frac{1}{2}\mathfrak{S}(x); \quad \frac{d\mathfrak{S}(x)}{dx} = \frac{1}{2x}\{\mathfrak{C}(x) - \mathfrak{S}(x)\}. \quad \text{...........(6)}$$

Ex. 1. Prove the result

$$\frac{d}{dx}\{\sqrt{x}\,.\,\mathfrak{S}(x)\} = \frac{1}{2\sqrt{x}}\mathfrak{C}(x). \quad \text{..............................(7)}$$

192. 1. *Introduction of special independent variable.* The epicene functions obtained by replacing the argument x by hx^2, where h is a constant, are of particular importance. We have the definitions

$$\left.\begin{aligned}\mathfrak{C}(hx^2) &= \begin{cases}\cosh(\sqrt{h}x), & (h > 0),\\ \cos(\sqrt{-h}x), & (h < 0),\end{cases}\\ \mathfrak{S}(hx^2) &= \begin{cases}\dfrac{1}{\sqrt{h}x}\sinh(\sqrt{h}x), & (h > 0),\\ \dfrac{1}{\sqrt{-h}x}\sin(\sqrt{-h}x), & (h < 0),\end{cases}\end{aligned}\right\} \quad \text{..............(8)}$$

and the corresponding infinite series are given by

$$\left.\begin{aligned}\mathfrak{C}(hx^2) &= 1 + \frac{1}{2!}hx^2 + \frac{1}{4!}h^2x^4 + \ldots + \frac{1}{(2n)!}h^nx^{2n} + \ldots,\\ \mathfrak{S}(hx^2) &= 1 + \frac{1}{3!}hx^2 + \frac{1}{5!}h^2x^4 + \ldots + \frac{1}{(2n+1)!}h^nx^{2n} + \ldots.\end{aligned}\right\} \quad \text{.....(9)}$$

We now have the following results :

(i) The fundamental relation between the functions $\mathfrak{C}(hx^2)$, $\mathfrak{S}(hx^2)$ is expressed by

$$\mathfrak{C}^2(hx^2) - hx^2\mathfrak{S}^2(hx^2) = 1. \quad \text{....................(10)}$$

(ii) The Addition Theorems for the functions are expressed by

$$\mathfrak{C}\{h(x_1 \pm x_2)^2\} = \mathfrak{C}(hx_1^2)\mathfrak{C}(hx_2^2) \pm hx_1x_2\mathfrak{S}(hx_1^2)\mathfrak{S}(hx_2^2), \quad \text{......(11)}$$

$$(x_1 \pm x_2)\mathfrak{S}\{h(x_1 \pm x_2)^2\} = x_1\mathfrak{S}(hx_1^2)\mathfrak{C}(hx_2^2) \pm x_2\mathfrak{S}(hx_2^2)\mathfrak{C}(hx_1^2), \quad \text{...(12)}$$

where the upper (lower) signs on the right-hand sides must be taken with the upper (lower) signs on the left-hand sides.

(iii) From (11), (12), we have, as particular cases,

$$\mathfrak{C}(4hx^2) = \mathfrak{C}^2(hx^2) + hx^2\mathfrak{S}^2(hx^2), \quad \text{................(13)}$$

$$\mathfrak{S}(4hx^2) = \mathfrak{S}(hx^2)\mathfrak{C}(hx^2). \quad \text{..........................(14)}$$

[1] As in the case of the trigonometrical and hyperbolic functions, $\mathfrak{C}^2(x)$ means the square of $\mathfrak{C}(x)$, etc.

(iv) With the aid of (10), the result (13) leads to the relations

$$\mathfrak{C}^2(hx^2)=\tfrac{1}{2}\{\mathfrak{C}(4hx^2)+1\}, \quad\text{.......................(15)}$$

$$hx^2\mathfrak{S}^2(hx^2)=\tfrac{1}{2}\{\mathfrak{C}(4hx^2)-1\}. \quad\text{.......................(16)}$$

(v) For the first derivatives of the functions $\mathfrak{C}(hx^2)$, $x\mathfrak{S}(hx^2)$, we find

$$\frac{d}{dx}\mathfrak{C}(hx^2)=hx\mathfrak{S}(hx^2),\quad \frac{d}{dx}\{x\mathfrak{S}(hx^2)\}=\mathfrak{C}(hx^2), \quad\text{.........(17)}$$

and, for the second derivatives,

$$\frac{d^2}{dx^2}\mathfrak{C}(hx^2)=h\mathfrak{C}(hx^2),\quad \frac{d^2}{dx^2}\{x\mathfrak{S}(hx^2)\}=hx\mathfrak{S}(hx^2). \quad\text{......(18)}$$

It follows from the last two results that the equation

$$\frac{d^2y}{dx^2}-hy=0 \quad\text{.............................(19)}$$

is satisfied when y is replaced by either of the functions $\mathfrak{C}(hx^2)$, $x\mathfrak{S}(hx^2)$. This important result is employed subsequently in Art. 358. 2.

Ex. 2. Establish the following results :

(i) $\mathfrak{C}(4x)=\mathfrak{C}^2(x)+x\mathfrak{S}^2(x)$, ...(20)

(ii) $\mathfrak{S}(4x)=\mathfrak{S}(x)\mathfrak{C}(x)$, ...(21)

(iii) $\mathfrak{C}^2(x)=\tfrac{1}{2}\{\mathfrak{C}(4x)+1\}$, ...(22)

(iv) $x\mathfrak{S}^2(x)=\tfrac{1}{2}\{\mathfrak{C}(4x)-1\}$. ....(23)

Ex. 3. Prove the results :

(i) $x\dfrac{d}{dx}\mathfrak{S}(hx^2)=\mathfrak{C}(hx^2)-\mathfrak{S}(hx^2)$, ...(24)

(ii) $hx^3\dfrac{d}{dx}\mathfrak{S}^2(hx^2)=2hx^2\mathfrak{S}(4hx^2)-\mathfrak{C}(4hx^2)+1$, ...(25)

(iii) $\dfrac{d}{dx}\mathfrak{C}^2(hx^2)=2hx\mathfrak{S}(4hx^2)$. ....(26)

SECTION VI. THE EXPOCYCLIC FUNCTIONS

193. Definitions of the Functions.

Products of exponential and sinoidal functions of the form

$$e^{mx}\sin(nx+\alpha), \quad\text{............................(1)}$$

where m, n, α are constants, are of such frequent occurrence in Analysis that it is convenient to standardize them as fundamental functions [1] and to introduce a special notation for them. These

[1] In the theory of *complex numbers* these functions appear in the generalization of the exponential and trigonometrical functions.

functions will be used continually in the sequel, especially when dealing with certain types of integrals and differential equations, and also in the investigation of fundamental series for functions of the types $e^x \sin x$, $\sinh x \cos x$.

The standard forms of the functions are led up to by observing that the coefficients m, n can be expressed in the forms

$$m = k\cos c, \quad n = k\sin c, \quad \text{......................(2)}$$

where k, c are constants such that

$$k^2 = m^2 + n^2, \quad \tan c = n/m. \quad \text{......................(3)}$$

We can take n to be positive in the expression (1) by adjustment of α, and the above relations can then always be satisfied by taking k positive and c in the range $(0, \pi)$; for this choice imposes no restriction on the value of $k\cos c$, or m, and keeps $k\sin c$, or n, positive. We thus arrive at the standard function

$$e^{kx\cos c}\sin(kx\sin c + \alpha). \quad \text{........................(4)}$$

The constant α, which is the phase-constant of the sinoidal term in this product, is also called the phase-constant of the function.

All the functions of the kind here considered are included in the form (4), but, as in the case of the trigonometrical functions, it is convenient to introduce a cosine function

$$e^{kx\cos c}\cos(kx\sin c + \alpha). \quad \text{........................(5)}$$

The functions (4), (5) are, from their form, called *expocyclic functions*, and the special notations introduced for them are combinations of the functional symbols for the exponential and trigonometrical (circular) factors. Thus we write

$$\operatorname{exsin}_c(kx, \alpha), \quad \operatorname{excos}_c(kx, \alpha)$$

for the respective functions (4), (5), so that

$$\operatorname{exsin}_c(kx, \alpha) \equiv e^{kx\cos c}\sin(kx\sin c + \alpha), \quad \text{.................(6)}$$

$$\operatorname{excos}_c(kx, \alpha) \equiv e^{kx\cos c}\cos(kx\sin c + \alpha). \quad \text{.................(7)}$$

If the phase-constant α vanishes, the notations for the functions become $\operatorname{exsin}_c(kx)$, $\operatorname{excos}_c(kx)$, simply.

The inclusion of the exponential and the trigonometrical functions among the expocyclic functions is expressed by the equations

$$e^{kx} = \operatorname{excos}_0(kx), \quad \sin kx = \operatorname{exsin}_{\frac{1}{2}\pi}(kx), \quad \cos kx = \operatorname{excos}_{\frac{1}{2}\pi}(kx). \quad (8)$$

The verification of these results is left to the student.

Expocyclic functions for which the parameter c lies in the range

$(\frac{1}{2}\pi, \pi)$, (so that $k \cos c < 0$), can, when desired, be expressed in terms of the corresponding functions for values of c lying in the range $(0, \frac{1}{2}\pi)$. Write $c = \pi - c'$, so that c' lies in the range $(0, \frac{1}{2}\pi)$. Then $\cos c = -\cos c'$, $\sin c = \sin c'$, and we have

$$\begin{aligned} \operatorname{exsin}_c(kx, \alpha) &= e^{kx\cos c}\sin(kx\sin c + \alpha) \\ &= e^{-kx\cos c'}\sin(kx\sin c' + \alpha) \\ &= -e^{-kx\cos c'}\sin(-kx\sin c' - \alpha) \\ &= -\operatorname{exsin}_{c'}(-kx, -\alpha). \qquad (9) \end{aligned}$$

Similarly we find

$$\operatorname{excos}_c(kx, \alpha) = \operatorname{excos}_{c'}(-kx, -\alpha). \qquad (10)$$

Ex. 1. Express $e^x \sin x$ in expocyclic form.

Writing $e^x \sin x$ in the form (6), we find

$$k\cos c = 1, \quad k \sin c = 1, \quad \alpha = 0,$$

so that $k = \sqrt{2}$, $c = \frac{1}{4}\pi$. Hence

$$e^x \sin x = \operatorname{exsin}_{\frac{1}{4}\pi}(\sqrt{2}x). \qquad (11)$$

Ex. 2. Show that

$$e^{-x}\sin x = \operatorname{exsin}_{\frac{3}{4}\pi}(\sqrt{2}x) = -\operatorname{exsin}_{\frac{1}{4}\pi}(-\sqrt{2}x). \qquad (12)$$

194. Properties and Derivatives of the Expocyclic Functions.

In investigating the general properties of the expocyclic functions it is sufficient to consider the sine function $y = \operatorname{exsin}_c(kx, \alpha)$, for this includes the cosine function by suitably adjusting the value of α; thus $\operatorname{excos}_c(kx, \alpha) = \operatorname{exsin}_c(kx, \alpha + \frac{1}{2}\pi)$.

The function y has a certain quasi-periodic character, for if we increase x by $\pi/(k \sin c)$, the value of y is simply multiplied by the constant factor $-e^{\pi \cot c}$.

A characteristic portion of the graph of $y = \operatorname{exsin}_c(kx, \alpha)$ for a negative value of $\cos c$ is shewn in Fig. 158, p. 453. The same figure obviously serves as a representation of the function for which $\cos c$ is positive merely by reversing the direction of the x-axis.

From the physical point of view, the graph may be regarded as representing an infinite train of waves with continually increasing (or decreasing) maximum displacement, or amplitude, the same general form repeating itself after the equal intervals $2\pi/(k \sin c)$, which we call the wave-length of the graph or of the function. The factor $e^{\pi \cot c}$ is called the *factor of growth* or the *factor of decay*, according as $\cot c$ (and hence also $\cos c$) is positive or negative. In the former case the amplitude continually increases and in the latter it continually decreases as x increases.

Ex. 1. Explain how the graph of the function $\operatorname{exsin}_c(kx, \alpha)$ may be employed as a representation of the function $\operatorname{excos}_c(kx, \alpha)$.

[The origin is to be transferred to the point $x = \pi/(2k \sin c)$, $y = 0$, and the scale along Oy increased in the ratio $e^{\frac{1}{2}\pi \cot c} : 1$.]

As in the case of the trigonometrical functions, we have Addition Theorems for the expocyclic functions. Some of these are set as exercises below; see Examples XXXVI, p. 467.

Passing now to the differentiation of the functions, we have

$$\begin{aligned} f(x) &= e^{kx \cos c} \sin(kx \sin c + \alpha), \\ f'(x) &= ke^{kx \cos c} \{\cos c \sin(kx \sin c + \alpha) + \sin c \cos(kx \sin c + \alpha)\} \\ &= ke^{kx \cos c} \sin(kx \sin c + \alpha + c) \\ &= k \operatorname{exsin}_c(kx, \alpha + c). \qquad \ldots\ldots(1) \end{aligned}$$

Stated in the form of a rule, the result is that *the derivative is formed by multiplying the function by k and increasing the phase-constant by c.*

The same rule applies at each successive differentiation, and the general derivative is thus given by

$$f^{(n)}(x) = k^n \operatorname{exsin}_c(kx, \alpha + nc). \qquad \ldots\ldots(2)$$

The case of the expocyclic cosine is included in this merely by replacing α by $\alpha + \frac{1}{2}\pi$. The general formula is found to be

$$\frac{d^n}{dx^n} \operatorname{excos}_c(kx, \alpha) = k^n \operatorname{excos}_c(kx, \alpha + nc). \qquad \ldots\ldots(3)$$

Ex. 2. Find the nth derivative of $e^x \sin x$.

Expressed in expocyclic form, this function becomes

$$y = \operatorname{exsin}_{\frac{1}{4}\pi}(\sqrt{2}x),$$

as in Ex. 1, Art. 193. Therefore we have

$$y^{(n)} = 2^{n/2} \operatorname{exsin}_{\frac{1}{4}\pi}(\sqrt{2}x, \tfrac{1}{4}n\pi). \qquad \ldots\ldots(4)$$

When $n = 1$, this reduces to $dy/dx = e^x (\sin x + \cos x)$.

Ex. 3. *The function* $e^{mx} \sin nx$, $(n > 0)$. Here we have, as on p. 447, $k = \sqrt{(m^2 + n^2)}$, $\tan c = n/m$, and the first derivative is given by

$$\frac{dy}{dx} = \sqrt{(m^2 + n^2)} e^{mx} \sin(nx + c),$$

which reduces to $e^{mx}(m \sin nx + n \cos nx)$, as in Ex. 6, Art. 161, p. 385.

195. Application of Taylor's Theorem to the Expocyclic Functions.

When $f(x) = \operatorname{exsin}_c(kx, \alpha)$, we have

$$f(0) = \operatorname{exsin}_c(0, \alpha) = \sin \alpha, \quad f^{(r)}(0) = k^r \operatorname{exsin}_c(0, \alpha + rc) = k^r \sin(\alpha + rc),$$

and therefore, by the $I.D_n.T'.$,

$$\operatorname{exsin}_c (kx, \alpha) = \sin\alpha + kx\sin(\alpha + c) + \frac{k^2x^2}{2!}\sin(\alpha+2c) + \dots$$
$$+ \frac{k^{n-1}x^{n-1}}{(n-1)!}\sin\{\alpha + (n-1)c\} + R_n(x), \quad \dots\dots\dots(1)$$

where $$R_n(x) = \frac{x^n}{n!}f^{(n)}(\theta x) = \frac{k^n x^n}{n!}\operatorname{exsin}_c(k\theta x, \alpha + nc)$$
$$= \frac{k^n x^n}{n!} e^{k\theta x \cos c}\sin(k\theta x\sin c + \alpha + nc). \quad \dots\dots(2)$$

Now $|\sin(k\theta x\sin c + \alpha + nc)| \leqslant 1$, and hence

$$|R_n(x)| \leqslant \frac{|kx|^n}{n!}e^{k\theta x\cos c} < \frac{|kx|^n}{n!}e^{|kx\cos c|}. \quad \dots\dots\dots\dots\dots(3)$$

The term $e^{|kx\cos c|}$ is independent of n, and since we can make $|kx|^n/n!$ as small as we please by taking n sufficiently large, it follows that $|R_n(x)| \to 0$ as $n \to \infty$. The conditions for Taylor's infinite series are therefore satisfied and we have, as the expression of Taylor's Theorem,

$$\operatorname{exsin}_c(kx, \alpha) = \sin\alpha + kx\sin(\alpha + c) + \dots + \frac{k^n x^n}{n!}\sin(\alpha + nc) + \dots . \quad \dots.(4)$$

In the case of the expocyclic cosine, Taylor's Theorem becomes

$$\operatorname{excos}_c(kx, \alpha) = \cos\alpha + kx\cos(\alpha + c) + \dots + \frac{k^n x^n}{n!}\cos(\alpha + nc) + \dots . \quad \dots.(5)$$

195. 1. *Series for $e^x \sin x$.* As in Ex. 1, p. 448, the expocyclic constants are given by $k = \sqrt{2}$, $c = \frac{1}{4}\pi$, $\alpha = 0$. The general term in the expansion is therefore $2^{n/2}\frac{x^n}{n!}\sin\frac{1}{4}n\pi$. The terms corresponding to values of n which are multiples of 4 thus vanish. We find

$$e^x \sin x = x + 2\frac{x^2}{2!} + 2\frac{x^3}{3!} - 2^2\frac{x^5}{5!} - 2^3\frac{x^6}{6!} - 2^3\frac{x^7}{7!} + \dots + 2^{n/2}\frac{x^n}{n!}\sin\tfrac{1}{4}n\pi + \dots . \quad \dots(6)$$

195. 2. *Series for $e^{-x}\sin x$.* Since $e^{-x}\sin x = -e^{-x}\sin(-x)$, the required series can be obtained from (6) by writing $-x$ for x and changing the sign of each term. We find

$$e^{-x}\sin x = x - 2\frac{x^2}{2!} + 2\frac{x^3}{3!} - 2^2\frac{x^5}{5!} + \dots + (-1)^{n+1}2^{n/2}\frac{x^n}{n!}\sin\tfrac{1}{4}n\pi + \dots . \quad \dots\dots(7)$$

The same result is obtained by the direct application of (4), where here we have $k = \sqrt{2}$, $c = \frac{3}{4}\pi$, as in Ex. 2, p. 448. The general term is found to be $2^{n/2}\frac{x^n}{n!}\sin\frac{3}{4}n\pi$, which we identify with the general term in (7) by observing that

$$\sin\tfrac{3}{4}n\pi = \sin(n\pi - \tfrac{1}{4}n\pi) = -\cos n\pi\sin\tfrac{1}{4}n\pi = (-1)^{n+1}\sin\tfrac{1}{4}n\pi.$$

195. 3. *Series for* $e^x \cos x$. Here we have $k=\sqrt{2}$, $c=\frac{1}{4}\pi$, $\alpha=0$, and (5) becomes

$$e^x \cos x = 1 + x - 2\frac{x^3}{3!} - 2^2\frac{x^4}{4!} - 2^2\frac{x^5}{5!} + 2^3\frac{x^7}{7!} + 2^4\frac{x^8}{8!} - \ldots + 2^{n/2}\frac{x^n}{n!}\cos\tfrac{1}{4}n\pi + \ldots \ldots (8)$$

195. 4. *Series for* $e^{-x}\cos x$. This is obtained from (8) merely by changing the sign of x throughout. We get

$$e^{-x}\cos x = 1 - x + 2\frac{x^3}{3!} - 2^2\frac{x^4}{4!} + \ldots + (-1)^n 2^{n/2}\frac{x^n}{n!}\cos\tfrac{1}{4}n\pi + \ldots \quad \ldots\ldots (9)$$

195. 5. *Series for* $\cosh x \sin x$, $\sinh x \sin x$, $\cosh x \cos x$, $\sinh x \cos x$. Series for these functions may be found by combinations of the above series using the equalities

$$\cosh x \sin x = \tfrac{1}{2}(e^x \sin x + e^{-x}\sin x), \quad \sinh x \sin x = \tfrac{1}{2}(e^x \sin x - e^{-x}\sin x),$$

and so on. The justification for the addition of infinite series in this manner is given in Art. 58. 1.

Thus we have, from (6), (7),

$$\cosh x \sin x = x + 2\frac{x^3}{3!} - 2^2\frac{x^5}{5!} - 2^3\frac{x^7}{7!} + 2^4\frac{x^9}{9!} + \ldots + (-1)^{[\frac{1}{2}(n-1)]}2^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \ldots, (10)$$

where $[\frac{1}{2}(n-1)]$ means the integral part of $\frac{1}{2}(n-1)$.

Similarly we find

$$\sinh x \sin x = 2\frac{x^2}{2!} - 2^3\frac{x^6}{6!} + 2^5\frac{x^{10}}{10!} - 2^7\frac{x^{14}}{14!} + \ldots + (-1)^{n-1}2^{2n-1}\frac{x^{4n-2}}{(4n-2)!} + \ldots, \quad (11)$$

$$\cosh x \cos x = 1 - 2^2\frac{x^4}{4!} + 2^4\frac{x^8}{8!} - 2^6\frac{x^{12}}{12!} + \ldots + (-1)^{n-1}2^{2n-2}\frac{x^{4n-4}}{(4n-4)!} + \ldots, \quad \ldots\ldots (12)$$

$$\sinh x \cos x = x - 2\frac{x^3}{3!} - 2^2\frac{x^5}{5!} + 2^3\frac{x^7}{7!} + \ldots + (-1)^{[\frac{1}{2}n]}2^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \ldots, \quad \ldots\ldots\ldots (13)$$

where, as before, $[\frac{1}{2}n]$ means the integral part of $\frac{1}{2}n$.

The values of the above functions are tabulated in parallel columns in Hayashi's tables to eight decimal places, the values of x ranging from 0 to 10. The student should verify the following results :

$$\cosh 1 \cos 1 = 0{\cdot}8337\ 3003, \quad \cosh 1 \sin 1 = 1{\cdot}2984\ 5758,$$
$$\sinh 1 \sin 1 = 0{\cdot}9888\ 9771, \quad \sinh 1 \cos 1 = 0{\cdot}6349\ 6391.$$

196. Further Properties and Graph of $e^{-kx\cos c}\sin(kx\sin c + \alpha)$.

We consider in detail the function $e^{-kx\cos c}\sin(kx\sin c + \alpha)$, where $k\cos c > 0$. This function, and the corresponding expocyclic cosine, are of special importance in Mechanics and Physics in connection with vibrations in which the amplitude continually diminishes (damped vibrations). This is referred to again in Chap. XX when dealing with *Differential Equations of the Second Order.*

Denoting the function by $f(x)$, we have, as in Art. 193,

$$f(x) = e^{-kx\cos c}\sin(kx\sin c + \alpha) = -\text{exsin}_c(-kx, -\alpha). \ldots\ldots\ldots (1)$$

(i) $f(x)=0$ when $\sin(kx\sin c+\alpha)=0$, i.e. when $kx\sin c+\alpha=n\pi$, so that the zeros of the function are given by

$$x=\frac{n\pi-\alpha}{k\sin c}. \qquad \text{(2)}$$

(ii) Since $|\sin(kx\sin c+\alpha)|\leqslant 1$, the graph of (1) must lie between the curves $y=\pm e^{-kx\cos c}$, and it meets them where $|\sin(kx\sin c+\alpha)|=1$, that is, where

$$x=\frac{\frac{2n+1}{2}\pi-\alpha}{k\sin c}=\frac{n\pi-\alpha}{k\sin c}+\frac{\frac{1}{2}\pi}{k\sin c}. \qquad \text{(3)}$$

These points are thus mid-way between the zeros of $f(x)$.

(iii) For the bounding curves we have

$$\frac{dy}{dx}=\mp k\cos c\,.\,e^{-kx\cos c}, \qquad \text{(4)}$$

while for the given function

$$f'(x)=k\,\mathrm{exsin}_c(-kx,\ -\alpha+c)=-ke^{-kx\cos c}\sin(kx\sin c+\alpha-c). \qquad \text{(5)}$$

At a point given by (3), this becomes

$$f'(x)=-ke^{-kx\cos c}\sin\left(\frac{2n+1}{2}\pi-c\right)=\mp ke^{-kx\cos c}\cos c, \qquad \text{(6}$$

the negative (positive) sign being taken if n is even (odd). The gradients are thus equal at the points of contact, so that the graph of (1) touches the bounding curves, which are described as *enveloping* the graph.

(iv) From (5) we have $f'(x)=0$ when $kx\sin c+\alpha-c=n\pi$, i.e. when

$$x=\frac{n\pi-\alpha}{k\sin c}+\frac{c}{k\sin c}. \qquad \text{(7)}$$

Comparing (2), (3), (7), we see that a stationary value occurs for a value of x lying between a zero and the abscissa of the next point of contact on the right with an enveloping curve.

(v) We have

$$f''(x)=-k^2\mathrm{exsin}_c(-kx,\ -\alpha+2c)=k^2e^{-kx\cos c}\sin(kx\sin c+\alpha-2c), \quad \text{(8)}$$

and employing the result (5), it is readily shewn that

$$f''(x)+2k\cos c\,.\,f'(x)+k^2f(x)=0. \qquad \text{(9)}$$

When $f'(x)=0$, this shews that $f''(x)$ has the opposite sign to $f(x)$ and thus the stationary values are maxima (minima) when $f(x)$ is positive (negative). The abscissae of the maxima (minima) are

given by (7) when n is even (odd). Since when $kx \sin c + \alpha - c = n\pi$, $\sin(kx \sin c + \alpha)$ is equal to $\sin c$ or to $-\sin c$ according as n is even or odd, it follows that $f(x)$ is equal to $e^{-kx \cos c} \sin c$ at the upper extremes, and to $-e^{-kx \cos c} \sin c$ at the lower extremes. Thus the upper extreme points of the curve $y = f(x)$ lie on the curve $y = e^{-kx \cos c} \sin c$ and the lower extreme points on the curve $y = -e^{-kx \cos c} \sin c$.

(vi) The ratio of successive amplitudes is constant throughout, and is equal to $e^{-\pi \cot c}$. This is the factor of decay (p. 448). The logarithm of its reciprocal λ, say, is given by

$$\log_e \lambda = \pi \cot c, \qquad (10)$$

and this is called the *logarithmic decrement* of the oscillation.

(vii) The abscissae of the points of contraflexure are found from (6) to be given by

$$x = \frac{n\pi - \alpha}{k \sin c} + \frac{2c}{k \sin c}. \qquad (11)$$

It is now seen that an extreme lies mid-way between a zero and the next point of contraflexure to the right of it.

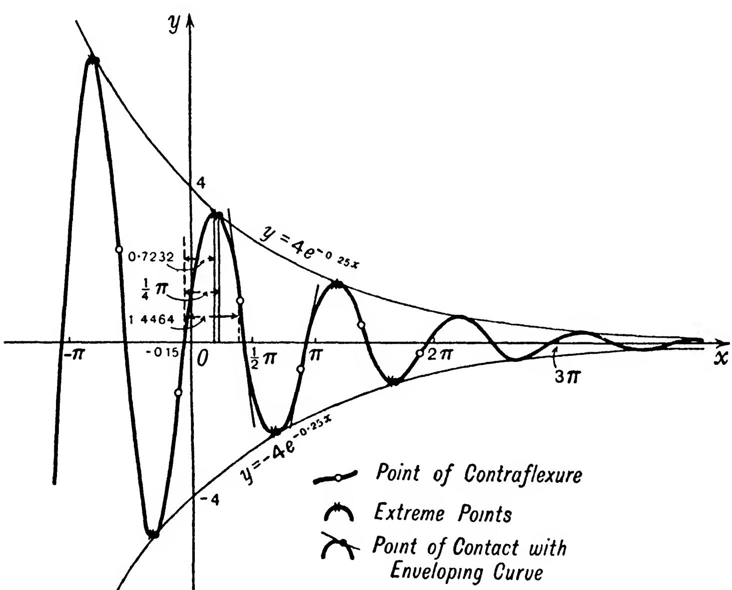

FIG. 158.
$y = 4e^{-0.25x} \sin(2x + 0{\cdot}3)$.

Some of the above features are illustrated in Fig. 158, which

represents the function $y = 4e^{-0\cdot25x} \sin(2x + 0\cdot3)$ in the range $(-4, 12)$. We have the following results :

(a) $k = \frac{1}{4}\sqrt{65}$, $c = \operatorname{artan} 8 = 1\cdot4464$, $\alpha = 0\cdot3$.

(b) Abscissae of points of intersection with Ox : $x = \frac{1}{2}n\pi - 0\cdot15$.

(c) Equations of enveloping curves : $y = \pm 4e^{-0\cdot25x}$.

(d) Abscissae of points of contact with enveloping curves : $x = \frac{1}{2}n\pi - 0\cdot15 + \frac{1}{4}\pi$.

(e) Abscissae of extreme points : $x = \frac{1}{2}n\pi - 0\cdot15 + \frac{1}{2} \operatorname{artan} 8$.

(f) Abscissae of points of contraflexure : $x = \frac{1}{2}n\pi - 0\cdot15 + \operatorname{artan} 8$.

Ex. 1. Shew that the equation (9) is satisfied when $f(x)$ is either of the functions

$$\text{(i) } \operatorname{excos}_c(-kx, -\beta),$$

$$\text{(ii) } A \operatorname{exsin}_c(-kx, -\alpha) + B \operatorname{excos}_c(-kx, -\beta),$$

where A, B are arbitrary constants.

EXAMPLES XXXI

EXPONENTIAL FUNCTIONS

1. Examine the continuity of the functions :

$$\text{(i) } a^x/x, \quad \text{(ii) } a^{\sin x}, \quad \text{(iii) } a^{\operatorname{arsin} x}, \quad \text{(iv) } \exp\sqrt{(1-x)}.$$

2. Prove that $\lim\limits_{x \to 0} \dfrac{e^{ax} - 1}{x} = a$.

3. Shew graphically that if c, k have opposite signs, there is one, and only one, solution of the equation $x = a - ce^{-kx}$.

[Draw the curves $y = a - x$, $y = ce^{-kx}$.]

4. In one American wire gauge the diameters of the successive gauge numbers are in the ratio $0\cdot890522$. Given that the diameter of No. 12 is $0\cdot080808$ inches, find the diameter of No. 20. [$0\cdot031961$ in.]

5. The distance x traversed by a body is given in terms of the velocity v by the formula $\exp \dfrac{x - x_0}{L} = \dfrac{1 + v_0^2/V^2}{1 + v^2/V^2}$, and the time t elapsed by

$$t - t_0 = T\{\operatorname{artan}(v_0/V) - \operatorname{artan}(v/V)\}.$$

Find the distance traversed and the time elapsed when the velocity vanishes.

[The formulae correspond to vertical upward motion under gravity, the air resistance being taken proportional to the square of the velocity.]

6. The masses of each of two substances which are mixed together decreases exponentially so that the total amount after time t is given by the law $Ae^{-at} + Be^{-bt}$, (A, B, a, b constants). If the whole mass is found to be m_1, m_2, m_3, m_4 at the times $t = 0$, τ, 2τ, 3τ, shew how to find the values of the constants.

If, in like manner, three substances are mixed and the total amount at

time t is $Ae^{-at}+Be^{-bt}+Ce^{-ct}$, shew how to find the constants from the observed values $m_1, m_2, \ldots, m_6$ of the whole mass at times $0, \tau, \ldots, 5\tau$.

[For the first part, write $e^{-a\tau}=u$, $e^{-b\tau}=v$. Then

$$m_1=A+B,\ m_2=Au+Bv,\ m_3=Au^2+Bv^2,\ m_4=Au^3+Bv^3.$$

It follows that $m_1uv-m_2(u+v)+m_3=0$, $m_2uv-m_3(u+v)+m_4=0$. These give $u+v$ and uv, and therefore u, v.]

7. Establish the result $\Delta^n a^x=a^x(a^{\Delta x}-1)^n$.

8. Find the first derivative of each of the following functions:

(i) xe^{-x}; (ii) e^{a-x}; (iii) $e^x(x^2-2x+2)$; (iv) $x^m e^{-kx^2}$; (v) $x^a a^x$;
(vi) $\dfrac{a^u+b^v}{a^u b^v}$; (vii) $\exp\sqrt{(ax)}$; (viii) $\exp\dfrac{x^2+a^2}{x}$; (ix) $\sqrt{\left(\dfrac{a+be^x}{a-be^x}\right)}$;
(x) $\cos e^x$; (xi) $2\operatorname{artan} e^x$; (xii) $e^{-x}\sin x$; (xiii) $e^{-|x|}$; (xiv) $(e^x)^n$;
(xv) $\exp(x^n)$; (xvi) $e^{\operatorname{artan} x}/\sqrt{(1+x^2)}$; (xvii) $x^n e^{kx}\cos mx$;
(xviii) $e^{-x^2}\cos 2x$.

9. Shew that the most general function whose derivative is e^{kx} is $\dfrac{1}{k}e^{kx}+C$, (C arbitrary).

10. If $(x-y)\exp\left(\dfrac{x}{x-y}\right)=c$, prove that $y\dfrac{dy}{dx}+x=2y$.

11. Find the value of c for which the line $y=2x+c$ is a tangent to the curve $y=e^x$. [0·614.]

12. If λ is a variable parameter, shew that the gradient of each curve of the family $y=\frac{1}{2}(\sin x+\cos x)-1+\lambda e^x$ at a point where it meets the curve $y=\sin x$ is 1.

13. Find the values of the constants A, B, C, D for which

$$\frac{d}{dx}\{e^x(A\cos x+B\sin x)\}=e^x\sin x,\qquad \frac{d}{dx}\{e^x(C\cos x+D\sin x)\}=e^x\cos x.$$

14. Shew that each of the functions e^{mx}, e^{-mx}, $ae^{mx}+be^{-mx}$, (a, b any real numbers), satisfies the equation $d^2y/dx^2=m^2y$.

15. Prove the following results:

(i) if $e^y+e^{-x}=2$, then $d^2y/dx^2+(dy/dx)^2+dy/dx=0$;

(ii) if $\sin(c-y/a)=be^{x/a}$, then $ad^2y/dx^2-dy/dx\{1+(dy/dx)^2\}=0$;

(iii) if $y=ae^{x^2}\cos(x\sqrt{2}+\alpha)$, then $y_0^{(3)}:y_0^{(1)}=4:1$.

16. Find the nth derivative of each of the following functions:

(i) e^{kx}; (ii) xe^{-x}; (iii) x^2e^{ax}; (iv) $e^x\cos x$.

17. If $f(x)=e^{-\tan x}$, shew that $f^{(2)}(\frac{1}{4}\pi)=0$, $f^{(3)}(\frac{1}{4}\pi)=0$. What is the nature of the graph of the function in the neighbourhood of the point for which $x=\frac{1}{4}\pi$? Sketch the curve for the range $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ of x.

18. If the independent variable x is changed to t by the transformation $x=e^{at}$, shew that $x\dfrac{dy}{dx}=\dfrac{1}{a}\dfrac{dy}{dt}$, $x^2\dfrac{d^2y}{dx^2}=\dfrac{1}{a^2}\left(\dfrac{d^2y}{dt^2}-a\dfrac{dy}{dt}\right)$.

19. (i) If $z=\exp\{q\sqrt{(x^2+xy)}\}$, prove that $\dfrac{\partial^2 z}{\partial x^2}-q^2z=\dfrac{y^2}{x^2}\dfrac{\partial^2 z}{\partial y^2}$.

(ii) If $z=e^{qx+ry}\cos pt$, where $q^2+r^2=1$, prove that

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}+\frac{1}{p^2}\frac{\partial^2 z}{\partial t^2}=0.$$

(iii) If $z=\dfrac{1}{\sqrt{x}}\exp\left(-hx-\dfrac{y^2}{4kx}\right)$, prove that $\dfrac{\partial z}{\partial x}+hz=k\dfrac{\partial^2 z}{\partial y^2}$.

20. If $y = \dfrac{\exp\{x\sqrt{(x^2-1)}\}}{x+\sqrt{(x^2-1)}}$, prove that $dy/dx > 0$ if $x > 1$. Hence shew that the equation $\exp\{x\sqrt{(x^2-1)}\} = m\{x+\sqrt{(x^2-1)}\}$ has only one real root if $m > 1$.

21. Shew that $\lim\limits_{x\to 0} \dfrac{2\sin x - e^x + e^{-x}}{x^3} = -\frac{2}{3}$.

[Employ the $I.D_3.T.$ for $\sin x$, e^x, e^{-x}.]

22. Investigate the range of convergence of the infinite series of term-sequence $\{e^{\sqrt{n}}x^n\}$, $(x > 0)$.

23. Shew that the extreme values of $x^5/(e^x-1)$ occur at $x=0$ and $x \approx 5(1-e^{-5})$.

24. Investigate the extreme values of (i) $x^2e^{-x/a}$, (ii) $xe^{1/x}$.

[(i) 0, $4a^2e^{-2}$; (ii) e.]

25. Investigate the extreme values of the ordinates and the points of contraflexure of the graphs of the following functions:

(i) e^{1/x^2}; (ii) e^{-1/x^2}; (iii) $\dfrac{1}{\sqrt{x}}e^{-1/x^2}$; (iv) $\exp(-\cot^2 x)$; (v) xe^{-x};

(vi) xe^{-x^2}; (vii) $6(e^{-x}-e^{-1\cdot 5x})$; (viii) $e^{-1/|x|}$; (ix) $\dfrac{1}{x^5e^{a/x}}$;

(x) $\dfrac{1}{x^4e^{a/x}}$; (xi) $\dfrac{\sqrt{x}}{x^4e^{-a/x}}$; (xii) $\dfrac{e^{1/x}-1}{e^{1/x}+1}$; (xiii) $\dfrac{a}{1+e^{b-x}}$.

Sketch the graphs of the functions.

26. Employ Newton's method to improve the approximation $0\cdot 85$ to the root of the equation $x = 2e^{-x}$. [$0\cdot 8526$.]

27. Shew that the curvature of the curve $y = e^x$ has the maximum value $\frac{2}{9}\sqrt{3}$ corresponding to the value $1/\sqrt{2}$ of e^x.

28. Find $n^{1/n}$ approximately when $n = 100$, given $\log_e 10 = 2\cdot 30259$.

[Write $n^{1/n} = \exp(\frac{1}{n}\log_e n) = \exp(\frac{1}{100}\log_e 100) = \exp(\frac{2}{100}\log_e 10)$, and use the exponential series. The result is $1\cdot 04713$.]

29. Employ the exponential series to obtain the results:

$$e^{1/10} = 1\cdot 10517\ 092, \qquad e^{-1/10} = 0\cdot 90483\ 742,$$
$$e^{1/2} = 1\cdot 64872\ 127, \qquad e^{-1/2} = 0\cdot 60653\ 066,$$
$$e^{2} = 7\cdot 38905\ 610, \qquad e^{-2} = 0\cdot 13533\ 528.$$

30. Find the Taylor polynomials of the sixth degree for the functions:

(i) $e^x\cos x$, (ii) $\exp x^2$; (iii) $e^{\sin x}$; (iv) $e^{m \operatorname{arsin} x}$.

31. Shew that $\dfrac{x}{e^x-1} + \frac{1}{2}x$ is an even function of x and find the Taylor polynomial of the eighth degree for $x/(e^x-1)$.

[Denote $x/(e^x-1)$ by y and write $y(e^x-1) = x$. Differentiate this equation successively and assume that the derivatives of y converge to finite limits as $x \to 0$. For the justification of this process, see Ex. 4, Set LXX, Chap. XVIII. Otherwise, assume the form $y = a - \frac{1}{2}x + bx^2 + cx^4 + dx^6 + ex^8$, use a Taylor polynomial for e^x and equate coefficients. The required polynomial is $1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \frac{1}{30240}x^6 - \frac{1}{1209600}x^8$. If we write this in the form $\sum\limits_{r=0}^{8} B_r x^r/r!$, where $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, $B_5 = 0$, $B_6 = \frac{1}{42}$, $B_7 = 0$, $B_8 = -\frac{1}{30}$, the numbers $B_0, B_1, \dots, B_8$ are the earlier *Bernoulli numbers*, referred to in Art. 147.]

32. If a_n is the coefficient of x^n in the expansion of $\dfrac{e^x \sin x}{1+x}$, prove that $a_n + a_{n-1} = \dfrac{2^{n/2} \sin \frac{1}{4}n\pi}{n!}$.

33. The formula $(-1)^n e^{x^2}\left(\dfrac{d}{dx}\right)^n (e^{-x^2})$ defines the general Hermite polynomial $H_n(x)$. Obtain the first four polynomials of the set and sketch the graphs of the functions $H_n(x)\, e^{-\frac{1}{2}x^2}$ for the values 0, 1, 2, 3 of n.

Shew that $H_n(x)$ satisfies the equation $d^2y/dx^2 - 2x dy/dx + 2ny = 0$.

34. The formula $e^x\left(\dfrac{d}{dx}\right)^n (x^n e^{-x})$ defines the general Laguerre polynomial $L_n(x)$. Employ Leibniz's Theorem to obtain the developed form of $L_n(x)$.

Shew that $L_n(x)$ satisfies the equation $x d^2y/dx^2 + (1-x)\, dy/dx + ny = 0$.

35. Establish the following results:

(i) $\left(\dfrac{d}{dx}\right)^n (e^{ax}u) = e^{ax}\left\{\dfrac{d^n u}{dx^n} + \dbinom{n}{1} a \dfrac{d^{n-1}u}{dx^{n-1}} + \ldots + \dbinom{n}{r} a^r \dfrac{d^{n-r}u}{dx^{n-r}} + \ldots + a^n u\right\}$;

(ii) $x^n \left(\dfrac{d}{dx}\right)^n (x^m e^x) = x^m \left(\dfrac{d}{dx}\right)^m (x^n e^x)$;

(iii) $x^{n-1}\left(\dfrac{d}{dx}\right)^n (xu) = \dfrac{d}{dx}\left(x^n \dfrac{d^{n-1}u}{dx^{n-1}}\right)$;

(iv) $x^n \left(\dfrac{d}{dx}\right)^n (x^m u) = x^m \left(\dfrac{d}{dx}\right)^m \left(x^n \dfrac{d^{n-m}u}{dx^{n-m}}\right)$, $(n \geqslant m)$.

36. Establish the result $\left(\dfrac{d}{dx}\right)^n f(e^x) = \sum\limits_{r=1}^{n} X_{nr} f^{(r)}(e^x)$, where

$$X_{nr} = e^{rx} \sum_{k=1}^{r} (-1)^{r+k} \frac{k^n}{k!(r-k)!}.$$

[Employ the general result of Ex. 11, (i), Set XVI, Chap. III.]

37. Prove that for the general Hermite polynomial, Ex. 33,

$$\left(\frac{d}{dx}\right)^n e^{-x^2} = (-1)^n e^{-x^2}\left\{(2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} - \ldots\right\},$$

the last term of the polynomial in the braces being $(-1)^{n/2} n!/(\frac{1}{2}n)!$ if n is even, and $(-1)^{(n-1)/2}\, 2x \,.\, n!/\{\frac{1}{2}(n-1)\}!$ if n is odd.

Establish the recurrence formula

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0.$$

38. Find the extreme values of the dimetric function

$$(ax^2 + by^2)\exp(-x^2 - y^2).$$

[The stationary values occur at the points (0, 0), (0, 1), (0, −1), (1, 0), (−1, 0).]

EXAMPLES XXXII

DIRECT HYPERBOLIC FUNCTIONS

1. Sketch the graphs of the functions $\sinh|x|$, $\tanh(1/x)$, $(\tanh x)/x$, $x \tanh(1/x)$.

2. Find the limiting value of the ratio $\sinh x_1 : \sinh x_2$ as x_1, x_2 each diverge to ∞ but in such a manner that their difference is constant. [$\exp(x_1 - x_2)$.]

3. Shew from the definition that $\coth x$ differs from 1 by less than 0·001 if $x > 4$. [Deduce $e^8 > 2001$ from $e > 2{\cdot}7$.]

4. Assuming that the velocity v of a falling body in a resisting medium is given by $v = V\tanh(t/T)$ at time t, where V, T are constants, shew that the velocity converges to V when t is a large multiple of T. Find the velocity when $t = T\log_e 2$ and when $t = 3T\log_e 2$. [$\frac{3}{5}V$, $\frac{63}{65}V$.]

5. Given $3\log_e 2 = 2{\cdot}0794$, shew that

$$\tanh 2{\cdot}0794 = 0{\cdot}96923 \text{ and } \operatorname{sech} 2{\cdot}0794 = 0{\cdot}24615.$$

6. Establish the following results:

$$\cosh x + \cosh y = 2\cosh \tfrac{1}{2}(x+y)\cosh \tfrac{1}{2}(x-y);$$
$$\cosh x - \cosh y = 2\sinh \tfrac{1}{2}(x+y)\sinh \tfrac{1}{2}(x-y);$$
$$\sinh x + \sinh y = 2\sinh \tfrac{1}{2}(x+y)\cosh \tfrac{1}{2}(x-y);$$
$$\sinh x - \sinh y = 2\cosh \tfrac{1}{2}(x+y)\sinh \tfrac{1}{2}(x-y);$$
$$\tanh x \pm \tanh y = \frac{\sinh(x \pm y)}{\cosh x \cosh y};$$
$$\cosh(x+y)\cosh(x-y) = \cosh^2 x + \sinh^2 y;$$
$$\frac{1+\tanh x}{1-\tanh x} = \cosh 2x + \sinh 2x;$$
$$\coth \tfrac{1}{2}x - \coth x = \operatorname{cosech} x;$$
$$(\cosh x + \sinh x)(\cosh y + \sinh y) = \cosh(x+y) + \sinh(x+y).$$

7. Shew that in a common catenary the ordinate of the middle point of the chord whose ends have abscissae $x-a$, $x+a$ is in a constant ratio to the ordinate of the curve corresponding to the abscissa x.

8. Find the first derivative of each of the following functions:

(i) $a\cosh(mx+c)$; (ii) $\sinh|x|$; (iii) $\sinh\sqrt{|x|}$; (iv) $\cosh^2 x + \sinh^2 x$; (v) $\cosh^m x$; (vi) $\sinh^m x$; (vii) $x^n \sinh^m(ax+b)$; (viii) $\tanh^2 x$; (ix) $e^{ax}\sinh bx$; (x) $\cosh x\cos x + \sinh x \sin x$; (xi) $x\sinh(1/x)$; (xii) $x - \tanh x$; (xiii) $\dfrac{1+\cosh x}{1-\cosh x}$; (xiv) $\dfrac{\cosh x - \cos x}{\sinh x + \sin x}$; (xv) $\sinh x + \frac{1}{3}\sinh^3 x$; (xvi) $\frac{1}{4}\sinh 2x + \frac{1}{2}x$; (xvii) $\cosh(k\cos mx)$; (xviii) $\tanh x - \frac{1}{3}\tanh^3 x$; (xix) $\tanh(m \operatorname{sech} x)$; (xx) $\operatorname{artan}(\tanh x)$; (xxi) $\cot x \coth x$; (xxii) $\tanh e^x$.

9. Verify the following results:

(i) if $\tanh \dfrac{y}{2a} = \tan^2 \dfrac{x}{2a}$, then $\dfrac{dy}{dx} = \tan \dfrac{x}{a}$;

(ii) if $x = \sqrt{(1+\cosh y)}$, then $dy/dx = 2/\sqrt{(x^2-2)}$, $(y > 0)$;

(iii) if $\sinh \varphi = \tan \theta$, $(-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi)$, then $\tanh \frac{1}{2}\varphi = \tan \frac{1}{2}\theta$, $\sin\theta = \tanh \varphi$ and $d\theta/d\varphi = \operatorname{sech} \varphi = \cos\theta$;

(iv) if $\sinh x \tan y = 1$, $(0 < y < \frac{1}{2}\pi)$, then $dy/dx = -\sin y$.

10. If $x = c\cosh\xi\cos\eta$, $y = c\sinh\xi\sin\eta$, where ξ, η are variable parameters, shew that the (x, y) curves corresponding to constant values of ξ are confocal ellipses and those corresponding to constant values of η, confocal hyperbolas. Shew further that the two families of curves intersect orthogonally.

11. Shew that if the law of variation with the time t of the variable x is given by $x = a + b\tanh kt$, $(a, b, k$ constants$)$, then the rate of variation of x is proportional to the product of the differences between x and two constants.

12. Shew that each of the functions $\cosh mx$, $\sinh mx$, $a\cosh mx + b\sinh mx$, (m, a, b constants), satisfies the equation $d^2y/dx^2 = m^2y$.

13. Find the nth derivative of each of the following functions :

(i) $\cosh^2 x$; (ii) $\sinh^2 x$; (iii) $e^x \cosh x$; (iv) $e^{-x}\sinh x$; (v) $x^2\cosh x$; (vi) $\cosh x \cos x$; (vii) $\sinh x \sin x$.

14. Shew that $\lim\limits_{x\to 0} \dfrac{x^2 - \sin x + \tan x}{\cosh x - 1} = 2.$

[Use the $I.D_2.T.$ for $\sin x$, $\tan x$, $\cosh x$.]

15. Shew that $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2} - \coth^2 x\right) = -2/3.$

[Write the function in the form $\dfrac{\sinh^2 x - x^2\cosh^2 x}{x^2\sinh^2 x}$ and employ the $I.D_3.T.$ for $\sinh x$ and the $I.D_2.T.$ for $\cosh x$ in the numerator.]

16. Sketch the graphs of the function $\operatorname{sech}(x-a) + \operatorname{sech}(x+a)$ for the values 1, 5 of a. Plot also the value of the maximum ordinate as a function of a.

[Use the results $\cosh 1 = 1{\cdot}54$, $\cosh 5 = 74{\cdot}2$.]

17. Shew that for catenaries of uniform section and of the same strength of material, the span is a maximum when the tangents at the ends intersect on the directrix and that this occurs when the ratio of half-span a to the parameter c of the catenary satisfies the relation $(a/c)\tanh(a/c) = 1$, or $a/c \approx 1{\cdot}200$.

Shew that if the greatest tension is 20 tons per square inch of section and the density is 500 lbs. per cu. ft., the greatest span is about 17,000 feet.

[The ratio of tension T in a common catenary to the weight w per unit length is equal to the height y above the directrix. Since $y = c\cosh(x/c)$, the half-span a is to be made a maximum by varying c, where $T = wc\cosh(a/c)$.]

18. If $\tanh x = \tanh^3 y$, find the maximum value of $y - x$.

[Maximize $\tanh(y - x)$. The result is $0{\cdot}54931$.]

19. A uniform chain hangs from two points at the same level at distance $2a$ apart and the sag in the middle is a. Find the parameter c of the catenary.

If, as an approximation, the form of the chain is taken to be parabolic, find the greatest difference of ordinates of the two curves.

[$c/a = 0{\cdot}619$; $a/20$, approximately.]

20. Shew that the points of contraflexure on the curve $y = \operatorname{sech} x$ occur where $\sinh x = \pm 1$.

21. Shew that the curvature at any point P of the curve $y = c\sinh(x/c)$ is $y^4/(c^2PN^3)$, where PN is the normal, taken to have the same sign as x.

22. Use the series for $\cosh x$, $\sinh x$ to shew that

$$\cosh \tfrac{1}{10} = 1{\cdot}00500\ 41681, \qquad \sinh \tfrac{1}{10} = 0{\cdot}10016\ 67500,$$
$$\cosh \tfrac{1}{2} = 1{\cdot}12762\ 59652, \qquad \sinh \tfrac{1}{2} = 0{\cdot}52109\ 53055.$$

23. Obtain power series for $(\sinh x)/x$, $(\cosh x - 1)/x$.

24. Shew that $\sum\limits_{r=1}^{n} x^r \sinh r\theta = \dfrac{x\{\sinh\theta - x^n\sinh(n+1)\theta + x^{n+1}\sinh n\theta\}}{1 - 2x\cosh\theta + x^2}$, and by differentiation with respect to θ find the value of $\sum\limits_{r=1}^{n} rx^r\cosh r\theta$.

[Express $\sinh r\theta$ in exponential form.]

25. Shew that for the catenary $y = c\cosh(x/c)$ the length of the perpendicular MQ from the foot M of the ordinate MP to the tangent PT at P is c. Hence shew that $MP = c\sec\psi$, where ψ is the slope at P.

Shew further that the coordinates (ξ, η) of Q are given by

$$\xi = c(u - \tanh u), \quad \eta = c\,\mathrm{sech}\, u, \quad (u = x/c).$$

Sketch the locus of Q when the position of P on the catenary varies.

[The locus is called the *tractrix*.]

26. Shew that the length PT of the tangent at any point P of the tractrix $x = c(u - \tanh u)$, $y = c\,\mathrm{sech}\, u$ is c.

Shew also that the curvature at any point is $1/(c\sinh u)$ and that the coordinates of the centre of curvature are $(cu, c\cosh u)$.

[The first result suggests a means of drawing the curve. The plane xOy being horizontal, if a string of length c is placed along Oy with one end at O and if a weight W is attached to the other end, the path traced out by W as the free end is slowly moved along Ox is a tractrix.]

27. A uniform heavy chain is suspended from two points at the same level and 100 feet apart, the sag in the middle being 1 foot. Use the Taylor polynomial of the second degree for $\cosh(x/c)$ to shew that the parameter c of the catenary is approximately 1250.

Employ Newton's method to shew that a closer approximation is

$$c \approx 1250{\cdot}167.$$

[Use $\cosh 0{\cdot}04 = 1{\cdot}00080\,0107$, $\sinh 0{\cdot}04 = 0{\cdot}04001\,0668$.]

28. Find the equations of the circle and parabola of closest fit to the catenary $y = c\cosh(x/c)$ at the vertex. Shew that the difference of ordinates between the curves and the catenary are of the fourth order in x, the first terms in the polynomial expressions for the differences being $\frac{1}{12}x^4/c^3$, $\frac{1}{24}x^4/c^3$.

If the catenary of the preceding Example is replaced by the above curves in turn, shew that the differences of ordinates at the positions of the supports are $0{\cdot}0032$, $0{\cdot}0016$ inch, respectively.

29. Find the real root of the equation $x^3 - 6x^2 + 15x - 13 = 0$.

[The condition for one real root, as given in Ex. 4, Art. 95. 1, is here satisfied. Its value is $1{\cdot}67784$.]

30. Find the real roots of the equation $2x^3 + 6x^2 - 1 = 0$.

[There are three real roots, one of which is $0{\cdot}38437$.]

EXAMPLES XXXIII

LOGARITHMIC FUNCTIONS. ILLUSTRATIONS OF THE 'INTEREST LAW'

1. Sketch the graphs of the functions $\log_e \sin x$, $\log_e|\tan\frac{1}{2}x|$, $x\log_e x$, $(1/x)\log_e x$, $\log_e|\log_e x|$, $\exp(x\log_e x)$.

2. Shew how the graph of the function $\log_e|x|$ can be made to represent the function $\log_e x^2$.

3. Investigate the continuity of the following functions, and sketch their graphs:
$$\text{(i) } \log_e \sqrt{\left(\frac{1+x}{1-x}\right)}; \qquad \text{(ii) } \log_e \sqrt{\left(\frac{x+1}{x-1}\right)};$$
$$\text{(iii) } \log_e \{x + \sqrt{(x^2+1)}\}; \qquad \text{(iv) } \log_e \{x + \sqrt{(x^2-1)}\}.$$
[Treat each as a function of a function.]

4. Discuss the continuity of the function $\log_e(1+x)\sin(1/x)$ in the range $(0, 1)$ of x.

5. Find the value of the sum $\sum_{r=1}^{n} rx^r$ and hence find for what values of x the infinite series of term-sequence $\{nx^n\}$ converges. Find the sum of this series when convergent.

6. If $\tanh \dfrac{y}{2a} = \tan^2 \dfrac{x}{2a}$, $\left(\left|\dfrac{x}{a}\right| < \tfrac{1}{2}\pi\right)$, prove that $y = a \log_e \sec(x/a)$.

7. Prove that $\lim\limits_{n\to\infty} \left(1+\dfrac{k}{n}\right)^n = e^k$, where k is any real number.

[Write $k/n = 1/m$ and use the result (8) of Art. 177.]

8. Investigate the convergence of the sequences whose general terms are
$$\text{(i) } \left(1+\frac{\sin n\theta}{n}\right)^n; \quad \text{(ii) } (\log_e n)^{1/n}; \quad \text{(iii) } \left(\frac{n^r + a_1 n^{r-1} + \dots + a_r}{n^r + b_1 n^{r-1} + \dots + b_r}\right)^n.$$

9. Find the first derivative of each of the following functions:

(i) $(\log_e x)/x$; (ii) $x^2 \log_e |x|$; (iii) $(\log_e |x|)^2$; (iv) $\log_e\{x + \sqrt{(a^2 - x^2)}\}$;

(v) $\log_{10} |f(x)|$; (vi) $\log_{10} |\sin^{\dagger} x|$; (vii) $x \operatorname{artan} x - \log_e \sqrt{(1+x^2)}$;

(viii) $\log_e \sqrt{\left(\dfrac{1+\sin x}{1-\sin x}\right)}$; (ix) $\log_e \cosh x$; (x) $\log_e |\tanh \tfrac{1}{2}x|$;

(xi) $\log_e \dfrac{x^2+x+1}{x^2-x+1} + \dfrac{2}{\sqrt{3}} \operatorname{artan} \dfrac{x\sqrt{3}}{1-x^2}$;

(xii) $\log_e [1 - \sqrt{\{1 - \exp(-n \operatorname{cosec} x)\}}]$; (xiii) $x \log_e \dfrac{a-x}{a+x}$;

(xiv) $\dfrac{x \arcsin x}{\sqrt{(1-x^2)}} + \tfrac{1}{2}\log_e(1-x^2)$; (xv) $\sin^2(\log_e x^2)$;

(xvi) $\log_e |\log_e(\cosh e^x)|$; (xvii) $\log_e \left\{e^x \left(\dfrac{x-2}{x+2}\right)^{3/2}\right\}$;

(xviii) $x^n e^x \log_e |x|$; (xix) $\dfrac{2}{n\sqrt{a}} \log_e \dfrac{\sqrt{(bx^n)}}{\sqrt{(a+bx^n)} + \sqrt{a}}$; (xx) x^x;

(xxi) $(\log_e x)^{mx}$.

10. Establish the following results:
$$\text{(i) if } y = x\log_e |xy|, \text{ then } \frac{dy}{dx} = \frac{y(1+\log_e|xy|)}{y-x};$$
$$\text{(ii) if } x = a \log_e \frac{\sqrt{(y+a)} + \sqrt{y}}{\sqrt{a}}, \text{ then } \frac{dy}{dx} = \sinh \frac{2x}{a}.$$

11. A function $f(x)$ is plotted as ordinate against x as abscissa and $\log_e |f(x)|$ is plotted against $\log_e |x|$ as abscissa. Find the relation between the gradients of the two graphs corresponding to the same value of x.

Find the most general function $f(x)$ for which the second graph is a straight line.

12. If $y = u^{vw}$, shew that $\dfrac{dy}{dx} = yv^w\left\{\log_e u \log_e v \dfrac{dw}{dx} + \log_e u \dfrac{w}{v}\dfrac{dv}{dx} + \dfrac{1}{u}\dfrac{du}{dx}\right\}$.

13. If $X_n = x^{X_{n-1}}$, where $X_1 = x$, shew that

$$x\frac{dX_n}{dx} = X_nX_{n-1} + X_nX_{n-1}X_{n-2}\log_e x + X_nX_{n-1}X_{n-2}X_{n-3}(\log_e x)^2 + \dots + X_nX_{n-1}\dots X_2(\log_e x)^{n-1}.$$

14. If $X_n = y_n^{X_{n-1}}$ and $X_1 = y_1$, prove that

$$\frac{dX_n}{dx} = \frac{dz_n}{dx}X_nX_{n-1} + z_n\frac{dz_{n-1}}{dx}X_nX_{n-1}X_{n-2} + z_nz_{n-1}\frac{dz_{n-2}}{dx}X_nX_{n-1}X_{n-2}X_{n-3} + \dots + \frac{dy_1}{dx}\prod_{r=2}^{n}(z_rX_r),$$

where $z_n \equiv \log_e y_n$.

Hence find dX_n/dx when $y_r = e^{ax}$, $(r = 1, 2, \dots, n)$.

15. A body in a room at 15° C. is initially at a temperature of 75° C. and 10 minutes later its temperature is 55° C. What is the temperature after another 5 minutes ?

[47·7° C. Assume Newton's law of cooling.]

16. A sum of money bears interest at the rate 5% per annum, added annually. Find the constant k of exponential increase which gives the same amount after any integral number of years.

[$k = \log_e(1 + 1/20)$, the unit of time being 1 year.]

17. The bacterial content of milk increases 1000 times in 12 hours when the temperature is 80° F., and this increase is sufficient to turn it sour. For every 10° F. fall in temperature the rate at which the bacteria multiply is halved. Find how long it takes milk to go sour at 70° F. and at 60° F.

[24 hours, 48 hours. Assume an exponential law of multiplying.]

18. The loss of light in passing through a small thickness of absorbing material is proportional to the amount that enters it and to the thickness. Find the fraction of light that penetrates a given thickness.

If one-half is lost in passing through 1 cm., how much is left after 10 cm. have been traversed ? [The fraction $1/2^{10}$ of the initial amount.]

19. A rope, which is used to moor a ship, is wound round a bollard on the ship, the coefficient of friction being 1/6. If there are 4½ complete turns on the bollard, find the strain in the rope that can be balanced by a pull of 30 lbs. wt. on the free end.

[The tension varies according to an exponential law, the ratio of the tensions for an angle of wrap of 1 radian being e^{μ}, where μ is the coefficient of friction. Result 3339·5 lbs. wt.]

20. A body moving in a straight line loses velocity at a rate proportional to the instantaneous velocity. Find the velocity at the end of time t.

Find also the position at any time and the total distance traversed.

21. If the distance x fallen in time t from rest by a body in a resisting medium is expressed by $x = (g/b^2)\log_e(\cosh bt)$, find the acceleration in terms of the velocity. Hence shew that the resistance varies as the square of the velocity.

22. If $y=a\cos(\log_e|x|)+b\sin(\log_e|x|)$, prove that $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+y=0$.

23. Find the nth derivative of each of the following functions:

(i) $x\log_e|x|$; (ii) $\log_e|ax+b|$; (iii) $\log_e|ax+x^2|$;
(iv) $x^3\log_e|x|$; (v) $x^n\log_e|x|$; (vi) $\log_e(1-2x\cos\alpha+x^2)$.

24. Employ Leibniz's Theorem to prove that

$$\left(\frac{d}{dx}\right)^n\frac{\log_e x}{x}=(-1)^n\frac{n!}{x^{n+1}}\left(\log_e x-1-\tfrac{1}{2}-\dots-\frac{1}{n}\right).$$

25. Prove that the function $x\cot x-\log_e|(\sin x)/x|$ is strictly down-way in each of the intervals $[n\pi, n\pi+\pi]$ of x, $(n=0, 1, 2, \dots)$.

26. Prove that $x< -\log_e(1-x)<x+\frac{1}{2}x^2/(1-x)$ if $0<x<1$.

27. Investigate the extreme values of the function $x^2\log_e(1/x)$.
[Maximum when $x\approx 0{\cdot}606$.]

28. Sketch the graph of the function $y=k\log_e\sec(x/k)$, $(k>0)$, shewing that it consists of branches lying in the ranges $\left[\dfrac{4n-1}{2}k\pi,\ \dfrac{4n+1}{2}k\pi\right]$.

Shew that for the branch in the range $[-\frac{1}{2}k\pi, \frac{1}{2}k\pi]$, $x=k\psi$, where ψ is the slope at the point (x, y).

[Any one of the above branches is called a *catenary of uniform strength*, for which the greatest span is $k\pi$.]

29. Find the values of the extreme ordinates and the points of contra-flexure of the graphs of the following functions:

(i) $(\log_e x)/x$; (ii) $\sin(\log_e|x|)$; (iii) $\log_e|\sin x|$;
(iv) $\log_e|\log_e x|$; (v) $x^m(\log_e x)^n$, $(x>0$ and $m, n>1)$.

30. Find the curvature at any point of the curve $y=a\log_e|\cos(x/a)|$.
[$-(1/a)\cos(x/a)$.]

31. Shew graphically that an approximate solution of the equation

$$\log_{10}\frac{1}{16}=-\frac{2116}{T}+0{\cdot}783\log_{10}T+0{\cdot}00043T$$

in T is 675. Use Newton's method to shew that a closer approximation is 676·2.

[The equation arises in the determination of the absolute temperature at which a mixture of steam, carbon monoxide, hydrogen and carbon dioxide, in equilibrium, contains 10% of carbon dioxide.]

32. Given $\log_{10}3=0{\cdot}47712\ 12547$ and $\log_{10}e=0{\cdot}43429\ 44819\ 03$, calculate $\log_{10}37$ correct to ten places of decimals by means of the equation

$$37\times 27=999=10^3-1.$$

33. Find $\log_{10}\pi$ correct to sixteen places of decimals by finding the Weddle factors for π and using Shortrede's *Tables* for their logarithms.
[0·49714 98726 94133 9.]

34. Find the principal at the end of one thousand years of £1 at 1% per annum, added annually, correct to the nearest penny. [£20,959 3s. 2d.]

35. Find the millionth power of 1·01 correct to six significant figures.
[$2{\cdot}36474\times 10^{4321}$.]

36. Shew that a sum of money, accumulating at compound interest $p\%$ per annum, added continuously, increases in the ratio $e:1$ in $100/p$ years, and that at compound interest $p\%$, added yearly, it increases in the same ratio in $100/p+\frac{1}{2}$ years, nearly.

37. Shew that $\log_{10} 13 = 1 + 3\log_{10} 11 - 10\log_{10} 2$ with an error of magnitude 0·0000653. [Shew that the error is $\log_{10}(6656/6655)$.]

38. Use the exponential and logarithmic series to shew that when n is large, $\left(1+\dfrac{1}{n}\right)^n \approx e\left(1-\dfrac{1}{2n}+\dfrac{11}{24n^2}-\dfrac{7}{16n^3}\right)$.

39. Shew that e is equal to $\frac{1}{2}\{(1\cdot 1)^{10}+(0\cdot 9)^{-10}\}$, with an error of about 0·46%.

40. Use the series (21), Art. 182, to obtain $\log_e 71$ correct to five decimal places. [4·26268.]

41. By writing $\dfrac{1+x}{1-x}=\dfrac{(N+2)(N-1)^2}{(N-2)(N+1)^2}$ in the series (11), Art. 182, shew that

$$\log_e(N+2)=\log_e(N-2)+2\log_e(N+1)-2\log_e(N-1)$$
$$+2\left\{\frac{2}{N^3-3N}+\frac{1}{3}\left(\frac{2}{N^3-3N}\right)^3+\dots\right\}.$$

Hence evaluate $\log_e 5$ correct to six decimal places, given $\log_e 2=0\cdot 693147$. [1·609438.]

42. Find the Taylor polynomials of the fourth degree for the functions

$$\text{(i) } \log_e \frac{x\sin x}{1-\cos x}, \quad \text{(ii) } \log_e\{1+x\sqrt{(1+x^2)}\}.$$

[(i) $\log_e 2-\dfrac{x^2}{12}-\dfrac{7x^4}{1440}$; (ii) $x-\frac{1}{2}x^2+\frac{5}{6}x^3-\frac{3}{4}x^4$.]

43. If $y=a+x\log_e(y/b)$, shew that the Taylor polynomial of the second degree for y is $a+x\log_e(a/b)+(x^2/a)\log_e(a/b)$.

44. A table of $\log_{10}\sinh x$ is given to six decimal places, the common difference of argument being 0·001. Investigate the range of the table in which it is complete.

45. Shew that a seven-figure table of logarithmic-tangents (to base 10), in which the common difference of argument is 1 minute, is complete only for angles between 17° and 73°, approximately.

46. Shew that the most general continuous function which satisfies the relation $f(x_1+x_2)=f(x_1)f(x_2)$, where x_1, x_2 are any two values of the variable, is given by $f(x)=a^{kx}$, $(a>0)$.

[Writing $\log_a f(x)=\varphi(x)$, the relation becomes $\varphi(x_1+x_2)=\varphi(x_1)+\varphi(x)_2$. Now employ the result of Ex. 7, Set III, Chap. I.]

47. Shew that the most general continuous function which satisfies the relation $f(x_1x_2)=f(x_1)+f(x_2)$ is given by $f(x)=k\log_a x$, $(a>0)$.

[Write $x=a^\xi$ and put $f(x)=f(a^\xi)=\varphi(\xi)$.]

48. Shew that the sequence $\{u_n\}$ defined by $u_n=1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\log_e n$ is down-way and convergent.

[Write $u_{n+1}-u_n$ in the form $\frac{1}{n+1}+\log_e\left(1-\frac{1}{n+1}\right)$ and use the $I.D_2.T.$ for $\log_e(1+x)$ to express this difference in the form $\frac{1}{(n+1)^2}R_{n+1}(\theta)$, shewing that $|R_{n+1}(\theta)|<2$. From this it follows that $|u_{n+1}-u_1|<\sum_{r=1}^{n+1}2/r^2$. Since the series $\sum_{r=1}^{\infty}1/r^2$ is convergent, u_n must converge to a definite limit as $n\to\infty$. This limit is called *Euler's Constant* and is denoted by the letter γ.]

49. Using Taylor's Theorem for $\log_e(1+x)$ for the values $1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots$ of x and summing the expansions, show that

$$s_1-\log_e(n+1)=\tfrac{1}{2}s_2-\tfrac{1}{3}s_3+\ldots+(-1)^{n-1}\frac{1}{n-1}s_{n-1}+(-1)^n\frac{1}{n}\theta s_n,\ (0<\theta<1),$$

where $s_r=\dfrac{1}{1^r}+\dfrac{1}{2^r}+\ldots+\dfrac{1}{n^r}$.

Hence shew that $\gamma=1-\log_e 2+\frac{1}{2}(S_2-1)-\frac{1}{3}(S_3-1)+\ldots$, where $S_r=\lim_{n\to\infty}s_r$.
Using the known values of S_r for $r=2, 3, \ldots, 11$, shew that $\gamma=0{\cdot}5772\ldots$.

[The values of S_r are given in the Smithsonian *Mathematical Formulae and Tables*, p. 140. The value of γ correct to ten places is 0·57721 56649.]

50. Shew that

$$\left(\frac{d}{dx}\right)^{m+1}\log_e(1+x^2)=-\frac{m!}{(1+x^2)^{(m+1)/2}}\cdot 2\cos\{(m+1)(\operatorname{artan}x+\tfrac{1}{2}\pi)\}.$$

Hence obtain the expansion of $\log_e\{1+(x+h)^2\}$ in powers of h, and, writing $h=k\surd(1+x^2)$, $x/\surd(1+x^2)=\cos\theta$, deduce the expansion of $\log_e(1+2k\cos\theta+k^2)$ in powers of k, $(|k|<1)$.

[For the first part write $\left(\dfrac{d}{dx}\right)^{m+1}\log_e(1+x^2)=\left(\dfrac{d}{dx}\right)^m\dfrac{2x}{1+x^2}$ and apply Leibniz's Theorem, using the result (13) of Art. 152. 2. The final expansion is

$$2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{k^n}{n}\cos nx.]$$

EXAMPLES XXXIV

INVERSE HYPERBOLIC FUNCTIONS

1. Prove that if the coordinates of a variable point P satisfy the equation $\operatorname{artanh}x-\operatorname{artanh}y=c$, then P lies on a rectangular hyperbola.

2. Sketch the graphs of the functions $\operatorname{arcosh}|x|$, $-\operatorname{arcosh}(-x)$, $\operatorname{arsinh}|x|$, $\operatorname{artanh}(-x)$.

3. Prove that (i) $\operatorname{arsech}x=\log_e\dfrac{1+\surd(1-x^2)}{x}$,

(ii) $\operatorname{arcosech}x=\log_e\dfrac{1\pm\surd(1+x^2)}{x}$,

the upper (lower) sign being taken when x is positive (negative).

4. Establish the results :

$$\text{(i) } \operatorname{arcosh}(x/a)=\begin{cases}\log_e\{x+\surd(x^2-a^2)\}-\log_e a,\ (x, a>0),\\ \log_e|x-\surd(x^2-a^2)|-\log_e|a|,\ (x, a<0);\end{cases}$$

$$\text{(ii) } \operatorname{arsinh}(x/a)=\begin{cases}\log_e\{x+\surd(x^2+a^2)\}-\log_e a,\ (a>0),\\ \log_e|x-\surd(x^2+a^2)|-\log_e|a|,\ (a<0).\end{cases}$$

5. Prove that the functions $(\operatorname{arsinh} x)/x$, $(\operatorname{artanh} x)/x$ are potentially continuous at $x=0$, the completing value being 1 in each case.

6. Find the first derivative of each of the following functions :

(i) $\operatorname{arsech}(x/a)$; (ii) $\operatorname{arcosech}(x/a)$; (iii) $\operatorname{arcoth}(x/a)$;

(iv) $\operatorname{arsinh}(\frac{1}{\sqrt{3}}\sec 2x)$; (v) $\operatorname{arsinh}\frac{x}{1+x}$; (vi) $x\sqrt{(1+x^2)}+\operatorname{arsinh} x$;

(vii) $x\sqrt{(x^2-a^2)}-a^2\operatorname{arcosh}(x/a)$; (viii) $x\sqrt{(x^2+a^2)}+a^2\operatorname{arsinh}(x/a)$;

(ix) $\operatorname{artanh}\frac{x\sqrt{2}}{1+x^2}$; (x) $2a\operatorname{artanh}\left(\tan^2\frac{x}{2a}\right)$;

(xi) $\frac{1}{2}\log_e(1-x^2)+x\operatorname{artanh} x$.

7. Shew that if ψ is the slope at any point (x, y) of the catenary

$$y=c\cosh(x/c),$$

then $x=c\log_e|\sec\psi\pm\tan\psi|$.

8. Expand $\sqrt{(x^2+1)}\,.\operatorname{arsinh} x$ in powers of x and shew that if a_n is the coefficient of x^n, $a_{n+2}=-\frac{n-1}{n+2}a_n$.

9. Establish the results $\operatorname{arsinh} 0{\cdot}1=0{\cdot}0998341$, $\operatorname{artanh} 0{\cdot}1=0{\cdot}1003353$.

10. Prove that $\left(\frac{d}{dx}\right)^m(x^2-1)^{m+\frac{1}{2}}=\frac{(2m+1)(2m-1)\dots 1}{m+1}\sinh\{(m+1)\operatorname{arcosh} x\}$

$$=(-1)^m\frac{(2m+1)(2m-1)\dots 1}{(m+1)!}(\lambda^2-1)^{(m+1)/2}\left(\frac{d}{d\lambda}\right)^m\frac{1}{\lambda^2-1},$$

where $\lambda=x/\sqrt{(x^2-1)}$.

EXAMPLES XXXV

EPICENE FUNCTIONS.

1. Shew that $\mathfrak{C}(x)$ satisfies the equation $4x\frac{d^2y}{dx^2}+2\frac{dy}{dx}-y=0$, and $\mathfrak{S}(x)$ the equation $4x\frac{y^2d}{dx^2}+6\frac{dy}{dx}-y=0$.

Shew also that $\sqrt{|x|}\,.\,\mathfrak{S}(x)$ satisfies the first equation and $\frac{1}{\sqrt{|x|}}\mathfrak{C}(x)$ the second.

2. Establish the results

(i) $\frac{x\mathfrak{S}(x)}{\mathfrak{C}(x)}=\frac{\mathfrak{C}(4x)}{\mathfrak{S}(4x)}-\frac{\mathfrak{C}(x)}{\mathfrak{S}(x)}$, (ii) $\frac{1}{\mathfrak{S}(x)}=\frac{\mathfrak{C}(\frac{1}{4}x)}{\mathfrak{S}(\frac{1}{4}x)}-\frac{x\mathfrak{S}(\frac{1}{4}x)}{4\mathfrak{C}(\frac{1}{4}x)}$.

3. Prove that $\frac{d}{dx}\log_e|\mathfrak{C}(x)|=\frac{\mathfrak{S}(x)}{2\mathfrak{C}(x)}$, $\frac{d}{dx}\log_e|\mathfrak{S}(x)|=\frac{1}{2x}\left\{\frac{\mathfrak{C}(x)}{\mathfrak{S}(x)}-1\right\}$.

4. Shew that the Taylor polynomial of the fourth degree for $1/\mathfrak{C}(x)$ is

$$1-\frac{1}{2!}x+\frac{5}{4!}x^2-\frac{61}{6!}x^3+\frac{1385}{8!}x^4.$$

Deduce the corresponding polynomials for $\operatorname{sech} x$, $\sec x$.

[The numerators in the coefficients of the above powers of x are the earlier *Euler numbers*, referred to in Art. 147.]

5. Shew that the Taylor polynomial of the fourth degree for $\mathfrak{C}(x)/\mathfrak{S}(x)$ is

$$1+\frac{2^2}{2!}B_2x+\frac{2^4}{4!}B_4x^2+\frac{2^6}{6!}B_6x^3+\frac{2^8}{8!}B_8x^4,$$

where the B's are Bernoulli's numbers.

Deduce the corresponding polynomials for $x\mathfrak{S}(x)/\mathfrak{C}(x)$, $1/\mathfrak{S}(x)$, $x \coth x$, $x \cot x$, $\tanh x$, $\tan x$, $x/(\sinh x)$, $x/(\sin x)$.

6. If $x=\mathfrak{C}(\xi)$ and $y=\mathfrak{C}(m\xi)$, shew that $(x^2-1)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-my=0$.

7. If $x=\mathfrak{C}(\xi)$ and $y=\mathfrak{S}(m\xi)/\mathfrak{S}(\xi)$, shew that

$$(x^2-1)\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+(1-m)y=0.$$

8. If $x=\xi\mathfrak{S}^2(\xi)$ and $y=\mathfrak{C}(m\xi)$, shew that

$$4x(1+x)\frac{d^2y}{dx^2}+2(1+2x)\frac{dy}{dx}-my=0.$$

EXAMPLES XXXVI

EXPOCYCLIC FUNCTIONS

1. Prove the results

$$\begin{aligned}
\operatorname{exsin}_c(kx,\alpha)&=A\operatorname{exsin}_c(kx)+B\operatorname{excos}_c(kx),\\
\operatorname{excos}_c(kx,\alpha)&=A\operatorname{excos}_c(kx)-B\operatorname{exsin}_c(kx),\\
\operatorname{exsin}_c(kx)&=A\operatorname{exsin}_c(kx,\alpha)-B\operatorname{excos}_c(kx,\alpha),\\
\operatorname{excos}_c(kx)&=A\operatorname{excos}_c(kx,\alpha)+B\operatorname{exsin}_c(kx,\alpha),
\end{aligned}$$

where $A=\cos\alpha$, $B=\sin\alpha$.

2. Prove that

$$\begin{aligned}
a\operatorname{exsin}_c(kx,\alpha)+b\operatorname{exsin}_c(kx,\beta)&=r\operatorname{exsin}_c(kx,\gamma),\\
a\operatorname{excos}_c(kx,\alpha)+b\operatorname{excos}_c(kx,\beta)&=r\operatorname{excos}_c(kx,\gamma),
\end{aligned}$$

where $r^2=a^2+b^2+2ab\cos(\alpha-\beta)$, $\tan\gamma=\dfrac{a\sin\alpha+b\sin\beta}{a\cos\alpha+b\cos\beta}$.

3. Prove that

$$\begin{aligned}
\operatorname{exsin}_c\{k(x+y)\}&=\operatorname{exsin}_c(kx)\operatorname{excos}_c(ky)+\operatorname{excos}_c(kx)\operatorname{exsin}_c(ky),\\
\operatorname{excos}_c\{k(x+y)\}&=\operatorname{excos}_c(kx)\operatorname{excos}_c(ky)-\operatorname{exsin}_c(kx)\operatorname{exsin}_c(ky).
\end{aligned}$$

4. Prove that

$$\begin{aligned}
\operatorname{exsin}_c(nk)&=2\operatorname{exsin}_c\{(n-1)k\}\operatorname{excos}_c k\\
&\qquad-\operatorname{exsin}_c\{(n-2)k\}.\{(\operatorname{excos}_c k)^2+(\operatorname{exsin}_c k)^2\},\\
\operatorname{excos}_c(nk)&=2\operatorname{excos}_c\{(n-1)k\}\operatorname{excos}_c k\\
&\qquad-\operatorname{excos}_c\{(n-2)k\}.\{(\operatorname{excos}_c k)^2+(\operatorname{exsin}_c k)^2\}.
\end{aligned}$$

5. A particle is moving with decaying S.H.M. so that the displacement is given by $x=a\operatorname{excos}_c\dfrac{2\pi t}{T}+b\operatorname{exsin}_c\dfrac{2\pi t}{T}$. Shew how to find the constants a, b, c, T by the observation of x at four equally spaced instants.

[Take the instants to be 0, $\dfrac{kT}{2\pi}$, $\dfrac{2kT}{2\pi}$, $\dfrac{3kT}{2\pi}$. In the elimination of a, b from the equations obtained, use the results of the preceding Example.]

6. Establish the results :

$$\operatorname{exsin}_{c_1}(k_1x,\ \alpha_1)\operatorname{exsin}_{c_2}(k_2x,\ \alpha_2) = -\tfrac{1}{2}\operatorname{excos}_{c'}(k'x,\ \alpha') + \tfrac{1}{2}\operatorname{excos}_{c''}(k''x,\ \alpha''),$$

$$\operatorname{excos}_{c_1}(k_1x,\ \alpha_1)\operatorname{excos}_{c_2}(k_2x,\ \alpha_2) = \tfrac{1}{2}\operatorname{excos}_{c'}(k'x,\ \alpha') + \tfrac{1}{2}\operatorname{excos}_{c''}(k''x,\ \alpha''),$$

where $k'^2 = k_1^2 + k_2^2 + 2k_1k_2\cos(c_1 - c_2)$, $\quad k''^2 = k_1^2 + k_2^2 + 2k_1k_2\cos(c_1 + c_2)$,

$$\tan c' = \frac{k_1\sin c_1 + k_2\sin c_2}{k_1\cos c_1 + k_2\cos c_2}, \qquad \tan c'' = \frac{k_1\sin c_1 - k_2\sin c_2}{k_1\cos c_1 + k_2\cos c_2},$$

$$\alpha' = \alpha_1 + \alpha_2, \qquad \alpha'' = \alpha_1 - \alpha_2.$$

Deduce the result

$$\operatorname{excos}_{c'}(k'x,\ \alpha') = \operatorname{excos}_{c_1}(k_1x,\ \alpha_1)\operatorname{excos}_{c_2}(k_2x,\ \alpha_2) - \operatorname{exsin}_{c_1}(k_1x,\ \alpha_1)\operatorname{exsin}_{c_2}(k_2x,\ \alpha_2).$$

7. Find the first derivative of each of the following functions :

(i) $kx\operatorname{exsin}_c(kx,\ \alpha + c) - \operatorname{exsin}_c(kx,\ \alpha)$; (ii) $x^m\operatorname{exsin}_c(kx,\ \alpha)$;
(iii) $k^2x^2\operatorname{exsin}_c(kx,\ \alpha + 2c) - 2kx\operatorname{exsin}_c(kx,\ \alpha + c) + 2\operatorname{exsin}_c(kx,\ \alpha)$;
(iv) $x^2e^{-3x}\sin 5x$.

8. Shew that if $y = A\operatorname{exsin}_c(kx) + B\operatorname{excos}_c(kx)$, then A, B can be found so that $\dfrac{d^2y}{dx^2} + 2k\cos c\dfrac{dy}{dx} + k^2y = \operatorname{excos}_c(kx)$.

9. If $y = e^{ax}\sin bx$, prove that $y^{(n+1)} - 2ay^{(n)} + (a^2 + b^2)y^{(n-1)} = 0$. [Express y in expocyclic form.]

10. If $y = e^x\sin x$, prove that $y^{(4)} = -4y$.

11. Find the nth derivative of each of the following functions :

(i) $e^{-x}\sin x$; (ii) $e^{mx}\cos nx$; (iii) $xe^x\sin x$;
(iv) $x^2e^{ax}\cos bx$; (v) $e^x\cos x\cos 2x$; (vi) $x^m\operatorname{exsin}(kx,\ \alpha)$.

12. Find the most general function whose derivative is $\operatorname{exsin}_c(kx,\ \alpha)$.

13. If $y = a\operatorname{exsin}_c(kx,\ \alpha)$, where c, k, α are variable parameters, find the values of $\partial y/\partial c$, $\partial y/\partial k$, $\partial y/\partial\alpha$.

CHAPTER VIII

CURVES VARIOUSLY SPECIFIED. POLAR COORDINATES, LINE COORDINATES, RADIUS-RECEDENT RELATION

SECTION I. POLAR COORDINATES

197. The Polar System of Coordinates.

The position of a point in a plane is completely specified when the length and direction (bearing) of the line drawn from a fixed point of the plane to it are assigned. Let P (Fig. 159) be the point whose position is to be specified, O the fixed point, and OA a line of fixed direction relative to which other directions in the plane are specified. Then the position of P is uniquely determined by the distance OP and the direction of OP relative to OA. The former is denoted by r, this being a *positive* number, and the latter is measured by the angle of rotation θ from OA to OP.[1] We adopt the convention that this angle is positive if OP is supposed reached by counter-clockwise rotation from OA, and negative if by clockwise rotation. This, without further restriction, is the ordinary convention for the measurement of an angle in Trigonometry.

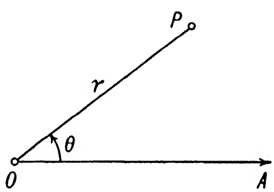

FIG. 159.

For some purposes it is important to ensure a one-one correspondence of plot-point P and coordinate pair r, θ, as in the Cartesian system of coordinates, and we may then restrict the range of values of θ to 2π. Thus we may take the range as $(0, 2\pi)$, or as $(-\pi, \pi)$, or generally as $(\alpha, \alpha+2\pi)$, where α is arbitrary, one of the terminals in each case being excluded from the range. From the analytical point of view, however, this restriction is often inconvenient as it leads to a discontinuity in the value of θ in passing from one side to the other of the line for which $\theta=\alpha$. We shall not, therefore, restrict the range of θ except in special cases for the purpose referred to. It is important to notice that with the general convention for the

[1] The measure θ is always in radians, unless otherwise stated.

measurement of θ, there is still one, and only one, plot-point corresponding to an assigned pair of values of r and θ (the former being positive [1]), but to any chosen point in the plane there corresponds an infinite number of values of θ. If α is any one of these values, the ambiguity in θ may be expressed by the relation $\theta = \alpha + 2n\pi$, where n is any integer.

The quantities r, θ are called the *polar coordinates* of the point P, and, following the similar practice in Cartesian coordinates, this point is referred to simply as *the point* [2] (r, θ). The line OA is called the *initial line* or *polar axis*, O is called the *pole*, r the *polar radius* or simply the *radius*, and θ the *polar angle*.

Other conventions for polar coordinates have been employed, but the system described above is sufficient for all practical purposes, and it is the one always used in the geometrical representation of complex numbers. We therefore keep to this system throughout.

198. Connection between Polar and Rectangular Coordinates.

The connection between the polar and rectangular coordinates of a point is simplest when the pole is taken as the origin and the polar axis [3] OA as the axis Ox, the axis Oy being obtained from it by counter-clockwise rotation through a right-angle. If the coordinates of P (Fig. 160) are (r, θ), (x, y) in the two systems, we have

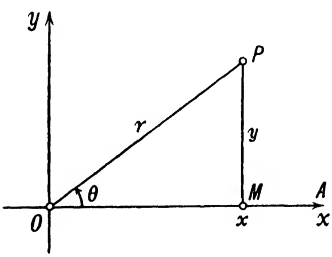

FIG. 160.

$$x = r\cos\theta, \quad y = r\sin\theta. \quad \text{......(1)}$$

These equations plainly determine the rectangular coordinates when the polar ones are given. Conversely, they determine the polar coordinates when the rectangular ones are given, except for the unimportant ambiguity of amount $2n\pi$ in the polar angle. For they give $r = \surd(x^2 + y^2)$, which determines r, and the relations

$$\cos\theta = x/r, \quad \sin\theta = y/r \quad \text{........................(2)}$$

then determine θ, except for the ambiguity mentioned. The relation $\tan\theta = y/x$ may be conveniently used to find the *orientation* of the

[1] The point O is exceptional; it corresponds to the value 0 of r and to any value whatever of θ.

[2] The practice is to give the value of r first in the parentheses.

[3] On account of the frequency of transformation from rectangular to polar coordinates, and vice versa, the polar axis OA is usually denoted by Ox. This practice will be adopted in the sequel.

polar radius, the *direction* of that line being then decided by the sign of $\cos\theta$ (or of $\sin\theta$). It should be observed that it is only when $x>0$ that $\operatorname{artan}(y/x)$ can be taken as the angle θ, since the inverse tangent is a number lying in the range $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. If $x<0$, we can take $\pi+\operatorname{artan}(y/x)$ for θ.

199. Curves in Polar Coordinates.

Suppose that there is a relation between the variables r, θ which defines r, say, as a positive [1] single-valued function of θ within, it may be, a limited range. Then to each value of θ in this range there will correspond a plot-point P, and the aggregate of these points forms the graph of the function r, or of the functional relation between r and θ. If the graph is a continuous curve, the (r, θ) relation is called the equation of the curve in polar coordinates, or more curtly, the polar equation of the curve.

Ex. 1. *The relation* $\qquad r=a\theta^2, \quad (a>0)$. ....(1)

This defines r as a positive function for all values of θ. Portion of the graph is shewn in Fig. 161, A being the point $(a, 1)$ and A' the point $(a, -1)$.

Ex. 2. *The equation* $\qquad r=a\theta$. ....(2)

When θ has the same sign as a, this defines r as a positive function of θ. The corresponding curve is known as the *Spiral of Archimedes* Portion of it

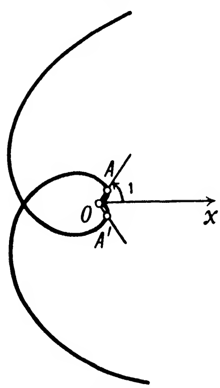

FIG. 161.
$r=a\theta^2$.

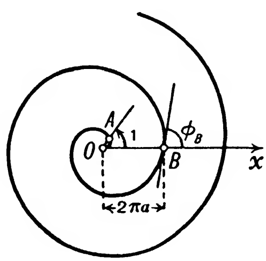

FIG. 162.
$r=a\theta$.

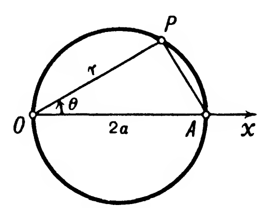

FIG. 163.
$r=2a\cos\theta$.

is shewn in Fig. 162 for a positive value of a (and hence positive values of θ). The coordinates of A are $(a, 1)$ and those of B are $(2\pi a, 2\pi)$. The form of the graph when $a<0$ is the geometrical image in Ox of the curve drawn.

Ex. 3. *The equation* $\qquad r=2a\cos\theta, \quad (a>0)$. ....(3)

Since $\cos\theta$ has the period 2π, it is not necessary to consider a range of θ

[1] The statement here, and in similar cases, is not intended to exclude a zero value of the function.

greater than 2π in order to obtain the complete range of values of r. Consider one such range, say $(-\pi, \pi)$. This must now be further restricted since $\cos\theta$, and hence r, is positive only when θ lies in the sub-range $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$. The curve is a circle with the pole O on the circumference and the polar axis Ox as a diameter. This is clear from the fact that if P (Fig. 163) is the point (r, θ) and A the point $(2a, 0)$, then $OP/OA = \cos\theta$, so that OPA is a right-angle; this is a characteristic property of the circle.

Ex. 4. *The equation* $\qquad r = ae^{k\theta}, \quad (a > 0).$ ....(4)

This defines r as a positive function for all values of θ. The graph is a spiral, r having the range $[0, \infty]$ corresponding to the range $[-\infty, \infty]$ of θ. It is known as the *logarithmic spiral*, or, for a reason explained in Ex. 3, Art. 202, the *equiangular spiral.* Portion of the curve is shewn in Fig. 164, corresponding to a positive value of k, A being the point $(a, 0)$.

The student will also observe that the same curve is specified by the equation $r = ae^{k(\theta+2n\pi)}$, where n is any integer.

Ex. 5. *The equation* $\qquad r = \dfrac{a\theta}{1+\theta}, \quad (a > 0).$ ....(5)

This defines r as a positive function when $\theta > 0$ and when $\theta < -1$. We can write (5) in the form $r = a\Big/\left(1 + \dfrac{1}{\theta}\right)$, which shews that as $|\theta| \to \infty$, $r \to a$ and the curve approximates indefinitely to the circle $r = a$. If $\theta > 0$, we have

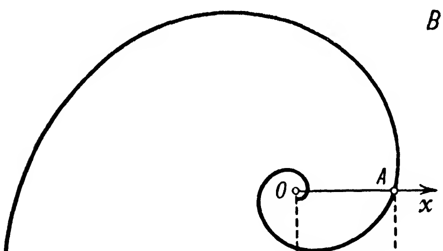

FIG. 164.

$r = ae^{k\theta}$.

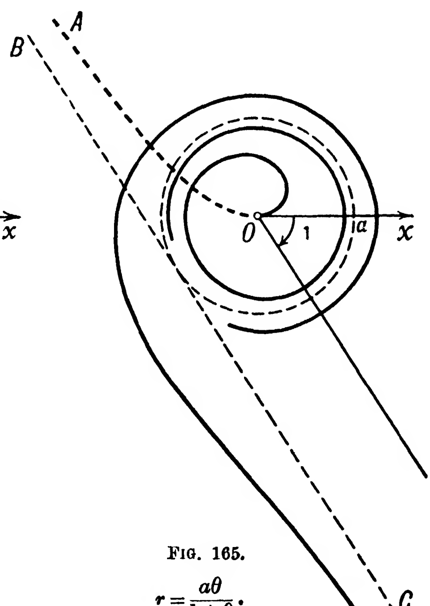

FIG. 165.

$r = \dfrac{a\theta}{1+\theta}$.

$r < a$, while if $\theta < -1$, $r > a$. Again, as $\theta \to -1-0$, $r \to \infty$. The general form of the curve is shewn in Fig. 165, the lightly dotted curve being the circle $r = a$. The significance of the heavily dotted portion markéd OA is explained in Ex. 2 of the following Article. The line BC is an asymptote to the curve.

Ex. 6. *The equation* $\qquad r = a + b\cos\theta.$ ....(6)

According to the relative values of a, b, we get a variety of curves which comprise the family of *limaçons*. The special curve for which $0<b=a$, namely,

$$r = a(1+\cos\theta), \qquad (7)$$

is called the *cardioid*.

The forms of the various curves of the family for positive values of b are shewn in Figs. 166-169, the lightly drawn curve in each case being the circle $r = b\cos\theta$. It is clear that the polar radius of a limaçon is the sum or difference of the polar radius of this circle and the constant distance a. It is by means of this property that the limaçon is usually defined, and it may be conveniently utilized in drawing the graph for chosen values of a, b.

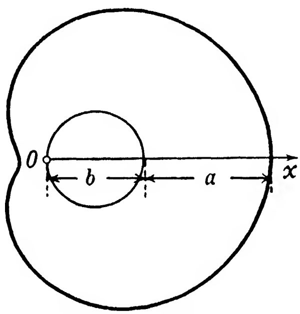

FIG. 166. $0 < b < a$.

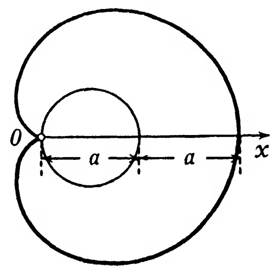

FIG. 167. $0 < b = a$.

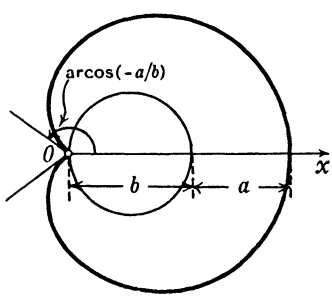

FIG. 168. $0 < a < b$.

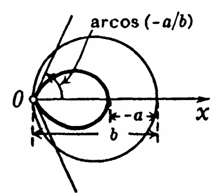

FIG. 169. $0 < -a < b$.

The curves $r = a + b\cos\theta$.

When $0 < b < a$ (Fig. 166) and when $0 < b = a$ (Fig. 167), equation (6) defines r as a positive function for all values of θ. Since $\cos\theta$ has the period 2π, it is sufficient to restrict θ to the range $(-\pi, \pi)$. When $0 < a < b$ (Fig. 168) and when $0 < -a < b$, $(a<0)$, (Fig. 169), we must further restrict θ to the range $\left(-\operatorname{arcos}\frac{-a}{b}, \operatorname{arcos}\frac{-a}{b}\right)$ in order to have r positive.

When $b<0$, the corresponding curves are reflections in Oy of those drawn. When $0<-b<a$, the curve is the reflection of Fig. 166; when $0<-b=a$, that of Fig. 167; when $0<a<-b$, that of Fig. 168; and when $0<-a<-b$, $(a<0)$, that of Fig. 169.

200. Polar Representation of a Functional Relation.

The preceding Article immediately suggests a method of constructing a *polar graph* of a functional relation. Let ξ, η be any two variables having a functional relation $\eta = f(\xi)$ between them. Then if, for a certain range of ξ, the function η is positive, we may identify η, ξ, respectively, with the polar coordinates r, θ of a point in the plane of representation. The relation between ξ and η then becomes the polar equation $r = f(\theta)$ of the graph. If, however, η is negative in a certain range of ξ, it cannot be identified with the positive coordinate r and a further convention must be introduced for this case. That which leads to the most natural connection between the graphs for positive and negative values of η is to take the plot-point P corresponding to ξ, η as the polar opposite of the point P' whose polar coordinates are $(|\eta|, \xi)$. Thus the polar coordinates (r, θ) of P are given by [1]

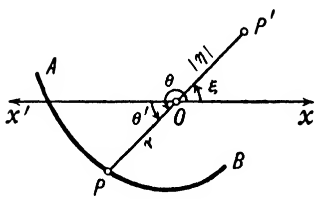

FIG. 170.

$$r = |\eta| = -\eta, \quad \theta = \xi + \pi, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

as illustrated in Fig. 170. The polar equation of the graph for a range in which η is negative is thus

$$r = -f(\theta - \pi). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

If we take θ' as the polar angle corresponding to an initial line Ox' in the opposite direction to Ox, this polar equation becomes

$$r = -f(\theta'). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

On this convention there is, in ordinary cases, no discontinuity of the gradient of the complete graph when η passes through the value zero and changes sign.

When the plots for $f(\xi)$ positive and for $f(\xi)$ negative are combined according to the above convention, we get a complete representation of the relation $\eta = f(\xi)$ in polar coordinates, and the curve is called the *extended graph* of the relation, the term *unextended* being taken as referring to the representation when $f(\xi)$ is positive. When a *name* is assigned to a graph representing a particular relationship

[1] In Analytical Geometry the radius for the plot-point P is in this case said to be *negative*, P being looked on as reached by measuring backwards (or in the *negative* direction) a distance $|\eta|$ along a line whose positive direction is OP' and bearing ξ. Radii are always treated as positive in this Book.

between η and ξ, it is largely a matter of tradition whether it applies to the extended or to the unextended graph.[1]

Ex. 1. *The equation* $\eta = a\xi$, $(a > 0)$. When $\xi > 0$, the polar representation is specified by $r = a\theta$, and when $\xi < 0$, by the relation

$$r = -a(\theta - \pi), \quad (\theta < \pi), \quad \text{.................(4)}$$

or, when referred to a reversed initial line, by

$$r = -a\theta', \quad (\theta' < 0). \quad \text{.................(5)}$$

The extended graph is shewn in Fig. 171 and is referred to as the *extended Spiral of Archimedes.*

Ex. 2. *The equation* $\eta = \dfrac{a\xi}{1+\xi}$, $(a > 0)$. The polar representation of this relation when ξ lies outside the range $(-1, 0]$ is shown in Fig. 165, p. 472. When ξ lies in the range $[-1, 0]$, η is negative and the polar representation is specified by

$$r = -\frac{a(\theta - \pi)}{1 + (\theta - \pi)}, \quad (\pi - 1 < \theta < \pi), \quad \text{.................(6)}$$

or by

$$r = -\frac{a\theta'}{1 + \theta'}, \quad (-1 < \theta' < 0), \quad \text{.................(7)}$$

when the initial line is Ox'. The unextended graph is shewn in Fig. 166 by

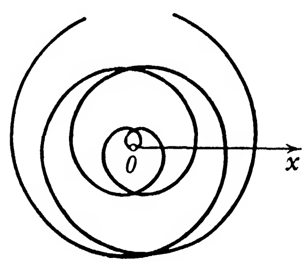

FIG. 171.
$r = a\theta$ and $r = a(\pi - \theta)$.

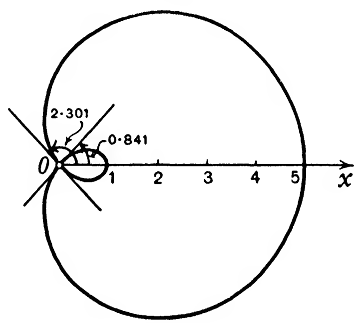

FIG. 172
$r = \pm 2 + 3 \cos \theta$.

the heavy continuous line, and the extended graph is the combination of this portion with the branch marked OA.

Ex. 3. *The equation* $\eta = a + b \cos \xi$. For the polar representation of this relation it is clearly sufficient to consider only a range of ξ of magnitude 2π.

If $a > 0$ and $|b| \leqslant a$, η is a positive function and the corresponding polar equation is

$$r = a + b \cos \theta, \quad \text{.................(8)}$$

where θ has the same restriction as ξ.

If, next, $|b| > |a|$, we have to consider separately the sub-ranges of ξ in which η is positive and in which η is negative. In the former case we may

[1] The extended graph for the function $\eta = f(\xi)$ forms the curve represented by the polar equation $r = f(\theta)$ when the negative radii of Analytical Geometry are introduced.

restrict ξ, and therefore θ, to the range $\left(-\operatorname{arcos}\frac{-a}{b}, \operatorname{arcos}\frac{-a}{b}\right)$, and the polar equation is the same as (8). In the latter case we may take the range of ξ to be $\left(\operatorname{arcos}\frac{-a}{b}, 2\pi-\operatorname{arcos}\frac{-a}{b}\right)$, and the corresponding polar equation is

$$r=-\{a+b\cos(\theta-\pi)\}=-a+b\cos\theta, \quad \text{.....................(9)}$$

where the range of θ is now $\left(-\pi+\operatorname{arcos}\frac{-a}{b}, \pi-\operatorname{arcos}\frac{-a}{b}\right)$.

If, finally, $a<0$ and $|a|>|b|$, η is negative for all values of ξ and the polar equation is now given by (9), where θ has a range of magnitude 2π.

The extended polar graph of the relation $\eta=2+3\cos\xi$ is shewn in Fig. 172. This is composed of the graph $r=2+3\cos\theta$ for the range $(-2{\cdot}301, 2{\cdot}301)$ of θ and the graph $r=-2+3\cos\theta$ for the range $(-0{\cdot}841, 0{\cdot}841)$ of θ.

201. Transformation of Equations of Curves.

We here illustrate the transformation of equations of curves from rectangular to polar coordinates, and vice versa. If the Cartesian equation is $f(x, y)=0$, the corresponding polar equation, with O as pole and Ox as initial line, is obtained by writing $x=r\cos\theta$, $y=r\sin\theta$, and is therefore given by

$$f(r\cos\theta, r\sin\theta)=0. \quad \text{..........................(1)}$$

This transformation is the one most frequently employed. If any other point O', (a, b), is chosen as pole and $O'x'$, at an inclination α to Ox, is taken as initial line, $x=a+r\cos(\theta+\alpha)$, $y=b+r\sin(\theta+\alpha)$, and the polar equation is

$$f\{a+r\cos(\theta+\alpha), b+r\sin(\theta+\alpha)\}=0. \quad \text{..............(2)}$$

In transforming from polars to rectangulars in the simple case above, we write

$$r=\sqrt{(x^2+y^2)}, \quad \cos\theta=x/\sqrt{(x^2+y^2)}, \quad \sin\theta=y/\sqrt{(x^2+y^2)}. \quad \text{...(3)}$$

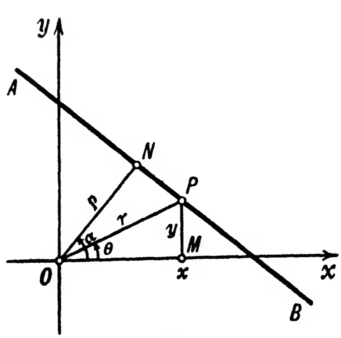

FIG. 173.
$r\cos(\theta-\alpha)=p$.

Ex. 1. *The straight line* $x\cos\alpha+y\sin\alpha=p$, $(p>0)$. Here we get, as the polar equation, $r\cos\theta\cos\alpha+r\sin\theta\sin\alpha=p$, or

$$r\cos(\theta-\alpha)=p. \quad \text{..............(4)}$$

Let AB (Fig. 173) be the given line, P being the point (x, y) on it and ON being perpendicular to it. Then, by projection, we have $ON=OM\cos\angle MON+MP\sin\angle MON$, where $OM=x$, $MP=y$. This equation is identified with the Cartesian equation of the line by taking $p=ON$, $\alpha=\angle MON$.

The polar equation is also obtained directly by observing that it is the equivalent of the equation $ON=OP\cos\angle PON$.

Ex. 2. *The parabola* $y^2 = 4a(x+a)$, $(a>0)$. The focus is here taken as origin. The corresponding polar equation is found from $r^2 = (r\cos\theta + 2a)^2$, which gives, on solving for r,

$$r = \frac{\pm 2a}{1 \mp \cos\theta}. \quad\text{.......(5)}$$

We must discard the solution corresponding to the lower signs in the numerator and denominator, since then r is negative. Thus the polar equation of the parabola with the focus at the pole is

$$r = \frac{2a}{1-\cos\theta} = a\operatorname{cosec}^2 \tfrac{1}{2}\theta. \quad\text{.......(6)}$$

Ex. 3. *The ellipse.* When the origin is at the centre, the (x, y) equation is $x^2/a^2 + y^2/b^2 = 1$, (a supposed greater than b), and the corresponding polar equation is given by

$$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}. \quad\text{....(7)}$$

It is often convenient to take the origin (pole) at a focus, O' say, (Fig. 174), whose coordinates, referred to the centre, are $(-ae, 0)$, where e, the eccentricity, is given by $e^2 = (a^2 - b^2)/a^2$. The Cartesian equation of the ellipse is now

$$\frac{(x-ae)^2}{a^2} + \frac{y^2}{b^2} = 1. \quad\text{....(8)}$$

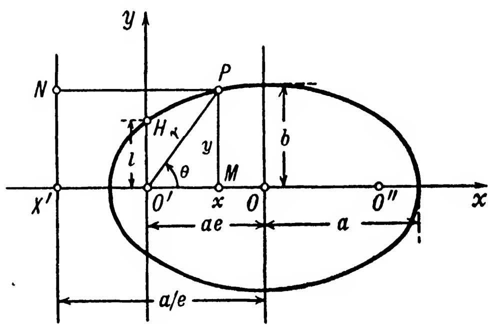

FIG. 174.
$l/r = 1 - e\cos\theta$.

If the length of the semi-latus rectum $O'H$ is l, we have $b^2 = l^2/(1-e^2)$, $a^2 = l^2/(1-e^2)^2$. We now find, on transforming (8) and inserting these values of a^2, b^2,

$$r^2 = l^2 + 2rel\cos\theta + r^2e^2\cos^2\theta, \quad\text{.......(9)}$$

or

$$r = \pm(l + re\cos\theta). \quad\text{.......(10)}$$

We must exclude the negative sign here as it leads to negative values of r. Taking the positive sign, we get, as the required equation,

$$\frac{l}{r} = 1 - e\cos\theta. \quad\text{.......(11)}$$

If the focus O'' is taken as pole and $O''x$ as polar axis, the equation is

$$\frac{l}{r} = 1 + e\cos\theta. \quad\text{.......(12)}$$

The student should obtain these equations directly from the focus-directrix definition of the ellipse.

The parabola $y^2 = 4a(x+a)$ is included in (11), being the special case where $e=1$ and $2a$ takes the place of l.

For the *hyperbola* $x^2/a^2 - y^2/b^2 = 1$, if we take as pole the focus $(-ae, 0)$, where now $e > 1$, we find, in place of (10), noticing that here $a^2 = l^2/(e^2-1)^2$, $b^2 = l^2/(e^2-1)$,

$$r = \pm(l - re\cos\theta), \quad\text{.......(13)}$$

so that
$$\frac{l}{r}=1+e\cos\theta \quad \text{or} \quad \frac{l}{r}=-1+e\cos\theta. \quad \text{......(14)}$$

In the first of these equations, therefore, we must restrict θ to the range $\left[-\operatorname{arcos}\frac{-1}{e}, \operatorname{arcos}\frac{-1}{e}\right]$; this gives the nearer branch. In the second we must restrict θ to the range $\left[-\operatorname{arcos}\frac{1}{e}, \operatorname{arcos}\frac{1}{e}\right]$; this gives the farther branch.

If we take as pole the focus $(ae, 0)$, we find, similarly,
$$\frac{l}{r}=1-e\cos\theta \quad \text{or} \quad \frac{l}{r}=-1-e\cos\theta. \quad \text{......(15)}$$

The first equation gives the nearer branch, θ being restricted to the range $\left[\operatorname{arcos}\frac{1}{e}, 2\pi-\operatorname{arcos}\frac{1}{e}\right]$, and the second gives the farther branch, θ being now restricted to the range $\left[\pi-\operatorname{arcos}\frac{1}{e}, \pi+\operatorname{arcos}\frac{1}{e}\right]$.

Ex. 4. Shew that the transform of the equation $(x^2+y^2)^2=a^2(x^2-y^2)$, $(a>0)$, in polar coordinates is
$$r=a\surd(\cos 2\theta). \quad \text{......(16)}$$

[The curve is called the *lemniscate of Bernoulli*, and is shewn in Fig. 175.]

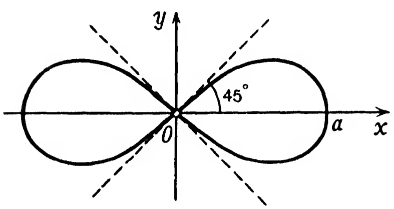

FIG. 175.
$r=a\surd(\cos 2\theta)$.

Ex. 5. *The equation* $r=a\sec\theta+b$, $(0<b<a)$. We here illustrate the transformation from polar to rectangular coordinates. We get
$$\surd(x^2+y^2)=a\frac{\surd(x^2+y^2)}{x}+b, \quad \text{...(17)}$$
or
$$\surd(x^2+y^2).\left(1-\frac{a}{x}\right)=b. \quad \text{......(18)}$$

If now we rationalize this equation to obtain
$$(x^2+y^2)\left(1-\frac{a}{x}\right)^2=b^2, \quad \text{......(19)}$$
we introduce the additional relation
$$\surd(x^2+y^2).\left(1-\frac{a}{x}\right)=-b, \quad \text{......(20)}$$
or
$$r=a\sec\theta-b. \quad \text{......(21)}$$

If, further, we multiply up by x^2 to obtain the *integral* equation
$$(x^2+y^2)(x-a)^2=b^2x^2, \quad \text{......(22)}$$
we must exclude the point (0, 0) from the corresponding graph, for the curves $r=a\sec\theta\pm b$ do not pass through the pole when $a>b$.

Portions of these curves are shewn in Figs. 176-178. They belong to a family of curves known as *Conchoids of Nicomedes*, which are most easily defined as follows: A straight line LM and a point O are taken in a plane, a being the distance of O from LM, and a variable line is drawn through O to cut the fixed line in R. From R equal distances b are measured along OR (in both

senses). The locus of either of the ends P, P' is a conchoid of the type considered. When $b<a$ we get the curve shewn in Fig. 176; when $b=a$, that in Fig. 177, and when $b>a$, that in Fig. 178.

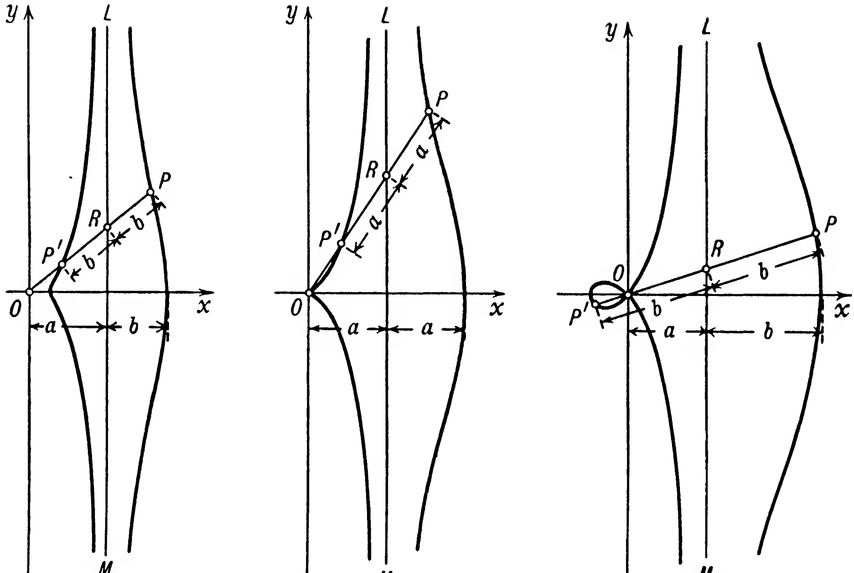

FIG. 176. $b<a$. FIG. 177. $b=a$. FIG. 178. $b>a$.

The curves $r=a\sec\theta\pm b$.

202. Geometrical Interpretation of the Derivative $dr/d\theta$. The Recedent.

We proceed to consider the application of the Differential Calculus to the investigation of certain properties of curves in polar coordinates, examining in the present Article a connection between the derivative $dr/d\theta$ and the inclination of the tangent at a point to the polar radius of the point.

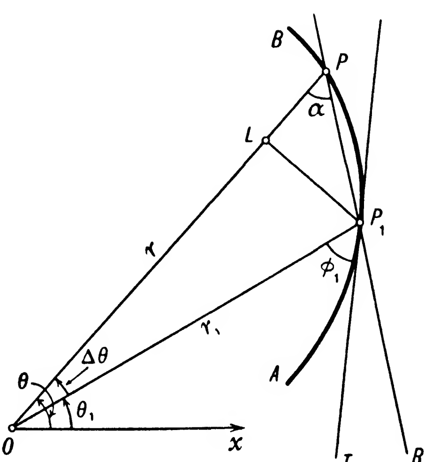

FIG. 179.

Let AB (Fig. 179) be the curve $r=f(\theta)$ in a certain range of θ, and let P_1, (r_1, θ_1), and P, (r, θ), be neighbouring points on the curve, where $r=r_1+\Delta r$, $\theta=\theta_1+\Delta\theta$. Also, let the secant PP_1R make an angle α with OP, and let P_1L be drawn perpendicular to OP.

Assuming the existence of a definite tangent at each point of the curve, let P_1T_1 be the tangent at P_1, making an angle φ_1 with the radius OP_1. The angle φ between the tangent at any point and the radius to the same point is here defined to be *the smallest angle of counter-clockwise rotation required to bring the radius into parallelism with the tangent*; it therefore lies in the range $(0, \pi)$.

Now the secant PP_1R converges to the tangent P_1T_1 as P moves up to P_1 (from either side) and hence α converges to φ_1 in this process. Thus $\tan\alpha$ is a function of θ which is continuous at P_1 provided we take $\tan\varphi_1$ as its value there.

Referring to Fig. 179, we have

$$\tan\alpha = \frac{P_1L}{LP} = \frac{r_1 \sin\Delta\theta}{r_1 + \Delta r - r_1\cos\Delta\theta}. \quad\text{.....................(1)}$$

The last expression in (1) is a continuous function[1] of θ except when $\theta = \theta_1$. It is potentially continuous when $\theta = \theta_1$, for it can be written $\dfrac{r_1\sin\Delta\theta}{2r_1\sin^2\frac{1}{2}\Delta\theta + \Delta r}$, or

$$\frac{r_1\dfrac{\sin\Delta\theta}{\Delta\theta}}{\frac{1}{2}r_1\left(\dfrac{\sin\frac{1}{2}\Delta\theta}{\frac{1}{2}\Delta\theta}\right)^2\Delta\theta + \dfrac{\Delta r}{\Delta\theta}}, \quad\text{.........................(2)}$$

where the terms in the numerator and denominator are all potentially continuous. The completing value is plainly

$$\frac{r_1}{(dr/d\theta)_1}, \quad\text{..................................(3)}$$

where, for the present, we suppose $(dr/d\theta)_1 \neq 0$.

Equation (1) shews that $\tan\alpha$ has the same completing value, so that

$$\tan\varphi_1 = \frac{r_1}{(dr/d\theta)_1}. \quad\text{..............................(4)}$$

Thus for any point of the curve at which $dr/d\theta \neq 0$, we have

$$\tan\varphi = \frac{r}{dr/d\theta} = r\frac{d\theta}{dr}. \quad\text{..........................(5)}$$

If at the point considered $dr/d\theta = 0$, the above argument can be carried out for $\cot\alpha$ to prove that

$$\cot\varphi = \frac{1}{r}\frac{dr}{d\theta}, \quad\text{................................(6)}$$

[1] Since $\Delta\theta$ is equal to $\theta - \theta_1$.

and this shews that when $dr/d\theta = 0$, we have $\cot\varphi = 0$. It follows that where r is stationary with respect to θ, the tangent is at right-angles to the radius, and conversely.

The angle φ is called the *angle of recedence* or simply the *recedent* of the curve at P, as being the angle at which the point P recedes from the pole. Equation (5) shews that it lies in the range $(0, \frac{1}{2}\pi)$ when $dr/d\theta > 0$, that is, when r is an increasing function of θ, and in the range $(\frac{1}{2}\pi, \pi)$ when $dr/d\theta < 0$, that is, when r is a decreasing function of θ.

We can write (6) in the form

$$\cot\varphi = \frac{d}{d\theta}(\log_e r), \qquad \text{.............................(7)}$$

and from (5) we obtain the following results for the other trigonometrical functions of the recedent:

$$\sec^2\varphi = 1 + \left(r\frac{d\theta}{dr}\right)^2, \quad \operatorname{cosec}^2\varphi = 1 + \left(\frac{1}{r}\frac{dr}{d\theta}\right)^2, \qquad \text{...........(8)}$$

$$\sin\varphi = \frac{1}{\sqrt{\left\{1 + \left(\frac{1}{r}\frac{dr}{d\theta}\right)^2\right\}}}, \qquad \text{........................(9)}$$

$$\cos\varphi = \pm\frac{1}{\sqrt{\left\{1 + \left(r\frac{d\theta}{dr}\right)^2\right\}}}. \qquad \text{....................(10)}$$

There is no ambiguity of sign in (9) since $\sin\varphi$ is always positive, but in (10) we must take the positive or the negative sign according as $d\theta/dr$ is positive or negative.

It will be observed that in introducing the above convention for the measurement of the recedent φ it is not necessary to prescribe a *direction* for the tangent. Subsequently (Art. 281) we shall introduce the notion of a *directed* tangent and with it a more general convention for the measurement of φ. The results (5), (6) are, however, unaffected.

202. 1. *Change of dependent variable.* It is often convenient to change the dependent variable from r to its reciprocal. Writing $u = 1/r$, so that the equation of the curve is $u = 1/f(\theta)$, we have

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}, \qquad \text{.............................(11)}$$

and hence, from (5) and (6),

$$\tan\varphi = -u\frac{d\theta}{du}, \quad \cot\varphi = -\frac{1}{u}\frac{du}{d\theta}. \qquad \text{.................(12)}$$

Ex. 1. Shew that if, in Fig. 179, P moves up to P_1, $P_1L/(r\Delta\theta)$ and $LP/\Delta r$ both converge to 1, and hence that $P_1P/\sqrt{\{(\Delta r)^2+(r\Delta\theta)^2\}}$ also converges to 1.

We have

$$\text{(i)}\quad \lim_{P\to P_1}\frac{P_1L}{r\Delta\theta}=\lim_{\Delta\theta\to 0}\frac{\sin\Delta\theta}{\Delta\theta}=1\,; \quad\text{.......(13)}$$

$$\text{(ii)}\quad \lim_{P\to P_1}\frac{LP}{\Delta r}=\lim_{\Delta\theta\to 0}\left\{1+\frac{r(1-\cos\Delta\theta)}{\Delta r}\right\}=\lim_{\Delta\theta\to 0}\left\{1+\frac{2r\sin^2(\frac{1}{2}\Delta\theta)}{\Delta r}\right\}$$

$$=\lim_{\Delta\theta\to 0}\left\{1+r\frac{\sin(\frac{1}{2}\Delta\theta)}{\frac{1}{2}\Delta\theta}\frac{\Delta\theta}{\Delta r}\sin(\tfrac{1}{2}\Delta\theta)\right\}=1, \quad\text{...(14)}$$

since $\dfrac{\sin(\frac{1}{2}\Delta\theta)}{\frac{1}{2}\Delta\theta}\to 1$, $\dfrac{\Delta\theta}{\Delta r}\to\dfrac{d\theta}{dr}$ and $\sin(\frac{1}{2}\Delta\theta)\to 0$ as $\Delta\theta\to 0$;

$$\text{(iii)}\quad \left|\frac{(P_1P)^2}{(\Delta r)^2+(r\Delta\theta)^2}-1\right|=\left|\frac{(LP)^2-(\Delta r)^2+(P_1L)^2-(r\Delta\theta)^2}{(\Delta r)^2+(r\Delta\theta)^2}\right|$$

$$<\left|\left(\frac{LP}{\Delta r}\right)^2-1\right|+\left|\left(\frac{P_1L}{r\Delta\theta}\right)^2-1\right|, \quad\text{.......(15)}$$

and hence
$$\lim_{P\to P_1}\frac{P_1P}{\sqrt{\{(\Delta r)^2+(r\Delta\theta)^2\}}}=1. \quad\text{.......(16)}$$

In the language of infinitesimals, the equivalent statements are that P_1L, LP, P_1P are ultimately equal to $r\,\delta\theta$, δr and $\sqrt{\{(\delta r)^2+(r\,\delta\theta)^2\}}$, respectively.

202. 2. *Working notions.* The formulae (5), (6) can be easily recovered with the use of infinitesimals. Referring to Fig. 180, where PL is drawn perpendicular to OP', the sides PL, LP' of the elementary triangle PLP' are taken as equal to $r\delta\theta$, δr, respectively, and the angle $PP'L$ is treated as the angle between the radius and the tangent, that is, as the recedent at P' (or at P). While thinking of the relations $\tan\varphi=r\delta\theta/\delta r$, $\cot\varphi=\delta r/(r\delta\theta)$, which apply for this triangle, we write down the corresponding accurate results

$$\tan\varphi=r\frac{d\theta}{dr},\quad \cot\varphi=\frac{1}{r}\frac{dr}{d\theta}, \quad\text{.......(17)}$$

which apply for the point P.

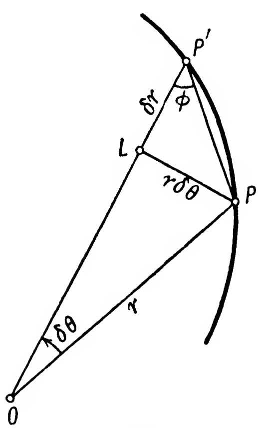

FIG. 180.

Ex. 2. *The spiral of Archimedes* $r=a\theta$. Here $dr/d\theta=a$ and therefore

$$\tan\varphi=\frac{r}{dr/d\theta}=\theta. \quad\text{.......(18)}$$

Hence $\tan\varphi\to\infty$ as $\theta\to\infty$ and the tangent is ultimately perpendicular to the radius. Referring to Fig. 162, p. 471, the value of φ at the point B is artan 2π, or about 1·413 radians, that is, about 80° 57′.

Ex. 3. *The equiangular spiral* $r=ae^{k\theta}$, $(a>0)$. Here $dr/d\theta=kr$, and hence

$$\tan\varphi=\frac{r}{kr}=\frac{1}{k}. \quad\text{.......(19)}$$

Thus the angle of recedence is the same for all points on the curve, which may

accordingly be called the *spiral of constant recedent.* The term *equiangular* is, however, traditionally employed in this connection.

Ex. 4. *The parabola* $\dfrac{2a}{r}=1-\cos\theta$, $(a>0)$. This is the equation of the curve when the focus is taken as the pole (Ex. 2, p. 477). Changing the dependent variable to $u(=1/r)$, the equation of the curve becomes

$$2au=1-\cos\theta, \quad\text{.................................(20)}$$

and so $2a\,du/d\theta=\sin\theta$. We now have, from equation (12),

$$\cot\varphi=-\frac{\sin\theta}{1-\cos\theta}=-\cot\tfrac{1}{2}\theta, \quad\text{.........................(21)}$$

and therefore

$$\varphi=n\pi-\tfrac{1}{2}\theta, \quad\text{..(22)}$$

where n is an integer chosen so that φ lies in the range $(0, \pi)$.

For the description of the complete curve it is sufficient to restrict θ to the range $(0, 2\pi)$, and we then have

$$\varphi=\pi-\tfrac{1}{2}\theta. \quad\text{...................................(23)}$$

Ex. 5. *The cardioid* $r=a(1+\cos\theta)$, $(a>0)$. Here $dr/d\theta=-a\sin\theta$, and hence, by equation (5),

$$\cot\varphi=-\frac{\sin\theta}{1+\cos\theta}=-\tan\tfrac{1}{2}\theta. \quad\text{.........................(24)}$$

Therefore

$$\varphi=\tfrac{1}{2}\theta+\tfrac{1}{2}\pi+n\pi, \quad\text{....................................(25)}$$

where n is an integer chosen so that φ lies in the range $(0, \pi)$. In order to obtain the complete graph it is sufficient to restrict θ to the range $(-\pi, \pi)$, starting from O (Fig. 181) where $\theta=-\pi$, and proceeding by way of A, B, C back to O where now $\theta=\pi$. On this convention we have, for all points on the curve,

$$\varphi=\tfrac{1}{2}\theta+\tfrac{1}{2}\pi. \quad\text{..................(26)}$$

It will be observed that for the lower half OAB, φ has the range $(0, \tfrac{1}{2}\pi)$, and for the upper half BCO, the range $(\tfrac{1}{2}\pi, \pi)$.

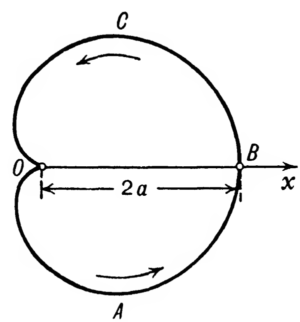

FIG. 181.
$r=a(1+\cos\theta)$.

Ex. 6. *The equation* $r^m=a^m\sin m\theta$, $(a>0)$. Here $r>0$ only if $m\theta$ lies in a range $(2n\pi, 2n\pi+\pi)$. We find

$$\cot\varphi=\frac{1}{r}\frac{dr}{d\theta}=\cot m\theta, \quad\text{..........(27)}$$

and hence, taking into account the above restrictions,

$$\varphi=m\theta+2n'\pi, \quad\text{.................................(28)}$$

where n' is an integer chosen so that φ lies in $(0, \pi)$.

The curves corresponding to different values of m are described, as a class, as *sine spirals*, and a similar description is applied to the curves given by the equation $r^m=a^m\cos m\theta$, which differ in position only from the former. The typical spiral character is, however, only exhibited by the curves for which m is a small fraction.

We note the following special forms.

m	Equation	Name	m	Equation	Name
$\frac{1}{2}$	$r = a(1+\cos\theta)$	Cardioid	$-\frac{1}{2}$	$\frac{2a}{r} = 1 - \cos\theta$	Parabola
1	$r = 2a\sin\theta$	Circle	-1	$r\cos\theta = a$	Straight line
2	$r^2 = a^2\cos 2\theta$	Lemniscate	-2	$r^2\cos 2\theta = a^2$	Rectangular hyperbola

202. 3. *The polar 'tangent', 'normal', 'subtangent', 'subnormal'.* Let PT, PN be the tangent and normal at P (Fig. 182), and let a line through O perpendicular to OP meet these lines in T, N. In books on Geometry, the lengths PT, PN, OT, ON are called respectively the *polar tangent, polar normal, polar subtangent, polar subnormal*, corresponding to the point P. We have the following results :

(i) *Polar tangent*: $$PT = r\sqrt{\left\{1 + \left(r\frac{d\theta}{dr}\right)^2\right\}}, \quad \text{.....................(29)}$$

(ii) *Polar normal* : $$PN = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}, \quad \text{.....................(30)}$$

(iii) *Polar subtangent*: $$OT = r^2\left|\frac{d\theta}{dr}\right| = \frac{1}{|du/d\theta|}, \quad \text{.....................(31)}$$

(iv) *Polar subnormal* : $$ON = \left|\frac{dr}{d\theta}\right|. \quad \text{.....................(32)}$$

Ex. 7. *The spiral of Archimedes* $r = a\theta$, $(a > 0)$. Here $dr/d\theta = a$, and (32) shows that the subnormal has the constant length a, so that the point N (Fig. 182) lies on a circle of radius a and centre O. This leads to a simple construction for the normal at any point P. For the point N is determined by drawing a line through O perpendicular to OP to cut the circle in question.

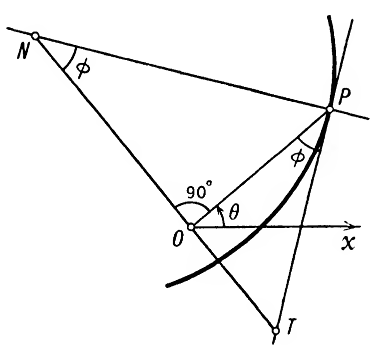

FIG. 182.

203. Concavity and Convexity in Polar Coordinates.

The notion of concavity (or convexity) of a curve with respect to an assigned direction has been introduced in Section III, Chap. IV, for curves in rectangular coordinates. Associating the positive side of a straight line with the half-plane entered by a point moving in the assigned direction when it crosses the line, we describe a curve

as concave (convex) with respect to the assigned direction at a point P on it when its deflection in the neighbourhood of P from the tangent there is on the positive (negative) side of this tangent.

In dealing with curves in polar coordinates, the radius to the point considered, drawn outwards (or else inwards), naturally suggests itself as a suitable direction for the purpose of describing the form relative to the tangent at the point. Throughout this Book the *outward drawn radius* is chosen as the assigned direction, so that a curve is convex at P if its deflection in the neighbourhood of P from the tangent there is on the same side of the tangent as the pole. In briefer terms, we may describe a curve which is convex with respect to the outward drawn radius at a point as *convex outwards* at the point. If a curve is convex outwards at all points in a range, we describe it as *convex outwards in the range.*[1]

We now investigate analytical conditions in polar coordinates for the determination of the local form of a curve relative to the tangent at the point considered. Let this be the point P_1, (r_1, θ_1). We suppose, at first, that the tangent at P_1 does not pass through the pole.

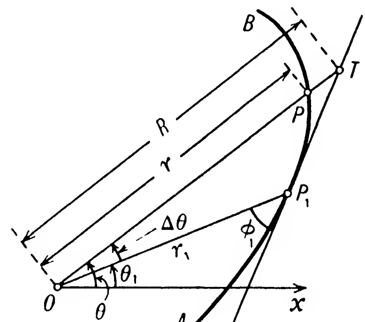

FIG. 183.

Let P, (r, θ), (Fig. 183) be any point on the curve, and let the radius OP be produced to meet, at T, the tangent P_1T to the curve at P_1. Denoting OT by R, the form of the curve in the neighbourhood of P_1 can be inferred if the nature of the variation of the difference $R - r$ is known as P passes through P_1. If this difference is positive (negative) on both sides of P_1, the curve is convex (concave) outwards at P_1; if it changes sign, the curve crosses the tangent at P_1, which is then a point of contraflexure.

In place of the difference $R - r$ we can work with the difference $\frac{1}{r} - \frac{1}{R}$, or $u - U$, say, where $u = 1/r$, $U = 1/R$, and this is found to be the most convenient procedure for the analytical discussion. This difference is expressed in terms of polar derivatives[2] (of the form

[1] Curves in polar coordinates are sometimes described as concave (convex) *with respect to the pole.* A curve which is concave outwards under the convention introduced above would thus be described as convex with respect to the pole.

[2] As in the corresponding case of rectangular coordinates, we assume that all the derivatives introduced are continuous.

$d^k u/d\theta^k$) by applying the general *I.D.T.*, and the investigation of the variation of the difference as P passes through P_1 is actually replaced by the investigation of the sign, *at* P_1, of an expression involving polar derivatives.

In the triangle OTP_1 we have $\angle P_1OT = \theta - \theta_1 = \Delta\theta$, $\angle OTP_1 = \varphi_1 - \Delta\theta$, and hence

$$\frac{R}{\sin(\pi - \varphi_1)} = \frac{r_1}{\sin(\varphi_1 - \Delta\theta)}. \quad \text{.......................(1)}$$

Therefore

$$u - U = u - u_1 \cos\Delta\theta + u_1 \cot\varphi_1 \sin\Delta\theta, \quad \text{..............(2)}$$

or, writing γ for the variable $\Delta\theta$, we have, on recalling the result (12), Art. 202. 1,

$$u - U = u - u_1 \cos\gamma - \left(\frac{du}{d\theta}\right)_{\theta_1} \sin\gamma = F(\gamma), \text{ say.} \quad \text{...........(3)}$$

Now by the $I.D_n.T.$ we have

$$F(\gamma) = F(0) + \gamma F^{(1)}(0) + \ldots + \frac{\gamma^n}{n!} F^{(n)}(\bar{\gamma}), \quad \text{..............(4)}$$

where $\bar{\gamma}$ lies in the range $(0, \gamma)$, and from (3) we have

$$F^{(n)}(\gamma) = \frac{d^n u}{d\gamma^n} - u_1 \cos\left(\gamma + \frac{n}{2}\pi\right) - \left(\frac{du}{d\theta}\right)_{\theta_1} \sin\left(\gamma + \frac{n}{2}\pi\right). \quad \text{......(5)}$$

Also[1] $\dfrac{d^n u}{d\gamma^n} = \dfrac{d^n u}{d\theta^n}$, and hence, if k is any positive integer, we have

$$F^{(k)}(0) = \left(\frac{d^k u}{d\theta^k}\right)_{\theta_1} - u_1 \cos\frac{k}{2}\pi - \left(\frac{du}{d\theta}\right)_{\theta_1} \sin\frac{k}{2}\pi, \quad \text{.................(6)}$$

$$F^{(n)}(\bar{\gamma}) = \left(\frac{d^n u}{d\theta^n}\right)_{\bar{\theta}} - u_1 \cos\left(\bar{\gamma} + \frac{n}{2}\pi\right) - \left(\frac{du}{d\theta}\right)_{\theta_1} \sin\left(\bar{\gamma} + \frac{n}{2}\pi\right). \quad \text{...(7)}$$

Observing that $F(0) = 0$, $F^{(1)}(0) = 0$, equation (4) thus leads to the result

$$u - U = \frac{\gamma^2}{2!}\left\{\left(\frac{d^2 u}{d\theta^2}\right)_{\theta_1} + u_1\right\} + \frac{\gamma^3}{3!}\left\{\left(\frac{d^3 u}{d\theta^3}\right)_{\theta_1} + \left(\frac{du}{d\theta}\right)_{\theta_1}\right\} + \ldots$$
$$+ \frac{\gamma^n}{n!}\overline{\left\{\frac{d^n u}{d\theta^n} - u_1 \cos\left(\gamma + \frac{n}{2}\pi\right) - \left(\frac{du}{d\theta}\right)_{\theta_1} \sin\left(\gamma + \frac{n}{2}\pi\right)\right\}}, \quad \text{...(8)}$$

where the bar indicates that the value of the expression in braces is to be taken at some intermediate point of the range (θ_1, θ).

Suppose now that $(d^2u/d\theta^2)_{\theta_1} + u_1 \neq 0$, and take $n = 2$, so that

$$u - U = \tfrac{1}{2}\gamma^2 \overline{\left\{\frac{d^2 u}{d\theta^2} + u_1 \cos\gamma + \left(\frac{du}{d\theta}\right)_{\theta_1} \sin\gamma\right\}}. \quad \text{............(9)}$$

[1] We have $\dfrac{du}{d\gamma} = \dfrac{du}{d\theta}\dfrac{d(\theta_1 + \gamma)}{d\gamma} = \dfrac{du}{d\theta}$, and similarly for the successive derivatives.

Since u and its derivatives with respect to θ are assumed continuous at θ_1, there is a neighbourhood of this point within which the expression in braces has the same sign as at the point θ_1, $(\gamma=0)$. It follows that in this neighbourhood the difference $u-U$ has the same sign as $(d^2u/d\theta^2)_{\theta_1}+u_1$. If this sign is positive, then $u>U$ or $R>r$ on both sides of P_1, and the curve is convex outwards at P_1. If the sign is negative, the curve is concave outwards at P_1.

Suppose next that $(d^2u/d\theta^2)_{\theta_1}+u_1=0$ but $(d^3u/d\theta^3)_{\theta_1}+(du/d\theta)_{\theta_1}\neq 0$. Employing the $I.D_3.T.$ we conclude that the difference $u-U$, and therefore $R-r$, changes sign with γ, i.e. as P passes through P_1 which is now a point of contraflexure. If the sign of the above non-vanishing term is positive (negative) at P_1, the curve is concave (convex) outwards for $\theta<\theta_1$, and convex (concave) outwards for $\theta>\theta_1$.

We may now state generally that if the first non-vanishing derivative of $F(\gamma)$ at P_1 is odd, P_1 is a point of contraflexure, while if even, the curve is convex or concave outwards according as the sign of the derivative there is positive or negative.

It is usually unnecessary to proceed beyond the third derivative criterion, and we have the following summary of results[1]:

(i) *Second Derivative Criterion.* The curve is convex or concave outwards at P according as

$$\frac{d^2u}{d\theta^2}+u, \quad \text{or} \quad r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}, \quad \text{or} \quad \frac{\dfrac{d\theta}{dr}\left\{2+r^2\left(\dfrac{d\theta}{dr}\right)^2\right\}+r\dfrac{d^2\theta}{dr^2}}{\left(\dfrac{d\theta}{dr}\right)^3} \quad \ldots(10)$$

is positive or negative at P.

(ii) *Third Derivative Criterion.* If the above expressions vanish the form of the curve changes from concave (convex) outwards to convex (concave) outwards with increase of θ if

$$\frac{d^3u}{d\theta^3}+\frac{du}{d\theta}, \quad \text{or} \quad \frac{dr}{d\theta}\left\{6r\frac{d^2r}{d\theta^2}-6\left(\frac{dr}{d\theta}\right)^2-r^2\right\}-r^2\frac{d^3r}{d\theta^3},$$

$$\text{or} \quad \frac{-\left(\dfrac{d\theta}{dr}\right)^2\left\{6+r^2\left(\dfrac{d\theta}{dr}\right)^2\right\}-3r\dfrac{d^2\theta}{dr^2}\left(r\dfrac{d^2\theta}{dr^2}+2\dfrac{d\theta}{dr}\right)+r^2\dfrac{d\theta}{dr}\dfrac{d^3\theta}{dr^3}}{\left(\dfrac{d\theta}{dr}\right)^5} \quad \ldots\ldots(11)$$

is positive (negative) at P. In either case there is a point of contraflexure at P.

[1] The third form in each case is obtained from the second by employing the results of Art. 89. 3.

Ex. 1. *The equation* $r = a/\sqrt{\theta}$, $(a > 0)$. Here we have $au = \sqrt{\theta}$, so that

$$\frac{du}{d\theta} = \frac{1}{2a\theta^{1/2}}, \quad \frac{d^2u}{d\theta^2} = -\frac{1}{4a\theta^{3/2}}, \quad \frac{d^3u}{d\theta^3} = \frac{3}{8a\theta^{5/2}}.$$

We now find

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{4a\theta^{3/2}}(1 - 4\theta^2), \quad \frac{d^3u}{d\theta^3} + \frac{du}{d\theta} = \frac{3}{8a\theta^{5/2}} + \frac{1}{2a\theta^{1/2}}. \quad \text{..........(12)}$$

The first of these expressions vanishes when $\theta = 1/2$, and then the sign of the second expression is positive. The point $(a\sqrt{2}, \frac{1}{2})$ is thus a point of contraflexure, the curve being convex outwards when $\theta > \frac{1}{2}$ and concave outwards when $\theta < \frac{1}{2}$.

The curve is of spiral form and is known as the *Lituus*. Portion of it is

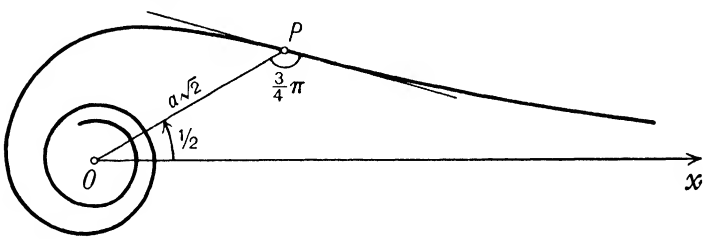

FIG. 184.
$r = a/\sqrt{\theta}$.

shewn in Fig. 184. We observe that as $\theta \to 0$, $r \to \infty$, while as $\theta \to \infty$, $r \to 0$. Also,

$$\tan\varphi = -u\frac{d\theta}{du} = -2\theta, \quad \text{............................(13)}$$

and at the point of contraflexure this gives $\tan\varphi = -1$, so that the value of the recedent there is $\frac{3}{4}\pi$.

203. 1. *Exceptional cases.* (i) *Tangent through the pole.* At the point P_1 concerned we now have $\varphi = 0$, $\tan\varphi = 0$, i.e. $\dfrac{r}{dr/d\theta} = 0$, and since $r \neq 0$, by supposition, we must have $|dr/d\theta| = \infty$. Two possible cases arise, as illustrated in Figs. 185, 186. In each there is a change from concavity to convexity with respect to the outward drawn radius as we pass through P, but in the second case the radius r is not a single-valued function of θ in the neighbourhood of P.

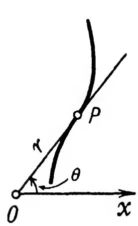

FIG. 185.

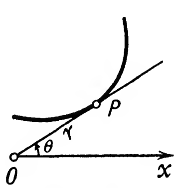

FIG. 186.

We can discuss the form of the curve at P directly by introducing r as the independent variable, θ being in each case a single-valued function of r. At P we have $d\theta/dr = 0$, and if $d^2\theta/dr^2 \neq 0$, the stationary value of θ corresponding to P is either a maximum or a minimum. In Fig. 186, $d^2\theta/dr^2 > 0$, and θ is a minimum. If, however, $d^2\theta/dr^2 = 0$, $d^3\theta/dr^3 \neq 0$, we have a point of contraflexure, as in Fig. 185. The discussion of higher derivative criteria in such cases is left to the student.

(ii) *Curve through the pole.* The curve is now always convex outwards in the neighbourhood of the pole, whether this is a point of contraflexure or not, as illustrated in Figs. 187, 188. This case is considered analytically in Art. 204. 2 below, where also a formula for the curvature at the pole is investigated.

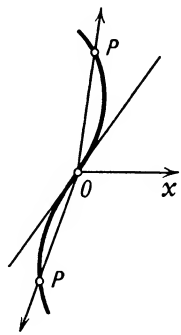

FIG. 187.

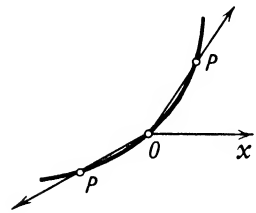

FIG. 188.

204. Curvature in Polar Coordinates.

If the equation of a curve is $r=f(\theta)$ and its curvature at any point is required, we might proceed to find the corresponding equation of the curve in rectangular coordinates and then use the formula for the curvature already obtained, Art. 133. It is, however, usually more convenient to employ a formula involving polar coordinates and derivatives, and we proceed to investigate the standard formula of this type.

The question of an appropriate convention for the sign of the curvature must first be considered. This is practically settled in the above discussion of convexity and concavity, for we can hardly avoid making the sign of the curvature the same at all points where the curve is convex (concave) outwards. It only remains to decide whether positive curvature is to correspond to convexity or to concavity outwards. We adopt the former alternative, that is, the curvature is to be positive or negative according as the curve is convex or concave with respect to the outward radius.

This convention is directly connected with the second main convention for the sign of the curvature of a curve, as explained in Art. 133. 1. On this latter convention we suppose a curve described by the continuous motion of a point on it in a definite sense, and we draw rectangular axes Px', Py', the former being along the tangent and in the sense of description of the curve at P, the latter being obtained by counter-clockwise rotation through a right-angle from Px'. The sign of the curvature is then determined relative to these axes, being positive (negative) when the curve is concave (convex) relative to the direction Py'. It is evident that these axes may be replaced by any system of parallel axes, so long as their directions are unchanged. In the present case, if we take the sense of description to be that corresponding to θ increasing,

and draw axes of x' and y' as described, a curve which is convex with respect to the outward drawn radius at a point is concave with respect to the axis of y', and will, therefore, have positive curvature.

The standard formula for κ in terms of the variables r, θ and their derivatives can be obtained in various ways. We may, for example, transform the ordinary formula $\kappa=\dfrac{dg/dx}{(1+g^2)^{3/2}}$ with the aid of the relations $x=r\cos\theta$, $y=r\sin\theta$. The treatment given here illustrates the use of the original measure of the curvature as the rate of change of gradient with respect to the tangent at the point considered.

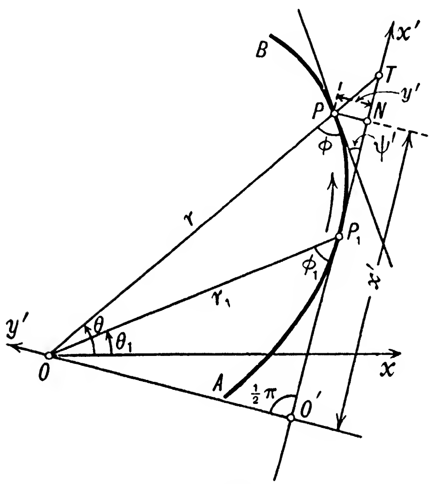

FIG. 189.

Let AB (Fig. 189) be a curve which is convex outwards in the neighbourhood of P_1, (r_1, θ_1), the point at which the curvature is to be investigated, and let P, (r, θ), be an adjacent point. The tangent P_1T is drawn at P_1, and OO' is the perpendicular to it from O. The sense of description of the curve being supposed counter-clockwise, as indicated, we draw the axes $O'x'$, $O'y'$ in the directions shewn, the point O' being chosen as origin instead of P_1 for convenience. The slope of the tangent at P relative to $O'x'$ is ψ', and if $O'N=x'$, where PN is perpendicular to $O'x'$, we have, from the definition of Art. 131,

$$\kappa_1=\left(\frac{d\tan\psi'}{dx'}\right)_{P_1}=\left(\frac{d\psi'}{dx'}\right)_{P_1}, \quad \text{.......................(1)}$$

since $\psi'=0$ and $\sec^2\psi'=1$ at P_1. Introducing the intermediary variable θ, we can write this in the form

$$\kappa_1=\left(\frac{d\psi'/d\theta}{dx'/d\theta}\right)_{P_1}. \quad \text{..................................(2)}$$

Now, from the Figure, we have $\varphi=\psi'+\angle PTN=\psi'+\varphi_1-(\theta-\theta_1)$, so that

$$\psi'=\theta-\theta_1+\varphi-\varphi_1. \quad \text{..........................(3)}$$

Also $$x'=OP\cos\angle PTN=r\cos(\varphi_1-\theta+\theta_1). \quad \text{...............(4)}$$

We therefore have

$$\frac{d\psi'}{d\theta}=1+\frac{d\varphi}{d\theta},\quad \frac{dx'}{d\theta}=\frac{dr}{d\theta}\cos(\varphi_1-\theta+\theta_1)+r\sin(\varphi_1-\theta+\theta_1). \quad \ldots.(5)$$

At P_1 we have $\theta=\theta_1$ and hence

$$\left(\frac{dx'}{d\theta}\right)_{\theta_1}=\left(\frac{dr}{d\theta}\cos\varphi+r\sin\varphi\right)_{P_1}=(r\cot\varphi\cos\varphi+r\sin\varphi)_{P_1}=r_1\operatorname{cosec}\varphi_1. \quad (6)$$

Thus equation (2) becomes

$$\kappa_1=\left(\frac{1+d\varphi/d\theta}{r\operatorname{cosec}\varphi}\right)_{P_1}. \quad \ldots\ldots\ldots\ldots(7)$$

Further we may write

$$\begin{aligned}1+\frac{d\varphi}{d\theta}&=1-\sin^2\varphi\frac{d\cot\varphi}{d\theta}=\sin^2\varphi\left(\operatorname{cosec}^2\varphi-\frac{d\cot\varphi}{d\theta}\right)\\&=\sin^2\varphi\left\{1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2-\frac{d}{d\theta}\left(\frac{1}{r}\frac{dr}{d\theta}\right)\right\}\\&=\frac{\sin^2\varphi}{r^2}\left\{r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}\right\}, \quad \ldots.(8)\end{aligned}$$

and the formula (7) becomes

$$\kappa_1=\left\{\frac{r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}}{r^3\operatorname{cosec}^3\varphi}\right\}_{P_1}=\left[\frac{r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}}{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}\right]_{P_1}. \quad \ldots\ldots(9)$$

Dropping the suffix, we thus have as the standard formula for the curvature at any point,

$$\kappa=\frac{r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}}{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}. \quad \ldots\ldots\ldots\ldots(10)$$

The radius of curvature ϱ at any point is the reciprocal of the magnitude of κ.

It should be noticed that if the curve AB is concave outwards at P_1, the slope ψ' at P, for which $\theta>\theta_1$, lies in the range $(\frac{1}{2}\pi, \pi)$. We now have $\psi'=\varphi+\theta-\theta_1+\pi-\varphi_1$, and $d\psi'/d\theta$ is negative, but the final result is unaffected.

If we introduce u as dependent variable, where $u=1/r$, then

$$\frac{dr}{d\theta}=-\frac{1}{u^2}\frac{du}{d\theta},\quad \frac{d^2r}{d\theta^2}=\frac{2}{u^3}\left(\frac{du}{d\theta}\right)^2-\frac{1}{u^2}\frac{d^2u}{d\theta^2},$$

and the formula (10) becomes

$$\kappa = \frac{u^3\left(u + \dfrac{d^2u}{d\theta^2}\right)}{\left\{u^2 + \left(\dfrac{du}{d\theta}\right)^2\right\}^{3/2}}, \quad \text{.........................(11)}$$

which is sometimes a more manageable form.

204. 1. *Working notions.* The formula (11) for the curvature is easily derived informally from Newton's formula, with the aid of the formula (9) of Art. 203, namely

$$u - U = \tfrac{1}{2}\gamma^2 \left\{\frac{d^2u}{d\theta^2} + u_1 \cos\gamma + \left(\frac{du}{d\theta}\right)_{\theta_1} \sin\gamma\right\}. \quad \text{..................(12)}$$

In Fig. 189 we take

$$PN = PT \sin \angle PTN \approx (R - r)\sin\varphi_1 \approx r_1{}^2(u - U)\sin\varphi_1$$

$$\approx \tfrac{1}{2} r_1{}^2\gamma^2 \sin\varphi_1 \left(\frac{d^2u}{d\theta^2} + u\right)_1, \quad \text{............(13)}$$

and $$(P_1N)^2 \approx (P_1P)^2 \approx r_1{}^2 \sin^2\gamma / \sin^2\varphi_1. \quad \text{..............................(14)}$$

Hence $$\kappa = \lim_{P_1N \to 0} \frac{2PN}{(P_1N)^2} = \frac{1}{\operatorname{cosec}^3\varphi_1}\left(\frac{d^2u}{d\theta^2} + u\right)_1, \quad \text{.................(15)}$$

where $\operatorname{cosec}^2\varphi_1 = \left\{1 + \dfrac{1}{u^2}\left(\dfrac{du}{d\theta}\right)^2\right\}_1$.

Ex. 1. *The spiral of Archimedes* $r = a\theta$, $(a > 0)$. Here $dr/d\theta = a$, $d^2r/d\theta^2 = 0$, and hence, by (10),

$$\kappa = \frac{r^2 + 2a^2}{(r^2 + a^2)^{3/2}}. \quad \text{...............................(16)}$$

Ex. 2. *The cardioid* $r = a(1 + \cos\theta)$, $(a > 0)$. Here we have $dr/d\theta = -a\sin\theta$, $d^2r/d\theta^2 = -a\cos\theta$, and the formula (10) leads to the result

$$\kappa = \frac{3}{2\sqrt{(2ar)}} = \frac{3}{4a\,|\cos\frac{1}{2}\theta\,|}. \quad \text{............................(17)}$$

If, as in Ex. 5, Art. 202. 2, p. 483, we restrict θ to the range $(-\pi, \pi)$, which is sufficient for the description of the complete graph, the modulus signs may be omitted in (17) since $\cos\frac{1}{2}\theta$ is positive throughout.

Ex. 3. *The equiangular spiral* $r = ae^{\theta\cot\alpha}$, $(a > 0)$. Here we have

$$dr/d\theta = r\cot\alpha, \quad d^2r/d\theta^2 = r\cot^2\alpha,$$

and (10) leads to the result

$$\kappa = \frac{\sin\alpha}{r}. \quad \text{................(18)}$$

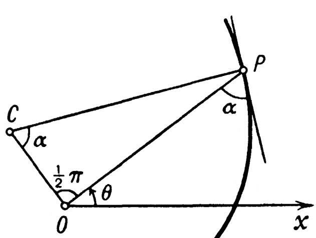

FIG. 190.

The radius of curvature is thus equal to $r\,|\operatorname{cosec}\alpha\,|$. Referring to Fig. 190, which represents a portion of the spiral for which α lies in the range $(0, \frac{1}{2}\pi)$, PC is the normal at P and OC the perpendicular to OP. Thus

$$CP = OP\operatorname{cosec}\alpha = r\operatorname{cosec}\alpha, \quad \text{......(19)}$$

so that C is the centre of curvature at P.

204. 2. *Failing cases.* (i) *Tangent through the pole.* In this case $dr/d\theta$ is infinite and the above formula (10) is not then applicable.

If we introduce r as independent variable, writing

$$\frac{dr}{d\theta}=\frac{1}{d\theta/dr}, \quad \frac{d^2r}{d\theta^2}=-\frac{d^2\theta/dr^2}{(d\theta/dr)^3},$$

the curvature at any point is given by

$$\kappa=\frac{r^2\left(\frac{d\theta}{dr}\right)^3+2\frac{d\theta}{dr}+r\frac{d^2\theta}{dr^2}}{\left\{1+r^2\left(\frac{d\theta}{dr}\right)^2\right\}^{3/2}}. \quad \text{.........................(20)}$$

In the case under discussion $d\theta/dr=0$, and (20) therefore gives

$$\kappa=r\frac{d^2\theta}{dr^2}, \quad \text{.................................(21)}$$

but it must now be noted that the sign of the curvature changes as we pass through the point ; see Figs. 185, 186. The formula (21) must therefore be taken as giving only the magnitude of the curvature at the point.

If $d^2\theta/dr^2=0$, $\kappa=0$ and the point is, in general, a point of contraflexure.

(ii) *Curve through the pole.* The angle θ is now undefined at the pole O and is, in general, discontinuous in passing through it. Also, the radius r is at the lower terminal 0 of its range there. In consequence we must consider separately each of the parts, such as OA, of the curve proceeding from O (Fig. 191). Considering this part, it is convenient to take r as independent variable, and we suppose that the angle θ converges to a definite value when the point P, (r, θ), moves along OA up to O so that there is a definite one-sided tangent at O to the branch OA. We suppose, further, that the derivatives of θ with respect to r which are introduced are continuous in a neighbourhood of the pole and that they converge to finite limits when P moves up to O.

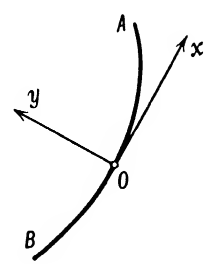

FIG. 191.

If we now introduce rectangular coordinates at O with Ox along the tangent to OA and take Ox as initial line in the polar coordinates, θ converges to 0 when O is approached. It is easy to see that, on the assumptions made, the derivatives of y with respect to x are continuous and converge to finite limits as O is approached. For, from the relations $x=r\cos\theta$, $y=r\sin\theta$, we have

$$\frac{dy}{dx}=\frac{dy/dr}{dx/dr}=\frac{\sin\theta+r\cos\theta\frac{d\theta}{dr}}{\cos\theta-r\sin\theta\frac{d\theta}{dr}}, \quad \text{.........................(22)}$$

and so on. The denominator of the expression for dy/dx does not vanish in a range terminating at O since $\cos\theta\to 1$ and $r\sin\theta\frac{d\theta}{dr}\to 0$. Thus dy/dx converges to a finite value, namely 0, and corresponding results will be arrived at for the higher derivatives.

We can now use the two $I.D_n.T.$'s

$$\theta = r\left(\frac{d\theta}{dr}\right)_0 + \frac{1}{2!}r^2\left(\frac{d^2\theta}{dr^2}\right)_0 + \ldots + \frac{1}{n!}r^n\overline{\left(\frac{d^n\theta}{dr^n}\right)}, \qquad (23)$$

$$y = \frac{1}{2!}x^2\left(\frac{d^2y}{dx^2}\right)_0 + \ldots + \frac{1}{n!}x^n\overline{\left(\frac{d^ny}{dx^n}\right)}, \qquad (24)$$

where, in each case, all the derivatives, except the last, have the values to which they converge when r, or x, converges to 0.

Since $\theta/r \to (d\theta/dr)_0$ and

$$\frac{y}{x^2} = \frac{\sin\theta}{r\cos^2\theta} = \frac{\theta}{r}\,\frac{\sin\theta}{\theta}\,\frac{1}{\cos^2\theta},$$

y/x^2 converges to the same value as θ/r, that is, to $(d\theta/dr)_0$. But from (24), $y/x^2 \to \frac{1}{2}(d^2y/dx^2)_0$, and hence we must have

$$\frac{1}{2}\left(\frac{d^2y}{dx^2}\right)_0 = \left(\frac{d\theta}{dr}\right)_0. \qquad (25)$$

In the same way, if $(d\theta/dr)_0 = 0$ we find that θ/r^2 and y/x^3 converge to $\frac{1}{2}(d^2\theta/dr^2)_0$, and hence that

$$\frac{1}{3}\left(\frac{d^3y}{dx^3}\right)_0 = \left(\frac{d^2\theta}{dr^2}\right)_0. \qquad (26)$$

So, generally, if $\left(\frac{d\theta}{dr}\right)_0 = \left(\frac{d^2\theta}{dr^2}\right)_0 = \ldots = \left(\frac{d^{n-1}\theta}{dr^{n-1}}\right)_0 = 0$ we find that

$$\frac{1}{n+1}\left(\frac{d^{n+1}y}{dx^{n+1}}\right)_0 = \left(\frac{d^n\theta}{dr^n}\right)_0. \qquad (27)$$

The results of investigations already made, in terms of rectangular coordinates, of the nature of a curve in the neighbourhood of a point can now be expressed in terms of polar coordinates. In particular,

(i) the curvature at O, which is measured by $\lim\limits_{x\to 0}\frac{2y}{x^2}$, is given by

$$\kappa_0 = 2\lim_{r\to 0}\frac{\theta}{r} = 2\left(\frac{d\theta}{dr}\right)_0; \qquad (28)$$

(ii) the deflection of the curve from the tangent at O is of the order of x^n if $d^{n-1}\theta/dr^{n-1}$ is the lowest derivative which does not converge to zero at O.

Throughout the preceding investigation we may, if convenient, take $\tan\theta$ or $\sin\theta$ as the dependent variable, the conditions stated taking the same forms as above with $\left(\frac{d^n\tan\theta}{dr^n}\right)_0$ or $\left(\frac{d^n\sin\theta}{dr^n}\right)_0$ substituted in place of $\left(\frac{d^n\theta}{dr^n}\right)_0$.

If the initial line makes an angle α with the tangent at O and θ' is the polar angle relative to a new initial line along the tangent, we have $\theta = \theta' + \alpha$, and therefore $\frac{d^n\theta}{dr^n} = \frac{d^n\theta'}{dr^n}$. The results involving derivatives with respect to r obtained above are therefore unchanged on introducing θ instead of θ' as dependent variable.

If the equation of the curve is $r = f(\theta)$, we have $\frac{d\theta}{dr} = \frac{1}{f'(\theta)}$. If the tangent at

the pole makes an angle α with the initial line, the curvature there of the branch considered is given by

$$\kappa_0 = \frac{2}{f'(\alpha)}. \qquad\qquad (29)$$

Ex. 4. Investigate the curvature of the circle $r = 2a\sin\theta$ at the pole.

We have $f(\theta) = 2a\sin\theta$, $f'(\theta) = 2a\cos\theta$.

Consider the branch OA (Fig. 192). For it $\theta \to 0$ as we approach the pole, and therefore the curvature there is given by

$$\kappa_0 = \frac{2}{f'(0)} = \frac{1}{a}. \qquad\qquad (30)$$

For the branch OB, $\theta \to \pi$ as we approach the pole, i.e. $\alpha = \pi$. We now have

$$\kappa_0 = \frac{2}{f'(\pi)} = -\frac{1}{a}. \qquad\qquad (31)$$

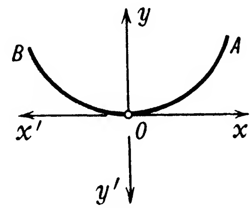

FIG. 192.

The difference in sign of these results is accounted for by the fact that, according to the method of treatment, the sign of the curvature is decided by the form of the curve relative to the direction of the corresponding y-axis. In the first case the y-axis is towards the centre of the circle and in the second in the opposite direction.

Ex. 5. Investigate the form of the lemniscate $r^2 = a^2\cos 2\theta$, $(a > 0)$, in the neighbourhood of the pole.

We have, on differentiating with respect to r,

$$r = -a^2\sin 2\theta\frac{d\theta}{dr}, \quad 1 = -2a^2\cos 2\theta\left(\frac{d\theta}{dr}\right)^2 - a^2\sin 2\theta\frac{d^2\theta}{dr^2}. \qquad (32)$$

As $r \to 0$, $\cos 2\theta \to 0$ and hence $\sin 2\theta \to \pm 1$. The orientations of the tangents at O are thus $\pm\frac{1}{4}\pi$. For each branch we have $(d\theta/dr)_0 = 0$, so that the curvature for each is zero at O.

Consider the branches along which $\sin 2\theta \to 1$. For each of them we have, from (32), $(d^2\theta/dr^2)_0 = -1/a^2$, and hence the deflection of either branch from the tangent at O is of the order of x^3, where x is a small distance along the tangent to either branch. The pole is thus a point of contraflexure for the combined branch consisting of the two portions described. A similar result holds for the other combined branch through O corresponding to $\sin 2\theta \to -1$.

Ex. 6. Investigate the nature of the curve $r^2 = \tan\theta\sec^2\theta$ at the pole.

[Write $r^2 = t(1 + t^2)$, where $t = \tan\theta$, and work with t as dependent variable. The curve is a cubic parabola.]

Ex. 7. Investigate the nature of the curve $r = a\cos^2\theta - b\sin^2\theta$ in the neighbourhood of the pole.

[At the pole each branch proceeding from it has a finite curvature, although the pole is a point of contraflexure for each of the two combined branches along which the gradient is continuous.]

205. Centre of Curvature in Polar Coordinates.

Let C, the centre of curvature at P, (Fig. 193), have coordinates (r_c, θ_c). By projection along and perpendicular to OP, we obtain the results

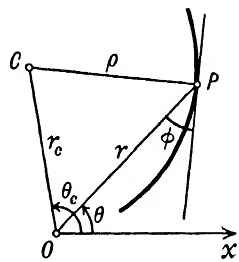

FIG. 193.

$$r = r_c \cos(\theta_c - \theta) + \varrho \sin \varphi, \quad \text{.............(1)}$$

$$0 = r_c \sin(\theta_c - \theta) - \varrho \cos \varphi, \quad \text{.............(2)}$$

and from these we obtain the equations

$$r_c^2 = r^2 + \varrho^2 - 2r\varrho \sin \varphi, \quad \tan(\theta_c - \theta) = \frac{\varrho \cos \varphi}{r - \varrho \sin \varphi}, \quad (3)$$

which give the required coordinates.

In the Figure, κ is positive and we can write, in place of equations (3),

$$r_c^2 = r^2 + \frac{1}{\kappa^2} - 2\frac{r}{\kappa} \sin \varphi, \quad \tan(\theta_c - \theta) = \frac{\cos \varphi}{\kappa r - \sin \varphi}. \quad \text{..............(4)}$$

Moreover, these equations, but not (3), can be shewn to be true when κ is negative.

206. Extreme Distances of a Point from a Curve.

Taking the point considered as the origin of coordinates, let $r = f(\theta)$ be the polar equation of the curve. The condition that the radius r_1, of polar angle θ_1, should be a stationary radius to the curve is $(dr/d\theta)_1 = 0$, or $f'(\theta_1) = 0$. Since $(dr/d\theta)_1 = r_1 \cot \varphi_1$, where φ_1 is the recedent at (r_1, θ_1), the condition is $\varphi_1 = \frac{1}{2}\pi$, so that the radius must be a normal to the curve at that point. The stationary distance r_1 is a maximum (minimum) if $(d^2r/d\theta^2)_1$ is negative (positive). Now, since $(dr/d\theta)_1 = 0$, the curvature of the curve at (r_1, θ_1) is given by

$$\kappa_1 = \frac{1}{r_1^2}\left(r - \frac{d^2r}{d\theta^2}\right)_1, \quad \text{...............................(1)}$$

so that $(d^2r/d\theta^2)_1 = r_1(1 - \kappa_1 r_1)$. The stationary distance is therefore a maximum if $1 - \kappa_1 r_1 < 0$. To satisfy this condition the curvature must be towards the pole and the radius of curvature must be less than r_1, that is, the centre of curvature must be between the pole and the curve. The stationary distance is a minimum if $1 - \kappa_1 r_1 > 0$. The curvature is then away from the pole or else the radius of curvature is greater than r_1, so that the pole lies between the centre of curvature and the curve.

If the centre of curvature is at the pole, so that $1 - \kappa_1 r_1 = 0$, $(d^2r/d\theta^2)_1 = 0$ and the stationary distance is neither a maximum nor a minimum unless $(d\kappa/d\theta)_1 = 0$. If $(d\kappa/d\theta)_1 = 0$, the distance is a maximum (minimum) if $(d^2\kappa/d\theta^2)_1$ is positive (negative). The proofs of these statements concerning the critical case are left to the student.

Ex. 1. Shew that the distance of a point on the major axis of an ellipse from the nearer apse is a minimum if the point lies at the corresponding centre of curvature or between it and the apse.

SECTION II. THE LINE-EQUATION OF A CURVE. CURVATURE IN LINE-COORDINATES

207. Line-Coordinates.

In elementary Analytical Geometry it is usual to regard the points of the plane as the fundamental elements, a straight line or a curve being regarded as defined by the points on it. From this standpoint, the coordinates[1] which are introduced are quantities which define the positions of points, and they are accordingly called *point-coordinates*. A straight line or a curve is then defined by a point-coordinate equation.

It is sometimes more convenient to treat the straight lines of the plane as the fundamental elements, a point being then defined by the intersection of two straight lines, and a curve by the straight lines which touch it, i.e. by its tangents, the curve being then referred to as the *envelope* of its tangent lines. From this standpoint, the coordinates which are introduced are quantities which define the positions of straight lines, and they are accordingly called *line-coordinates*. A point or a curve is then defined by a *line-coordinate equation*, as will be seen presently.

The coordinates of a line have been chosen in various ways. If rectangular axes are drawn, we note, in particular, that the following sets of coordinates may be taken as specifying uniquely the position of a line in the plane : (i) The intercepts a, b, say, cut off the axes Ox, Oy by the line. It is, however, more convenient to employ as coordinates the *reciprocals* l, m, say, of these intercepts,[2] the point-equation of the line then becoming $lx+my=1$. (ii) The above intercept b and the gradient g of the line with respect to Ox. The point-equation is then $y=gx+b$. (iii) The slope ψ of the line relative to Ox and the perpendicular p from the origin on it. The point-equation is then $x\sin\psi-y\cos\psi=p$. We shall take ψ to lie, as usual, in the range $(0, \pi)$, and p will then be positive (negative) if O is on the negative (positive) side of the line with respect to the direction Ox, i.e. if O is to the left (right) of the line.

Suppose next that the position of the line is to be specified relative to a pole O and an initial line Ox, the point-coordinates in the plane being of the polar type. Suitable line-coordinates, corresponding to

[1] The meaning of the word *coordinates* is here extended to cover quantities of any nature by means of which the position of a configuration is specified.

[2] In Analytical Geometry, l, m are usually taken as the negative reciprocals of the intercepts chiefly for the sake of symmetry when proceeding to other types of coordinates (e.g. homogeneous coordinates).

the above pair (p, ψ), are then obtained if we treat the perpendicular p as a positive radius and replace ψ by the polar angle, ω say, of that radius.

The connections between the above sets of coordinates are shewn

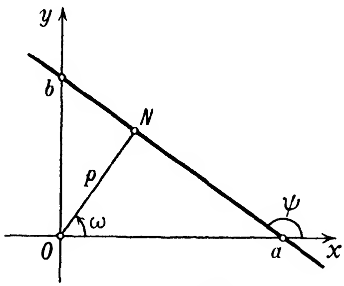

FIG. 194.

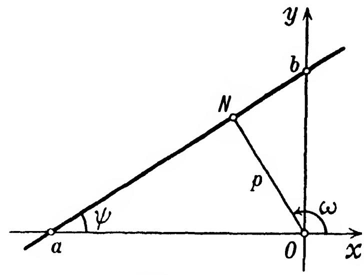

FIG. 195.

in Figs. 194, 195. In Fig. 194, $a>0$, $b>0$ and in Fig. 195, $a<0$, $b>0$. We have

$$l=1/a, \quad m=1/b, \quad g=\tan\psi=-b/a=-l/m,$$
$$p=a\sin\psi=-b\cos\psi.$$

From these we find

$$l=(\sin\psi)/p, \quad m=-(\cos\psi)/p. \quad \text{......................(1)}$$

We observe that p has the sign of a, which is in agreement with the convention laid down in (iii) above.

For the polar type of coordinate-pair (p, ω), p being now *positive*, we observe that the difference of the angles ω, ψ is always an *odd* multiple of $\frac{1}{2}\pi$. We now have $p=a\cos\omega=b\sin\omega$, and hence

$$l=(\cos\omega)/p, \quad m=(\sin\omega)/p, \quad (p>0). \quad \text{...............(2)}$$

The one-one correspondence of line and line-coordinate-pair (l, m) is lost if the line passes through the origin, i.e. if one or both of the coordinates l, m becomes indefinitely great. There are three cases. (i) A line along Ox is definitely specified by coordinates (l, ∞), where l may have *any* finite value; (ii) a line along Oy is definitely specified by the coordinates (∞, m), where m may have *any* finite value; (iii) a line in any other direction through O corresponds to coordinates (∞, ∞) but its direction is not defined by such (infinite) coordinates. A knowledge of the (limit of the) ratio l/m is here required to determine the direction. A line along an axis may be supposed defined in this way, the ratio l/m being 0 in case (i) and ∞ in case (ii).

It should be observed that when line-coordinates (p, ψ) are employed, there is no loss of one-one correspondence in the above cases since in them $p=0$ and the value of ψ specifies the slope of the line.

208. The Line-Equations of a Point and of a Curve.

We may now employ the line-coordinates (l, m) in order to explain the definition of a point or a curve by a line-equation.

The (l, m) line-equation of a point is the relation that must hold between l and m in order that the line (l, m) may pass through it. Let (x_1, y_1) be the point considered. The point-equation of the line being $lx+my=1$, this passes through (x_1, y_1) if, and only if,

$$lx_1+my_1=1. \quad\text{.............................(1)}$$

This is therefore the line-equation of the point (x_1, y_1). Conversely, a relation of the form

$$lh+mk=1 \quad\text{.............................(2)}$$

defines a point (h, k) through which the line (l, m) passes if the relation is satisfied. Thus the line-equation of a point is linear in the line-coordinates.

Consider next the case of a curve. Here the (l, m) line-equation is the condition that the line may touch the curve.[1] If we take any value of l, corresponding to a definite intercept of the line (l, m) on Ox, we shall have to adjust the intercept $1/m$ on Oy in order to make the line touch the curve, as in Fig. 196. In other words, m is a function of l. Conversely, a relation between l and m, say $f(l, m)=0$, defines a curve, for, corresponding to each value of l (in a range) the relation determines a definite value, or set of values, of m, i.e. a definite set of lines, and a definite curve is specified by the condition that it is touched (or enveloped) by all the lines so found for all the values of l (in the range).

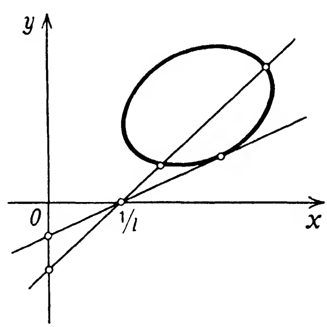

FIG. 196.

Ex. 1. *The circle* $x^2+y^2=c^2$. This is touched by the line $lx+my=1$ if the perpendicular from O on this line is equal to c, that is, if $1/\sqrt{(l^2+m^2)}=c$, or

$$l^2+m^2=1/c^2. \quad\text{.................................(3)}$$

This is therefore the (l, m) line-equation of the circle.

Ex. 2. *The equation* $lm=2$. If we give to l a succession of values, the corresponding values of m can be determined at once and the system

[1] The preceding case is to be regarded as a limiting one in which the curve shrinks up to a point.

of lines drawn. This has been done in Fig. 197 corresponding to the values $\pm\frac{1}{3}$, $\pm\frac{1}{2}$, ± 1, ± 2, ± 3, ± 4 of l. It is shewn in Art. 210, Ex. 1, that these lines envelop a rectangular hyperbola, and the given equation is thus the (l, m) line-equation of this hyperbola.

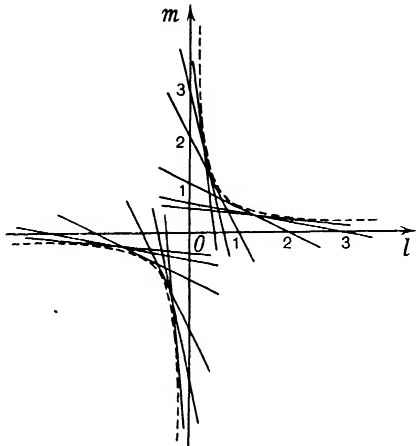

FIG. 197.
$lm=2$.

209. Determination of the Line-Equation of a Curve.

209. 1. *Cartesian equation given.* Let the point-equation of the curve be $y=f(x)$. If (l, m) are the line-coordinates of the tangent at (x, y), we have $lx+my=1$, that is,

$$lx+mf(x)=1. \quad \text{.......(1)}$$

Also the gradient g at this point is the same as that of the tangent, or

$$f'(x)=-l/m. \quad \text{........(2)}$$

The required (l, m) equation of the curve is found by eliminating x between these equations.

If the Cartesian equation of the curve is given in the form $f(x, y)=0$, the required (l, m) equation is to be found from the equations

$$lx+my=1, \quad l+m\,dy/dx=0, \quad f(x, y)=0, \quad \text{............(3)}$$

the derivative dy/dx being obtained from the equation $\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial y}\dfrac{dy}{dx}=0$, (Art. 82).

Ex. 1. *The ellipse* $x^2/a^2+y^2/b^2=1$. Here we have $\dfrac{dy}{dx}=-\dfrac{b^2x}{a^2y}$, and, corresponding to the first two equations in (3), we have

$$lx+my=1, \quad mb^2x-la^2y=0. \quad \text{........................(4)}$$

These give

$$\frac{x}{la^2}=\frac{y}{mb^2}=\frac{1}{l^2a^2+m^2b^2}, \quad \text{..............................(5)}$$

and on substituting the values of x, y given by (5) in the original equation, we find, as the required (l, m) line-equation of the ellipse,

$$l^2a^2+m^2b^2=1. \quad \text{....................................(6)}$$

With the aid of the relations (1), (2), Art. 207, we find as the (p, ψ) equation,

$$a^2\sin^2\psi+b^2\cos^2\psi=p^2, \quad \text{..............................(7)}$$

and as the (p, ω) equation, (p now positive),

$$a^2\cos^2\omega+b^2\sin^2\omega=p^2. \quad \text{..............................(8)}$$

Ex. 2. Obtain analytically the following line-equations for the circle $x^2+y^2-2ax=0$, $(a>0)$: (i) $a^2(l^2+m^2)=(1-al)^2$, (ii) $p=a(\sin\psi \pm 1)$, (iii) $p=a(\cos\omega+1)$, $(p>0)$, where in (ii) the positive (negative) sign is to be taken when the origin is on the negative (positive) side of the tangent relative to Ox.

[The (p, ω) equation is easily obtained geometrically. Let C be the centre and draw OT perpendicular to the tangent PT at P. Then if CT' is perpendicular to OT, we have

$$p=OT=OT'+T'T=OT'+CP=a\cos\omega+a.]$$

209. 2. *Polar equation given.* Let the point-equation be now $r=f(\theta)$. In obtaining the line-equation it is natural to employ the line-coordinates (p, ω) of polar type. Referring to Fig. 198 and recalling the result (9) of Art. 202, we have

$$p=r\sin\varphi=\frac{\{f(\theta)\}^2}{\sqrt{[\{f(\theta)\}^2+\{f'(\theta)\}^2]}}, \quad \ldots(9)$$

and, since $\omega=\psi+\dfrac{2n+1}{2}\pi$, ($n$ an integer), we have $\tan\omega=-\cot\psi$. From the relations $x=r\cos\theta$, $y=r\sin\theta$ we find

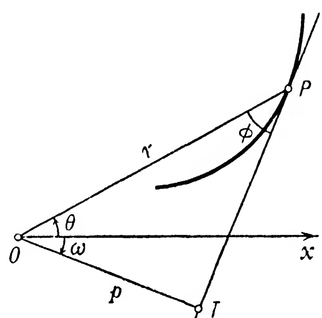

FIG. 198.

$$\tan\psi=\frac{dy/d\theta}{dx/d\theta}=\frac{\dfrac{dr}{d\theta}\tan\theta+r}{\dfrac{dr}{d\theta}-r\tan\theta}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

and, therefore, writing $f(\theta)$, $f'(\theta)$ for r, $dr/d\theta$,

$$\tan\omega=\frac{f(\theta)\tan\theta-f'(\theta)}{f(\theta)+f'(\theta)\tan\theta}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

The required (p, ω) equation is the result of eliminating θ between the equations (9) and (11).

Ex. 3. *The circle* $r=2a\cos\theta$, $(a>0)$. Here we find, from (9), (11),

$$p=a(1+\cos 2\theta), \quad \tan\omega=\tan 2\theta. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

The required equation is therefore $p=a(1+\cos\omega)$. See Ex. 2, (iii), above.

Ex. 4. *The cardioid* $r=a(1+\cos\theta)$, $(a>0)$. We now find

$$p=2a\cos^3\tfrac{1}{2}\theta, \quad \tan\omega=\tan\tfrac{3}{2}\theta. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

As before, if we restrict θ to the range $(-\pi, \pi)$, which is sufficient for the description of the complete curve, and if we take $\omega=\frac{3}{2}\theta$, the range of ω is $(-\frac{3}{2}\pi, \frac{3}{2}\pi)$, and the (p, ω) equation becomes

$$p=2a\cos^3\tfrac{1}{3}\omega. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

210. Determination of the Point-Equation of a Curve from the Line-Equation. Envelopes.

Let the line-equation of the curve be $m=f(l)$. Consider two tangents (l, m), $(l+\Delta l, m+\Delta m)$ to the curve. Their point-equations are

$$lx+f(l)y=1, \quad (l+\Delta l)x+\{f(l)+\Delta f(l)\}y=1, \quad \text{............(1)}$$

and these are satisfied by the coordinates, (x^*, y^*) say, of their point of intersection P^* (Fig. 199). We thus obtain the equations

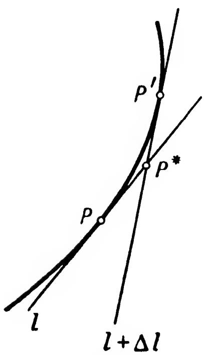

FIG. 199.

$$lx^*+f(l)y^*=1, \quad \Delta l \,.\, x^*+\Delta f(l)\,.\,y^*=0, \quad \text{...(2)}$$

the latter of which can be written in the form $x^*+\dfrac{\Delta f(l)}{\Delta l}y^*=0$, or, on employing the $I.D_1.T.$, in the form

$$x^*+f'(\bar{l})y^*=0, \quad \text{................(3)}$$

where $\bar{l}$ lies in the range $(l, l+\Delta l)$.

Suppose now that the second tangent moves up to coincidence with the first, so that P^* moves up to P on the curve. Then $f'(\bar{l}) \to f'(l)$, and we thus have the equations

$$lx+f(l)y=1, \quad x+f'(l)y=0, \quad \text{....................(4)}$$

which hold for any point P, (x, y), on the curve. On eliminating l between them we get the required (x, y) equation of the curve.

It will be observed that the second equation in (4) is obtained from the first by differentiating with respect to l, keeping x, y constant.

If the line-equation of the curve is given in the implicit form $f(l, m)=0$, the (x, y) point-equation is obtained by eliminating l, m between the equations

$$lx+my=1, \quad x+\frac{dm}{dl}y=0, \quad f(l, m)=0, \quad \text{...............(5)}$$

the derivative dm/dl being given by the equation $\dfrac{\partial f}{\partial l}+\dfrac{\partial f}{\partial m}\dfrac{dm}{dl}=0.$

The curve, regarded as defined by the system of lines which touch it, is called the *envelope* of the system. The argument applied above shews that the curve may also be described as the locus of the points of intersection of coalescing lines of the system.

If the relation between l and m is expressed in the parametric form $l=\varphi(\lambda)$, $m=\chi(\lambda)$, the equation $lx+my-1=0$ becomes

$\varphi(\lambda)x+\chi(\lambda)y-1=0$, where the left-hand side is a function of x, y and λ, say $f(x, y, \lambda)$. The application of the above argument now leads to the equations

$$f(x, y, \lambda)=0, \quad \frac{\partial}{\partial\lambda}f(x, y, \lambda)=0, \quad \text{..................(6)}$$

and the point-equation of the envelope follows on the elimination of λ.

This process can be generalized to cover the case in which the system of straight lines is replaced by a system of curves. Let this system be given by an equation $f(x, y, \lambda)=0$ for a range of the parameter λ, the particular curve corresponding to a value λ_1 of λ in this range being thus given by $f(x, y, \lambda_1)=0$. Then the equations (6) define the locus of the points of intersection of coalescing curves of the system. As in the case of a system of straight lines, this locus may be shewn to be touched by each of the curves of the system, and it is called the *envelope* of the system.

Ex. 1. Shew that if the line-equation of a curve is $p=f(\psi)$, the point-equation is found by eliminating ψ between the equations

$$x\sin\psi-y\cos\psi=f(\psi), \quad x\cos\psi+y\sin\psi=f'(\psi). \quad \text{..............(7)}$$

Ex. 2. *The equation* $lm=2$. Here $m+l\,dm/dl=0$ and the equations (4) become

$$lx+my=1, \quad lx-my=0. \quad \text{..............................(8)}$$

Therefore $lx=my=\frac{1}{2}$ and the point-equation is $8xy=1$. Thus the curve (envelope) is a rectangular hyperbola; see Ex. 2, Art. 208.

Ex. 3. *The equation* $\frac{1}{l^2}+\frac{1}{m^2}=1$. Here we have $\frac{1}{l^3}+\frac{1}{m^3}\frac{dm}{dl}=0$, and the equations (4) become

$$lx+my=1, \quad l^3x-m^3y=0. \quad \text{............................(9)}$$

Hence $\frac{x}{m^3}=\frac{y}{l^3}=\frac{1}{lm(l^2+m^2)}=\frac{1}{l^3m^3}$. Thus $x=\frac{1}{l^3}$, $y=\frac{1}{m^3}$, and substitution in the line-equation gives, for the point-equation,

$$x^{2/3}+y^{2/3}=1. \quad \text{..................................(10)}$$

This represents an astroid.

Ex. 4. Find the envelope of the parabolas

$$x^2-2\lambda hy-\lambda^2k^2-l^2=0, \quad \text{............................(11)}$$

where h, k, l are constants and λ is a parameter.

Here the equation $\partial f/\partial\lambda=0$ becomes

$$hy+\lambda k^2=0, \quad \text{or} \quad \lambda=-hy/k^2. \quad \text{......................(12)}$$

On substituting in (11) we get, for the envelope, the ellipse

$$\frac{x^2}{h^2}+\frac{y^2}{k^2}=\frac{l^2}{h^2} \quad \text{...................................(13)}$$

211. The Projection of the Radius on the Tangent.

The equations (7) of the preceding Article apply to any point (x, y) on the curve. On squaring and adding they give

$$x^2+y^2=\{f(\psi)\}^2+\{f'(\psi)\}^2,$$

or

$$r^2=p^2+\left(\frac{dp}{d\psi}\right)^2 \quad \text{.................................(1)}$$

$$=p^2+\left(\frac{dp}{d\omega}\right)^2, \quad \text{.................................(2)}$$

since ω and ψ differ only by a constant. It follows that the magnitude of the projection PT (Fig. 198) of the radius on the tangent is given by

$$PT=\left|\frac{dp}{d\psi}\right|=\left|\frac{dp}{d\omega}\right|. \quad \text{.........(3)}$$

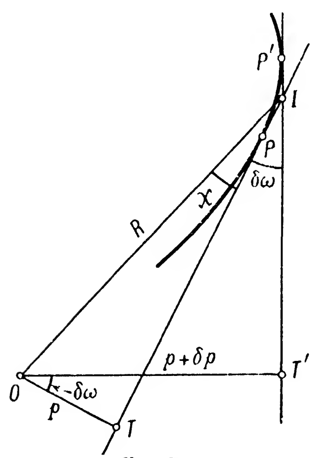

FIG. 200.

In some cases it is convenient to consider the algebraic measure of this projection, which is taken positive (negative) if the recedent φ is less (greater) than $\frac{1}{2}\pi$. We then have, simply,

$$PT=dp/d\psi=dp/d\omega. \quad \text{........(4)}$$

211. 1. *Working notions.* The above formulae are easily recovered with the use of infinitesimals. Let two adjacent tangents PT, $P'T'$, at an inclination $\delta\omega$, meet at I (Fig. 200), and let $OI=R$. Then if p, $p+\delta p$ are the lengths of the perpendiculars OT, OT' on the tangents and $\angle OIT=\chi$, we have

$$p=R\sin\chi, \quad p+\delta p=R\sin(\chi+\delta\omega)\approx R\sin\chi+R\cos\chi\,.\,\delta\omega. \quad \text{..........(5)}$$

Thus

$$\delta p\approx R\cos\chi\,.\,\delta\omega\approx IT\,.\,\delta\omega. \quad \text{.............................(6)}$$

When $\delta\omega$ is indefinitely decreased the point of intersection I of the tangents converges to P. Hence $PT=dp/d\omega$.

212. Curvature of a Curve specified in Line-Coordinates.

Let the line-equation of the curve be $m=f(l)$, so that, according to Art. 210, the (x, y) point-equation is given by the equations

$$lx+my=1, \quad x+\frac{dm}{dl}y=0, \quad \text{.......................(1)}$$

on eliminating l, m. The required formula for the curvature of the curve in terms of l, m is found by writing the ordinary formula in the form

$$\kappa=\frac{\dfrac{dg/dl}{dx/dl}}{(1+g^2)^{3/2}}, \quad \text{..............................(2)}$$

y being equal to $-l/m$ and the derivative dx/dl being obtained from the equations (1). On eliminating y between these equations, we get

$$\left(m - l\frac{dm}{dl}\right)x = -\frac{dm}{dl}; \quad \text{.........................(3)}$$

whence we find

$$\frac{dx}{dl} = -\frac{m\dfrac{d^2m}{dl^2}}{\left(m - l\dfrac{dm}{dl}\right)^2}. \quad \text{.........................(4)}$$

Also

$$\frac{dg}{dl} = -\frac{m - l\dfrac{dm}{dl}}{m^2}, \quad (1+g^2)^{3/2} = \frac{(l^2+m^2)^{3/2}}{|m^3|}, \quad \text{................(5)}$$

and the formula (2) thus becomes

$$\kappa = \pm\frac{\left(m - l\dfrac{dm}{dl}\right)^3}{\dfrac{d^2m}{dl^2}(l^2+m^2)^{3/2}}, \quad \text{.........................(6)}$$

the alternative sign being positive (negative) if m is positive (negative).

If the line-equation is given in terms of p and ψ, we make the transformation

$$\kappa = \frac{\dfrac{dg/d\psi}{dx/d\psi}}{(1+g^2)^{3/2}}, \quad \text{..............................(7)}$$

where now

$$\left.\begin{aligned} &\frac{dg}{d\psi} = \sec^2\psi, \quad x = p\sin\psi + \frac{dp}{d\psi}\cos\psi, \\ &\frac{dx}{d\psi} = \left(p + \frac{d^2p}{d\psi^2}\right)\cos\psi, \quad (1+g^2)^{3/2} = |\sec\psi|^3. \end{aligned}\right\} \quad \text{...........(8)}$$

We thus find

$$\kappa = \pm\frac{1}{p + \dfrac{d^2p}{d\psi^2}}, \quad \text{..............................(9)}$$

the positive (negative) sign being taken if $\sec\psi$ is positive (negative), i.e. if ψ is less (greater) than $\frac{1}{2}\pi$.

The convention for the curvature throughout the above discussion is that relating to axes-type of point-coordinates. If line-coordinates (p, ω) are introduced, the formula (9) is immediately transformed to

$$\kappa = \frac{1}{p + \dfrac{d^2p}{d\omega^2}}, \quad \text{..............................(10)}$$

but p is now positive and the polar convention for the curvature applies, κ being positive if the curve is convex outwards.

Ex. 1. *The ellipse* $p^2 = a^2 \sin^2 \psi + b^2 \cos^2 \psi$. Here we find

$$\frac{d^2p}{d\psi^2} = \frac{(a^2 - b^2)(b^2 \cos^4 \psi - a^2 \sin^4 \psi)}{p^3}, \quad\ldots\ldots(11)$$

and hence

$$p + \frac{d^2p}{d\psi^2} = \frac{a^2b^2}{p^3}. \quad\ldots\ldots(12)$$

Therefore

$$\kappa = \pm \frac{p^3}{a^2b^2}, \quad\ldots\ldots(13)$$

the positive (negative) sign being taken if ψ is less (greater) that $\frac{1}{2}\pi$.

The student should reconcile this result with the results in Ex. 5, Art. 133, p. 302, (where, it will be noted, the sign of p was taken as positive throughout).

Ex. 2. *The cardioid* $p = 2a \cos^3 \frac{1}{3}\omega$, $(a > 0, \ |\omega| \leqslant \frac{3}{2}\pi)$. Here we find

$$\frac{d^2p}{d\omega^2} = \tfrac{4}{3}a \cos \tfrac{1}{3}\omega \sin^2 \tfrac{1}{3}\omega - \tfrac{2}{3}a \cos^3 \tfrac{1}{3}\omega, \quad p + \frac{d^2p}{d\omega^2} = \tfrac{4}{3}a \cos \tfrac{1}{3}\omega, \quad\ldots\ldots(14)$$

and hence

$$\kappa = \frac{1}{\frac{4}{3} a \cos \frac{1}{3}\omega}. \quad\ldots\ldots(15)$$

Further, the relation $r^2 = p^2 + (dp/d\omega)^2$ here gives $r^2 = 4a^2 \cos^4 \frac{1}{3}\omega$, and hence we have the result

$$\kappa = \frac{3r}{4ap}. \quad\ldots\ldots(16)$$

SECTION III. THE RADIUS-RECEDENT RELATION OF A CURVE. THE (p, r) RELATION

213. The (r, φ) and (p, r) Relations.

A point- or line-coordinate equation of a curve possesses the property of defining the curve relative to axes or points of reference, that is, each defines not only the *form* of the curve but also its *position* relative to the reference axes or points. Each of the equations involves implicitly all the properties of the curve.

In the discussion of curves it is often convenient to employ a derived relation or property instead of one of the equations of the curve. Such a derived relation may completely define the curve, apart from position and orientation, or it may define it as one of a more or less extensive class of curves. A relation of this type is often referred to as an *equation* of the curve, but it seems preferable to retain the term *relation*, adding the names or symbols of the quantities which occur in the relation.

It is easy to shew that a relation exists between *any* two variable

measures associated with the points of a curve. We may take, as an example, the radius r and the recedent φ as the measures in question. If we start with the (x, y) point-equation of the curve, we can find r and φ in terms of the position on the curve of the point considered, i.e. in terms of x and y. The equation of the curve gives a relation between x and y, and therefore the quantities r, φ are expressible in terms of x alone, say $r=f(x), \varphi=g(x)$. The elimination of x between these gives a relation between r and φ, which is the *radius-recedent*, or (r, φ), *relation* for the curve.

The same argument would apply to the determination of a relation between the perpendicular p and the radius r, or, as a third example, to the determination of a relation between the curvature κ and the slope ψ of the tangent.

The reverse process of determining the form of the curve from a relation of the kind indicated requires, in general, the aid of the Integral Calculus[1] or of Differential Equations, and is often not at all simple.

In the present Article we consider the formation of the radius-recedent or (r, φ) relation and of the (p, r) relation for some simple cases of curves whose equations are given in terms of rectangular or polar coordinates. These relations are frequently employed in Dynamics when a knowledge of the form only of a curve is required, its orientation in the plane being a matter of indifference.

It may be remarked here that it is most convenient to treat p as *positive throughout*, the connection between p, r and φ being always $p=r \sin \varphi$, where the range of φ is, as usual, $(0, \pi)$.

213. 1. *Polar equation given.* The (r, φ) relation is most simply found from the polar equation of the curve when expressed in the form $\theta=f(r)$. We then have, from (5), Art. 202,

$$\tan \varphi=rf'(r), \qquad \text{(1)}$$

which expresses the relation in question.

To find the (p, r) relation, we observe that since $\sin \varphi=p/r$, we have $\tan^2\varphi=p^2/(r^2-p^2)$. We thus have, from (1),

$$\frac{p^2}{r^2-p^2}=r^2\left(\frac{d\theta}{dr}\right)^2, \qquad \text{(2)}$$

which is usually written in the form

$$\frac{1}{p^2}=\frac{1}{r^2}+\frac{(dr/d\theta)^2}{r^4}, \qquad \text{(3)}$$

[1] See, for example, Art. 235, Chap. IX.

here the derivative $dr/d\theta$ is supposed expressed in terms of r alone.

If we introduce the reciprocal of r as dependent variable, writing $=1/r$, (3) becomes

$$\frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2, \quad \text{.............................(4)}$$

hich is sometimes more convenient.

213. 2. *Cartesian equation given.* Let the equation of the curve be $=f(x)$. If PT (Fig. 201) is the tangent at P, (x, y), and ON is the ɜrpendicular p from O to it, then, by projection on ON, we have

$$ON=|\,OM\sin\psi-MP\cos\psi\,|,$$

ıat is,
$$p=|\,x\sin\psi-y\cos\psi\,|=\frac{|\,x\tan\psi-y\,|}{\sqrt{(1+\tan^2\psi)}},$$

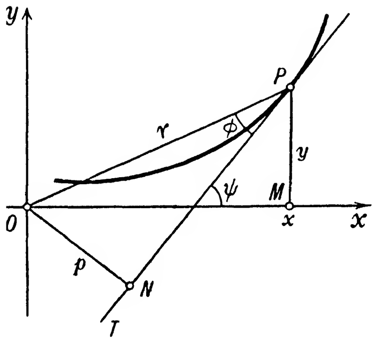

FIG 201.

so that

$$p^2=\frac{\left(x\dfrac{dy}{dx}-y\right)^2}{1+\left(\dfrac{dy}{dx}\right)^2}, \quad \text{.......(5)}$$

where $y=f(x)$, $dy/dx=f'(x)$. Also $OP^2=OM^2+MP^2$, that is,

$$r^2=x^2+\{f(x)\}^2, \quad \text{.......(6)}$$

and the (p, r) relation is obtained by eliminating x between (5) and (6).

The (r, φ) relation follows by writing $p=r\sin\varphi$.

In many cases the (r, φ) and (p, r) relations can be written down ɔm a knowledge of special properties of the curve considered. The alytical method is then unnecessary, except as a check.

Ex. 1. *The circle* $r=2a\cos\theta$, $(a>0)$, (Fig. 163, p. 471). Here we have $\cdot/d\theta)^2=4a^2\sin^2\theta=4a^2-r^2$, and (3) gives

$$\frac{1}{p^2}=\frac{1}{r^2}+\frac{4a^2-r^2}{r^4},$$

ıence we obtain the desired (p, r) relation in the form

$$2ap=r^2. \quad \text{..(7)}$$

Γhis result also follows geometrically by observing that if C is the centre, $'A$ the diameter through O and OT the perpendicular from O to the tangent P, the triangles OTP, OPA are similar, so that $OT/OP=OP/OA$.

Ex. 2. *The equiangular spiral* $r=ae^{\theta\cot\alpha}$, $(a>0,\ 0<\alpha<\pi)$. Since here $=\alpha$, by Ex. 3, p. 482, this is the (r, φ) relation for the curve. The (p, r) ation is therefore

$$p=r\sin\alpha. \quad \text{.....................................(8)}$$

Ex. 3. *The cardioid* $r=a(1+\cos\theta)$, $(a>0)$. Here we have

$$(dr/d\theta)^2=a^2\sin^2\theta=2ar-r^2,$$

and the (p, r) relation is thus given by

$$\frac{1}{p^2}=\frac{1}{r^2}+\frac{2ar-r^2}{r^4}=\frac{2a}{r^3},$$

or by

$$2ap^2=r^3. \qquad\qquad (9)$$

Ex. 4. *The ellipse* $x^2/a^2+y^2/b^2=1$. In this case (5) leads to the equation $1/p^2=x^2/a^4+y^2/b^4$, and the required (p, r) relation is obtained by eliminating x^2, y^2 between these equations and the equation $r^2=x^2+y^2$. We obtain

$$\begin{vmatrix} 1/a^2 & 1/b^2 & 1 \\ 1/a^4 & 1/b^4 & 1/p^2 \\ 1 & 1 & r^2 \end{vmatrix}=0$$

or

$$\frac{a^2b^2}{p^2}=a^2+b^2-r^2. \qquad\qquad (10)$$

This relation also follows geometrically. If OP, OQ are conjugate semi-diameters, of lengths r, r', and if p is the perpendicular distance from the centre to the tangent at P, then we have the well-known results

$$r^2+r'^2=a^2+b^2, \quad pr'=ab,$$

and (10) follows on eliminating r'.

Ex. 5. *The ellipse with pole at a focus.* The tangent at any point makes equal angles with the focal radii, so that $\angle SPT=\angle S'PT'$ (Fig. 202), where S, S' are the foci. This gives $p/r=p'/r'$.

Also we have the fundamental results $pp'=b^2$, $r+r'=2a$, and, on eliminating p', r' between these three equations, we get

$$\frac{p^2}{r}=\frac{b^2}{2a-r}, \qquad\qquad (11)$$

or

$$\frac{l}{p^2}=\frac{2}{r}-\frac{1}{a}, \qquad\qquad (12)$$

if we write l for the length b^2/a of the semi-latus rectum.

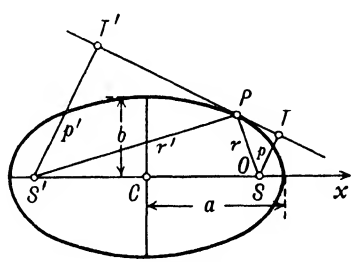

FIG. 202.
$l/p^2=2/r-1/a$.

The student should obtain the same result analytically, starting with the equation $l/r=1-e\cos\theta$, (Ex. 3, Art. 201).

214. Curvature in terms of r and φ, and in terms of p and r.

For simplicity, suppose the curve specified in terms of polar coordinates. According to (7), Art. 204, p. 491, the curvature at any point is given by

$$\kappa=\frac{1+\dfrac{d\varphi}{d\theta}}{r\operatorname{cosec}\varphi}, \qquad\qquad (1)$$

being positive (negative) if the curve is convex (concave) outwards. We may write this in the form

$$\kappa=\frac{1}{r}\sin\varphi+\frac{1}{r}\sin\varphi\frac{d\varphi}{dr}\frac{dr}{d\theta}$$

$$=\frac{1}{r}\sin\varphi+\cos\varphi\frac{d\varphi}{dr}, \quad\text{.....(2)}$$

$$=\frac{1}{r}\frac{d}{dr}(r\sin\varphi)$$

$$=\frac{1}{r}\frac{dp}{dr}. \quad\text{.....(3)}$$

214. 1. *Curvature in terms of p and ω deduced from that in terms of p and r.* From the result (2), Art. 211, namely,

$$r^2=p^2+(dp/d\omega)^2, \quad\text{.....(4)}$$

we find, on differentiating with respect to p,

$$2r\frac{dr}{dp}=2p+\frac{d}{dp}\left(\frac{dp}{d\omega}\right)^2=2p+2\frac{d^2p}{d\omega^2}. \quad\text{.....(5)}$$

With the aid of (3) above, we thus get $\frac{1}{\kappa}=p+\frac{d^2p}{d\omega^2}$, as in Art. 212.

Ex. 1. *The equiangular spiral* $\varphi=\alpha$. From (2) we conclude

$$\kappa=\frac{1}{r}\sin\alpha. \quad\text{.....(6)}$$

See Ex. 3, Art. 204.

Ex. 2. *The cardioid* $2ap^2=r^3$. Here we have $4ap\,.\,dp/dr=3r^2$, and from (3) we have

$$\kappa=\frac{3r}{4ap}=\frac{3}{2\sqrt{(2ar)}}. \quad\text{.....(7)}$$

See Ex. 2, Art. 204.

215. Plotting with the aid of the (r, φ) Relation.

If the radius-recedent relation is given, the analytical determination of the curves for which it holds requires the use of integration, as we have already remarked. A curve can, however, be *plotted* from its (r,φ) relation by using a step-by-step process. Suppose that a starting point P_1 on the curve is chosen consistently with the relation, i.e. such that the relation gives a real value of φ for the chosen distance r_1 of P_1 from the origin O. If, now, we draw a short length of tangent P_1P_2 in the direction assigned by the value of φ, we can, to a first approximation, regard P_2 as another point on the curve. The distance OP_2, (r_2), being known, the (r,φ) relation will

give the direction of the tangent at P_2, and we can draw a short length of tangent P_2P_3 there and take P_3 as a third point on the curve, and so on. However, the errors in the determination of the successive points caused by ignoring the deflection of the curve from its tangent at each step will, in general, accumulate. We can reduce these errors by using Newton's formula for the deflection of the curve from the tangent. The point P_2' of the curve will then be taken at a distance $\frac{1}{2}\kappa_1(P_1P_2)^2$ from P_2 in the sense of the curvature, κ_1 being the curvature at P_1. The curvature at any point is given by the formula (2) of the preceding Article, and its value is calculated by the use of the (r, φ) relation and its derivative.

EXAMPLES XXXVII

POLAR COORDINATES

1. Sketch the following curves :
(i) $r=a\sin m\theta$, $(m=1, 2, 3, 4)$; (ii) $r=a\exp(-\theta^2/c^2)$;
(iii) $r=a\exp(\theta^2/c^2)$; (iv) $\theta=2a\cos r$, $(r>0)$; (v) $a/r=\sinh n\theta$.

2. Plot the curves $\tan(r/a)=k\sec\theta$, $\tanh(r/a)=k\sec\theta$.

3. Obtain the polar equations of the conic-sections directly from the focus-directrix definitions.

4. Transform the following equations to polar coordinates :
(i) $(x-b)^2+y^2=a^2$; (ii) $x^2-y^2=a^2$; (iii) $(x^2+y^2-3x)^2=4(x^2+y^2)$;
(iv) $y^2(a-x)=x^3$; (v) $x^3+y^3=3axy$.

5. Transform the following equations to rectangular coordinates :
(i) $r=2+3\cos\theta$; (ii) $r=ae^{k\theta}$; (iii) $r=a\cos^2\theta-b\sin^2\theta$;
(iv) $r=a\theta/\sin\theta$; (v) $r=(a\sin\theta)/\theta$.
Sketch the graphs of the last two equations.
[The graph (iv) is called the *quadratrix*.]

6. Sketch the curve $r=a\sqrt{(\cos\theta)}$. Find the length of the longest chord of the loop perpendicular to the axis of symmetry, and shew that if this chord cuts the axis in M, $OM=a/27^{1/4}$.

7. If a circle touches a given curve at the point (r, θ) and passes through the pole, shew that its diameter is $\sqrt{\{r^2+(dr/d\theta)^2\}}$ and hence that its equation is

$$R+r^2\frac{d}{d\theta}\left\{\frac{\sin(\vartheta-\theta)}{r}\right\}=0,$$

where (R, ϑ) are current coordinates on the circle.

If the given curve is $r^m=a^m\sin m\theta$, shew that the locus of the centre of the circle described is the curve $(2r)^n=a^n\sin n\theta$, where $n=m/(1-m)$.

8. Find the recedent at any point of the following curves :
(i) $r=a/\sqrt{\theta}$; (ii) $r=a\theta/(1+\theta)$; (iii) $r=a\theta^2/(\theta^2-1)$; (iv) $r\sin m\theta=a$;
(v) $r=ab/(ae^{\theta}+be^{-\theta})$; (vi) $u=\dfrac{\cos^3\theta+\sin^3\theta}{3a\sin\theta\cos\theta}$; (vii) $r=a(1-\cos\theta)$.

9. For the reciprocal spiral $r=a/\theta$, shew that $\tan\varphi=-\theta$. Shew also that the subtangent is equal to a.

10. Shew that if a point P describes an equiangular spiral of recedent $\pi-\alpha$, (greater than $\frac{1}{2}\pi$), so that the radius OP rotates with uniform angular velocity $\frac{2\pi}{T}\sin\alpha$, the projection of P on Ox executes a decaying S.H.M., the time in which the amplitude decreases in the ratio $1:e$ being $\frac{T}{2\pi}\sec\alpha$.

11. For the curve $r^m=a^m\sin m\theta$ find (i) the curvature, and (ii) the length of the normal. [(i) $(m+1)r^{m-1}/a^m$, (ii) $(m+1)/\kappa$.]

12. If P is a variable point on a given curve and OP is produced to P', where PP' is constant, prove that the polar subnormal of the locus of P' is equal to the polar subnormal of the given curve at the corresponding point.

Shew, further, that if the tangents at P, P' meet at T, the projection of $P'T$ on OP is equal to OP.

[The locus of P' is called a *conchoid* of the given curve relative to the centre O.]

13. Shew that if two curves are related in such a way that the product of their radii for the same polar angle is constant, the corresponding recedents are supplementary. [Each curve is said to be the *inverse* of the other.]

14. Prove the following results for the equiangular spiral $r=ae^{k\theta}$:

(i) subtangent $=r/k$; (ii) subnormal $=kr$;

(iii) tangent $=\frac{r}{k}\sqrt{(1+k^2)}$; (iv) normal $=r\sqrt{(1+k^2)}$.

Prove, further, that if a straight line drawn through the pole cuts the spiral in the consecutive points $P_1, P_2, \ldots, P_n, \ldots$, and κ_n is the curvature at P_n, then

$$\log_e(\kappa_m/\kappa_n)=2\pi k(n-m).$$

15. Investigate the points of contraflexure and the curvature at any point of the following curves:

(i) $r=a/\theta$; (ii) $r=a\theta/(1+\theta)$; (iii) $r=a\theta^n$;

(iv) $r=a\theta^2/(\theta^2-1)$; (v) $r=a+b\cos\theta$; (vi) $r=a(\theta+\sin\theta)$.

16. Shew that for the spiral of Archimedes, $r=a\theta$, the coordinates (r_c, θ_c) of the centre of curvature are given by

$$r_c^2=a^2\frac{r^4+3a^2r^2+a^4}{r^4+4a^2r^2+4a^4}, \quad \tan(\theta_c-\theta)=\frac{1+\theta^2}{\theta}.$$

17. Obtain the formula for the curvature in polar coordinates by transforming the formula

$$\kappa=\frac{d^2y/dx^2}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}.$$

If the coordinates of a point on the curve are expressed in terms of a parameter t, transform the formula (45), Ex. 7, Art. 133, to obtain the formula

$$\kappa=\pm\frac{r\frac{dr}{dt}\frac{d^2\theta}{dt^2}-r\frac{d\theta}{dt}\frac{d^2r}{dt^2}+2\left(\frac{dr}{dt}\right)^2\frac{d\theta}{dt}+r^2\left(\frac{d\theta}{dt}\right)^3}{\left\{\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2\right\}^{3/2}},$$

the positive (negative) sign being taken when $d\theta/dt$ is positive (negative).

18. If $r = a + bf(\theta)$, where b is small compared with a, shew that
$$1/\kappa \approx a + bf(\theta) + bf''(\theta).$$

19. A particle moves in a plane, the component velocities u, v at any instant along and perpendicular to the radius being given as functions of the position of the particle. Find the curvature of the path of the particle in the position (r, θ). $\left[\text{Put } \frac{1}{r}\frac{dr}{d\theta} = \frac{u}{v}.\right]$

20. Shew that for the conic $r = l/(1 + e\cos\theta)$ the curvature is given by
$$\kappa = \frac{(1 + e\cos\theta)^3}{l(1 + 2e\cos\theta + e^2)^{3/2}} = \frac{\sin^3\varphi}{l}.$$

21. Prove that
$$\frac{d\varphi}{d\theta} = \frac{u\dfrac{d^2u}{d\theta^2} - \left(\dfrac{du}{d\theta}\right)^2}{u^2 + \left(\dfrac{du}{d\theta}\right)^2}.$$

22. Shew that if $f(x, y) = 0$ is the equation of a curve relative to axes of obliquity ω, the equation in polar coordinates with the same origin and the polar axis along the axis of x is
$$f\left\{r\frac{\sin(\omega - \theta)}{\sin\omega}, r\frac{\sin\theta}{\sin\omega}\right\} = 0.$$

If the equation of a conic in oblique coordinates is $ax^2 + 2hxy + by^2 = 1$, find the directions of its axes. $\left[\tan 2\theta = \frac{2\sin\omega(a\cos\omega - h)}{a\cos 2\omega - 2h\cos\omega + b}.\right]$

23. Shew that the following functions V satisfy the corresponding equations:

(i) $V = \log_e r$, then $\dfrac{d}{dr}\left(r\dfrac{dV}{dr}\right) = 0$;

(ii) $V = (Ar^n + Br^{-n})\cos n\theta$, then $\dfrac{\partial^2 V}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial V}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 V}{\partial\theta^2} = 0$;

(iii) $V = \dfrac{f(at - r)}{r}$, then $\dfrac{\partial^2 V}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial V}{\partial r} - \dfrac{1}{a^2}\dfrac{\partial^2 V}{\partial t^2} = 0$;

(iv) $V = r^2\log_e r$, then $\dfrac{d}{dr}\left[r\dfrac{d}{dr}\left\{\dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{dV}{dr}\right)\right\}\right] = 0.$

24. Verify that the equation $\dfrac{\partial^2 U}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial U}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 U}{\partial\theta^2} = 0$, where
$$U = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2},$$
is satisfied by putting $V = r\theta\sin\theta$ or $V = r\log_e r\cos\theta$.

25. Verify that the function $V = a\ \text{exsin}_\theta(mr, \alpha)$ satisfies the equation
$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} = 0.$$

EXAMPLES XXXVIII

LINE-COORDINATES. ENVELOPES

1. Shew that the (l, m) equation of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ is $a^2l^2 - b^2m^2 = 1$, and of the parabola $y^2 = 4ax$, $l + am^2 = 0$.

2. For the following equations draw a sufficient number of lines to define the corresponding envelopes and identify the envelope in each case :

(i) $2l+3m=6$; (ii) $m=l^2$; (iii) $m=-l^3$;
(iv) $l^2+m^2=(1-l)^2$; (v) $2l^2+m^2=1$; (vi) $lm+l+m=0$.

3. Find the (p, ω) line-equation for the parabola $y^2=4a(x+a)$.
[$p=-a\sec\omega$.]

4. Find the (l, m) equation for the general conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

(i) when the centre is at the origin, (ii) when the centre is at any point.

If there are tangents through the origin, shew that their gradients γ_1, γ_2 satisfy the equations

$$b\gamma^2+2h\gamma+a=0, \qquad \gamma^2(bc-f^2)-2\gamma(fg-ch)+ac-g^2=0$$

in the respective cases.

[The general form of the equation is $Al^2+2Hlm+Bm^2+2Gl+2Fm+C=0$, where $A, B, \ldots, H$ are expressible in terms of $a, b, \ldots, h$.]

5. Shew that for the astroid $x^{2/3}+y^{2/3}=a^{2/3}$,

$$1/l^2+1/m^2=a^2, \quad 4p^2=a^2\sin^2 2\psi.$$

6. Shew that for the cycloid $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$, the (p, ψ) equation is $p=2a\psi\sin\psi$.

7. Shew how to find the (x, y) point-equation from the (g, b) line-equation.

8. Find the point-equations of the curves for which (i) $p=a\tan\frac{1}{2}\omega$, (ii) $p=a\cos\omega$. [In (i) put $\tan\frac{1}{2}\omega=t$.]

9. If the line-coordinates (l, m) of a curve are specified in terms of a parameter t, shew that

$$\kappa=\pm\frac{\left(m\dfrac{dl}{dt}-l\dfrac{dm}{dt}\right)^3}{\left(\dfrac{dm}{dt}\dfrac{d^2l}{dt^2}-\dfrac{dl}{dt}\dfrac{d^2m}{dt^2}\right)(l^2+m^2)^{3/2}},$$

the upper (lower) sign being taken if m is positive (negative).

Find the curvature of the envelope of the lines $Lx+My+N=0$, where L, M, N are given as functions of a parameter t.

10. Find the curvature at any point of the ellipse

$$Al^2+2Hlm+Bm^2+2Gl+2Fm+C=0.$$

11. If a curve is specified in line-coordinates (g, b), shew that

$$\kappa=\frac{(dg/db)^3}{(1+g^2)^{3/2}d^2g/db^2}=-\frac{1}{(1+g^2)^{3/2}d^2b/dg^2}.$$

12. If C is the centre of curvature corresponding to a point P on a curve, and if OU is the perpendicular from O to CP, prove that $OU^2=(dp/d\omega)^2$.

Applying this result to the evolute, where it is to be observed that OU is the perpendicular to the tangent of the evolute at the centre of curvature, establish the result $UC^2=(d^2p/d\omega^2)^2$.

Hence shew that if (r_c, θ_c) is the centre of curvature,

$$r_c^2=\left(\frac{dp}{d\omega}\right)^2+\left(\frac{d^2p}{d\omega^2}\right)^2, \quad \cot(\omega-\theta_c)=\frac{d}{d\omega}\log_e\left|\frac{dp}{d\omega}\right|, \quad \kappa^2 r_c\frac{dr_c}{d\kappa}=-\frac{d^2p}{d\omega^2}.$$

Find OU for the cardioid $r = a(1 + \cos\theta)$ corresponding to the point for which $\theta = \frac{1}{4}\pi$.

13. Circles are described having their centres on Ox and radii equal to the corresponding ordinates of the parabola $y^2 = 4ax$. Shew that their envelope is the parabola $y^2 = 4a(x + a)$.

14. Prove that the envelope of the normals of a curve is the locus of the centre of curvature (the evolute).

15. Find the evolute of (i) the parabola $y^2 = 4ax$; (ii) the ellipse $x = a\cos\theta$, $y = b\sin\theta$; (iii) the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$; (iv) the tractrix $x = c(u - \tanh u)$, $y = c\,\mathrm{sech}\,u$.

16. Circles are drawn with their centres on (i) the hyperbola $xy = a^2$, (ii) the parabola $y^2 = 4ax$, and in each case pass through the origin. Find the corresponding envelopes.

17. Ellipses are drawn with their centres at the origin and their axes along Ox, Oy. If the sum of the semi-axes is constant, find the envelope of the family. [An astroid.]

18. Lines are drawn from the foot M of the ordinate MP perpendicular to the tangent at P to a curve. Find their envelope when the curve is specified by the equations (i) $x = a\cos\theta$, $y = b\sin\theta$, (ii) $y = ax^m$, (iii) $y = c\cosh(x/c)$.

19. Rays of light emanating from a point C are reflected from a circular cylindrical mirror. Find the envelope of the reflected rays which lie in the plane through C at right-angles to the axis of the cylinder.

In particular, for the cases where C is (i) on the reflecting surface, (ii) at a great distance (so that the rays are parallel), shew that the envelopes are epicycloids.

[Let O be the centre of the circular section, a the radius and c the distance of C from O. Let P be any point on the mirror, where $\angle COP = \theta$, and draw the incident ray CP and the reflected ray PQ, where $\angle CPO = \angle QPO$. If CL is perpendicular to PQ, express the coordinates of P, L in terms of a, c, θ, and hence obtain the equation of PQ in terms of the parameter θ.]

20. Rays of light from a point O are reflected from parallel infinitesimal facets lying along a straight line AB, the planes of the facets being perpendicular to the plane OAB. Show that the reflected rays envelop a parabola.

EXAMPLES XXXIX

THE (p, r) RELATION

1. Shew that the (p, r) relation for the parabola $y^2 = 4a(x + a)$, the focus being at the pole, is $p^2 = ar$.

2. Shew that if a focus of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ is taken as pole, the (p, r) relations for the branches which are convex and concave relative to the outward drawn radius are $l/p^2 = 2/r + 1/a$ and $l/p^2 = -2/r + 1/a$, respectively.

3. Obtain the following (p, r) relations for the curves given :

(i) spiral of Archimedes $r = a\theta$, $p^2 = r^4/(a^2 + r^2)$;

(ii) reciprocal spiral $r = a/\theta$, $1/p^2 = 1/r^2 + 1/a^2$;

(iii) rectangular hyperbola $r^2 \cos 2\theta = a^2$, $pr = a^2$;

(iv) lemniscate $r^2 = a^2 \cos 2\theta$, $a^2 p = r^3$;

(v) Cotes's spiral $r = a \sec m\theta$, $1/p^2 = (1 - m^2)/r^2 + m^2/a^2$;

(vi) $r = a/\cosh m\theta$, $1/p^2 = (1 + m^2)/r^2 - m^2/a^2$;

(vii) $r = a/\sinh m\theta$, $1/p^2 = (1 + m^2)/r^2 + m^2/a^2$;

(viii) $r^2 = a^2 \cos \theta$, $p^2 = 4r^6/(a^4 + 3r^4)$;

(ix) $r^n = a^n \cos n\theta$, $a^n p = r^{n+1}$.

4. If $f(p, r) = 0$ is the (p, r) relation for the curve $F(r, \theta) = 0$ and if r is changed to r^n and θ to $n\theta$, prove that the corresponding (p, r) relation is obtained from $f(p, r) = 0$ by replacing r by r^n and p by pr^{n-1}.

5. Find the curvature at any point of the following curves :

(i) $p^2 = ar$; (ii) $a^n p = r^{n+1}$; (iii) $p = a - b/r$; (iv) $a^2 b^2/p^2 = a^2 + b^2 - r^2$.

END OF VOLUME I

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